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EQUILIBRIUM FOR PERTURBATIONS OF MULTIFUNCTIONS BY CONVEX PROCESSES

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Abstract. We present a general equilibrium theorem for the sum of an upper hemicontinuous convex valued multifunction and a closed convex process defined on a noncompact subset of a normed space. The lack of compactness is compensated by inwardness conditions related to the existence of viable solutions of some differential inclusion.

1. Introduction and Preliminaries

Equilibrium theorems provide sufficient conditions for the existence of an *equilibrium* (or a zero) for a given multifunction Φ under certain constraints, that is, a solution to the inclusion $0 \in \Phi(x)$ required to belong to a certain constraint set X.

Many important problems in nonlinear analysis can be reduced to equilibrium problems (for example, the problem of existence of critical points for smooth and non-smooth functions, the problem of existence of stationary solutions to differential inclusions, etc.). In the mid-seventies B. Cornet [1] derived an equilibrium theorem for multifunctions defined on compact convex constraint sets from classical results of Ky Fan and F. Browder by using the celebrated inf-sup inequality of Ky Fan [2] (see, for instance, Aubin [3] and Aubin and Frankowska [4] and references therein).

The purpose of this paper is to present an equilibrium theorem for the sum of an upper hemicontinuous multifunction with closed convex values and a closed convex process defined on a closed convex subset of a normed space (Theorem 3.1 below). The lack of compactness of the domain X is compensated by tangency conditions of the Ky Fan-type on a compact subset K of X and outside of it. These tangency conditions are necessary for the

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existence of viable solutions to a general differential inclusion. Our equilibrium theorem generalizes a classical result of G. Haddad [5] on the existence of a stationary point of differential inclusions to noncompact viability domains. It contains as particular cases results in [1], [4], Ben-El-Mechaiekh [6], Bae and Park [7], Simons [8], and others.

All topological spaces in this paper are assumed to be Hausdorff spaces. The *closure* and the *boundary* of a subspace A of a topological space are denoted by $\overline{A}$ and ∂A respectively. The *open ball* of radius $\epsilon > 0$ around a subset A of a normed space E is denoted by $B_E(A, \epsilon)$.

Locally convex real topological vector spaces are simply called *locally convex spaces.* The *topological dual* of a topological vector space E is denoted by E^* and the *convex hull* of a subset K of E is denoted by $\mathrm{co}(K)$.

Given a subset X of a topological vector space E and an element $x \in E$, $S_X(x)$ denotes the cone $\bigcup_{t>0} \frac{1}{t}(X-x)$ spanned by $X-x$, and $N_X(x)$ denotes the set $\{\varphi \in E^* : \langle\varphi, x\rangle \geq \sup_{v\in X}\langle\varphi, v\rangle\}$. Observe that if E is a normed space, X is convex in E, and $x \in X$, then $\overline{S_X(x)}$ is precisely the tangent cone to X at x and $N_X(x)$ is the negative polar cone of $\overline{S_X(x)}$, called the *normal cone* to X at x.

Given a subset K of a topological vector space E, the *support function* of K is the function $\sigma(K, .) : E^* \longrightarrow R \cup \{+\infty\}$ defined as

$$\sigma(K, \varphi) := \sup_{y\in K}\langle\varphi, x\rangle, \quad \varphi \in E^*. \tag{1}$$

We recall that the *negative* (respectively *positive*) *polar cone* of K is the set

$$K^- := \{\varphi \in E^* : \sigma(K, \varphi) \leq 0\} \quad (K^+ = -K^- \text{ respectively}). \tag{2}$$

Observe that given two subsets K_1, K_2 of E and $\varphi \in E^*$,

$$\sigma(K_1 + K_2, \varphi) = \sigma(K_1, \varphi) + \sigma(K_2, \varphi). \tag{3}$$

In particular, if K_2 is a cone, then $\sigma(K_1 + K_2, \varphi) = \sigma(K_1, \varphi)$ if $\varphi \in K_2^-$ and $+\infty$ otherwise.

By the Hahn–Banach separation theorem, the closed convex hull of K is characterized by

$$\overline{\mathrm{co}(K)} = \{x \in E : \langle\varphi, x\rangle \leq \sigma(K, \varphi), \forall\varphi \in E^*\}. \tag{4}$$

A *multifunction* Φ from a set X into a set F is a map from X into the family $\mathcal{P}(F)$ of all subsets of F. The multifunction Φ is said to be *strict* if all of its values are nonempty. The *domain* of Φ is the set of all those elements $x \in X$ for which $\Phi(x)$ is nonempty. If F is a vector space, an element $x_0 \in X$ is said to be an *equilibrium* for a multifunction $\Phi : X \longrightarrow \mathcal{P}(F)$ if $0 \in \Phi(x_0)$.

This paper is particularily concerned with two types of multifunctions, namely *upper hemicontinuous* multifunctions with convex values and *closed convex processes*. Upper hemicontinuity is a weak form of upper semicontinuity, and closed convex processes (introduced by Rockafellar [9]) are multivalued analogues of linear operators. For details concerning these two concepts, we refer to [4].

Definition 1.1. A multifunction $\Phi : X \longrightarrow \mathcal{P}(F)$ from a topological space X into a topological vector space F is said to be upper hemicontinuous at $x_0 \in X$ if for any $\psi \in F^*$, the function $x \longmapsto \sigma(\Phi(x), \psi)$ is upper semicontinuous at x_0. It is said to be upper hemicontinuous on X, if and only if it is upper hemicontinuous at each point of X.

Any upper semicontinuous multifunction from X into F supplied with the weak topology is upper hemicontinuous. Conversely, if Φ is upper hemicontinuous at a point x_0 and if $\Phi(x_0)$ is convex and weakly compact, then Φ is also upper semicontinuous at x_0. We recall that a finite sum and a finite product of upper hemicontinuous multifunctions is upper hemicontinuous.

Definition 1.2. (i) A closed convex process Λ from a locally convex space E into a locally convex space F is a multifunction $\Lambda : E \longrightarrow \mathcal{P}(F)$ whose graph is a closed convex cone.

(ii) The transpose of a closed convex process $\Lambda : E \longrightarrow \mathcal{P}(F)$ is the closed convex process $\Lambda^* : F^* \longrightarrow \mathcal{P}(E^*)$ defined by

$$\varphi \in \Lambda^*(\psi) \iff \langle \varphi, x\rangle \leq \langle \psi, y\rangle,\ \forall x \in E, \forall y \in \Lambda(x). \tag{5}$$

Important examples of closed convex processes are provided by contingent derivatives of multifunctions.

If the domain of Λ is the whole space E, then the domain of Λ^* is the positive polar cone $\Lambda(0)^+$.

A *linear process* is a multifunction whose graph is a vector subspace.

The support functions of a closed convex process $\Lambda : E \longrightarrow \mathcal{P}(F)$ and of its transpose Λ^* are related by the following

Lemma 1.3 (see [4]). *For every ψ_0 in the interior of the domain of Λ^* and for every x_0 in the barrier cone $b(\Lambda^*(\psi_0)) := \{x \in E : \sigma(\Lambda^*(\psi_0), x) < +\infty\}$ of $\Lambda^*(\psi_0)$, there exists $y_0 \in \Lambda(x_0)$ such that*

$$\langle \psi_0, y_0\rangle = \sigma(\Lambda^*(\psi_0), x_0) = -\sigma(\Lambda(x_0), -\psi_0). \tag{6}$$

Let E and F be two normed spaces. Denote by $\Lambda(E, F)$ the normed space of all closed convex processes from E into F equipped with the norm

$$\|\Lambda\| := \sup_{u \in E\setminus\{0\}} \inf_{v \in \Lambda(u)} \frac{\|v\|}{\|u\|}.$$

The uniform boundedness principle holds true for convex processes between normed spaces (again, see [4]); it implies the following crossed convergence property.

Let us recall first that a mapping $\Lambda : X \longrightarrow \Lambda(E,F)$ is said to be *pointwise bounded* if and only if

$$\forall u \in E, \exists v_x \in \Lambda(x)(u) \text{ with } \sup_{x \in X} \|v_x\| < +\infty. \tag{7}$$

Lemma 1.4 ([10]). *Let X be a topological space, E, F be normed spaces, and $\Lambda : X \longrightarrow \Lambda(E,F)$ be pointwise bounded. Then the following conditions are equivalent:*

(a) *the multifunction $x \longmapsto \operatorname{graph}(\Lambda(x))$ is lower semicontinuous;*

(b) *the multifunction $(x,u) \longmapsto \Lambda(x)(u)$ is lower semicontinuous.*

2. Tangency in the sense of Ky Fan

Let E, F be two topological vector spaces, X be a subset of E, $\Phi : X \longrightarrow \mathcal{P}(F)$ be a multifunction with domain X, and $\Lambda : X \longrightarrow \Lambda(E,F)$ be a mapping.

Definition 2.1. Given a subset X_1 of X and a subset X_2 of E, we say that the multifunction Φ satisfies the condition of Ky Fan on (X_1, X_2) with respect to Λ if and only if

$$\overline{\Lambda(x)(-S_{X_2}(x))} \cap \Phi(x) \neq \varnothing, \quad \forall x \in X_1. \tag{8}$$

In other terms, the directions of Φ on X_1 are controlled by a family of cones depending on X_1 and X_2. Weaker forms of this tangency condition were considered by various authors (e.g., [3], [6, [11], [8]). In the case where $E = F$, $\Lambda(x) \equiv -Id_E, X_1 = X_2 = X$, (8) reads as

$$\Phi(x) \cap \overline{S_X(x)} \neq \varnothing, \quad \forall x \in X,$$

that is, the multifunction Φ is *inward* in the sense of Ky Fan [12].

Lemma 2.2. *If the multifunction Φ satisfies the condition of Ky Fan on (X_1, X_2) with respect to Λ, then*

$$\inf\{\sigma(\Phi(x), \psi) : \psi \in F^*, \Lambda(x)^*(\psi) \cap N_{X_2}(x) \neq \varnothing\} \geq 0. \tag{9}$$

The converse holds true whenever the values of Φ are compact.

Proof. Let $x \in X_1$ be arbitrary and let $\psi \in F^*$ be such that $\Lambda(x)^*(\psi) \cap N_{X_2}(x) \neq \varnothing$. Choose an element $y \in \Phi(x) \cap \overline{\Lambda(x)(-S(x))}$, that is, y is the limit of a net $\{y_i\}$ with $y_i \in \Lambda(x)(v_i), -v_i = \frac{1}{t_i}(x_i - x) \in S_{X_2}(x), x_i \in X_2, t_i > 0$.

Clearly, $\forall \varphi \in \Lambda(x)^*(\psi) \cap N_{X_2}(x)$, $\langle \varphi, x \rangle \geq \langle \varphi, x_i \rangle$ and thus $\langle \varphi, v_i \rangle \geq 0$, $\forall i$. By (5) $\langle \psi, y_i \rangle \geq \langle \varphi, v_i \rangle \geq 0$, $\forall i$. Hence $\sigma(\Phi(x), \psi) \geq \langle \psi, y \rangle \geq 0$.

Conversely, given $x \in X_1$, if $\Phi(x)$ is compact, then the set $\Phi(x) - \overline{\Lambda(x)(-S_{X_2}(x))}$ is closed and convex. Assume that (9) is satisfied and let $\psi \in F^*$ be such that $\Lambda(x)^*(\psi) \cap N_{X_2}(x) \neq \varnothing$ and $\sigma(\Phi(x), \psi) \geq 0$. Observe that $\Lambda(x)^*(\psi) \cap N_{X_2}(x) \neq \varnothing$ is equivalent to $\psi \in \Lambda(x)^{*-1}(N_{X_2}(x)) = \Lambda(x)^{*-1}(S_{X_2}(x)^-) = \Lambda(x)^{*-1}(-S_{X_2}(x)^+)$ which, by the bipolar theorem (Theorem 2.5.7 in [4]), equals $-[\Lambda(x)(-S_{X_2}(x))]^- = -\overline{\Lambda(x)(-S_{X_2}(x))}^-$. By the remark following (3), $\sigma(\Phi(x) - \overline{\Lambda(x)(-S_{X_2}(x))}, \psi) = \sigma(\Phi(x), \psi)$. The characterization (4) ends the proof. $\square$

Condition (8) is necessary for the solvability of the following differential inclusion:

$$\Lambda(x'(t)) \subset \Phi(x(t)), \quad t \in [0, T], \tag{10}$$

where $\Lambda : R^n \longrightarrow \mathcal{P}(R^m)$ is a linear process, and $\Phi : R^n \longrightarrow \mathcal{P}(R^m)$ is an upper hemicontinuous multifunction with closed convex values. Let K be a subset of R^n containing the initial point $x(0) = x_0$.

Proposition 2.3. *If $x(.)$ is a solution of problem* (10) *satisfying the following condition:*

$$\forall T' \in (0, T], \exists t \in (0, T'] \text{ such that } x(t) \in K, \tag{11}$$

then $\overline{\Lambda(S_K(x_0))} \cap \Phi(x_0) \neq \varnothing$, that is, $-\Phi$ satisfies the condition of Ky Fan on $(\{x_0\}, K)$ with respect to Λ.

Proof. By (11) there exists a sequence of positive reals $\{t_k\}_{k \in N}$ converging to 0^+ such that $x(t_k) \in K$. Since $\forall \psi \in R^m$, the real function $x \longmapsto \sigma(\Phi(x), \psi)$ is upper semicontinuous, then $\forall \epsilon > 0, \exists \delta_\psi > 0$ such that

$$\sigma(\Lambda(x'(\tau)), \psi) \leq \sigma(\Phi(x(\tau)), \psi) < \sigma(\Phi(x_0), \psi) + \epsilon \|\psi\|, \quad \forall \tau \in [0, \delta_\psi].$$

The definition of the transpose of a closed convex process (5) implies that

$$\forall \tau \in [0, \delta_\psi], \langle \varphi, x'(\tau) \rangle < \sigma(\Phi(x_0), \psi) + \epsilon \|\psi\|, \quad \forall \varphi \in \Lambda^*(\psi).$$

Hence,

$$\begin{aligned} \forall k, \forall \varphi \in \Lambda^*(\psi), \frac{1}{t_k} \int_0^{t_k} \langle \varphi, x'(\tau) \rangle d\tau &\leq \frac{1}{t_k} \int_0^{t_k} [\sigma(\Phi(x_0), \psi) + \epsilon \|\psi\|] d\tau \\ &= \sigma(\Phi(x_0), \psi) + \epsilon \|\psi\|. \end{aligned}$$

We conclude that $\sigma(\Lambda^*(\psi), v_k\rangle \leq \sigma(\Phi(x_0), \psi) + \epsilon \|\psi\|, v_k = \frac{1}{t_k}(x(t_k) - x_0)$, $\forall k$. By Lemma 1.3 and since the domain of $\Lambda^* = \Lambda(0)^+ = \{0\}^+ = R^m, \forall k$, $\exists y_k \in \Lambda(v_k)$ such that $\langle \psi, y_k \rangle = \sigma(\Lambda^*(\psi), v_k\rangle$. Being bounded by the uniform boundedness principle, the sequence $\{y_k\}_{k \in N}$ converges to some $y \in \overline{\Lambda(S_K(x_0))}$ satisfying the inequality $\langle \psi, y \rangle \leq \sigma(\Phi(x_0), \psi) + \epsilon \|\psi\|$. Since

ϵ and ψ are arbitrary, (4) implies that $y \in \Phi(x_0)$. Finally, observe that since $\Lambda(0) = \{0\}$, then $\overline{\Lambda(S_K(x_0))} \cap \Phi(x_0) \neq \varnothing$ implies that $\overline{\Lambda(-S_K(x_0))} \cap -\Phi(x_0) \neq \varnothing$. □

Remark. If for all $x_0 \in K$, there is a viable trajectory $x(.)$ of the differential inclusion (10) in K (that is, $x(t) \in K$, $\forall t \in [0,T]$) starting at x_0, then (11) is obviously satisfied and therefore $-\Phi$ satisfies the condition of Ky Fan on (K, K) with respect to Λ.

We will see later (Corollary 3.2 below) that one of the consequences of our main theorem is the existence of a stationary solution of (10) for noncompact viability domains.

3. The Main Theorem

We state now our main theorem.

Theorem 3.1. *Let X be a convex subset of a normed space E, F be a normed space, $\Phi : X \longrightarrow \mathcal{P}(F)$ be a strict upper hemicontinuous multifunction with closed convex values, and $\Lambda : X \longrightarrow \Lambda(E, F)$ be a continuous mapping satisfying the boundedness condition:*

(i) *$\exists M > 0$ such that $\forall x \in X$, $\forall u \in E$ with $\|u\| = 1$, $\exists v \in \Lambda(x)(u)$ such that $\|v\| \leq M$.*

Furthermore, assume that there exists a compact subset K of X such that:

(ii) *for each finite subset N of X, there exists a compact convex subset C_N of X containing N such that Φ satisfies the condition of Ky Fan on $(C_N \setminus K, C_N)$ with respect to Λ;*

(iii) *Φ satisfies the condition of Ky Fan on (K, X) with respect to Λ.*

Then,

(A) *Φ has an equilibrium;*

(B) *$\forall x_0 \in X$, the multifunction $\Phi(.) + \Lambda(.)(. - x_0)$ has an equilibrium.*

Remarks. (**1**) This theorem remains valid in the context of spaces having separating duals (e.g., locally convex spaces) or convex spaces in the sense of [13].

(**2**) If $\forall x \in X$, $\Lambda(x)$ is a linear process, then (B) is the coincidence property:

$$\forall x_0 \in X, \exists \hat{x} \in X \text{ such that } \Lambda(\hat{x})(x_0) \cap \Phi(\hat{x}) + \Lambda(\hat{x})(\hat{x}) \neq \varnothing.$$

(**3**) In the case where X is compact, (i) follows from the continuity of the operator Λ, and putting $K = \varnothing$ and $C_N = X$ for any N, (ii) and (iii) reduce to:

(ii)′ Φ satisfies the condition of Ky Fan on (X, X) with respect to Λ; that is the direction of the multifunction Φ is controlled by a family of convex cones. In this case, if for each interior point $x \in X$, $\Lambda(x)$ is a surjective process, (ii)$'$ is simply the tangency condition:

Φ satisfies the condition of Ky Fan on $(\partial X, X)$ with respect to Λ.

When X is compact and $\Lambda(x) = \ell(x), x \in X$, is a bounded linear operator from E into F this result is precisely the solvability theorem in [4].

(**4**) In the case where $\forall x \in X$, $\Lambda(x) \equiv \ell$ is a bounded linear operator from E into F, and $C_N = C$ is the same for all finite subsets N of X, this result can be found in [6]. In this case (iii) is equivalent to:

Φ satisfies the condition of Ky Fan on $(K \cap \partial X, X)$ with respect to ℓ,

and (B) guarantees the surjectivity of the perturbation $\Phi + \ell$ onto $\ell(X)$.

(**5**) In order to formulate our next result, we will consider a weaker form of condition (ii) involving the concept of c-compactness of [13] in our main theorem (for a particular case of this particular instance, see [7]). Recall that a subset C of X is said to be c-compact in X if for each finite subset N of X there exists a compact convex subset C_N of X such that $C \cup N \subset C_N$. Note that any bounded subset of a finite dimensional space is c-compact. Condition (ii) in Theorem 8 may be replaced by

$$\text{(ii)}'' \ \exists C \ c\text{-compact } \subseteq X \text{ such that } \overline{\Lambda(-S_{\mathrm{co}(\{x\}\cup C)}(x)} \cap \Phi(x) \neq \varnothing,$$
$$\forall x \in X \setminus K.$$

As an immediate consequence of Remark 5 above, we obtain that existence of viable solutions in a noncompact domain implies the existence of a stationary solution (or rest point) of the inclusion (10). More precisely, we have

Corollary 3.2. *Let X be a closed convex subset of R^n, K a compact subset of X, C a c-compact subset of X, $\Lambda : R^n \longrightarrow \mathcal{P}(R^m)$ a strict linear process, and $\Phi : X \longrightarrow \mathcal{P}(R^m)$ a strict upper hemicontinuous multifunction with closed convex values.*

Assume that the following properties are satisfied:

(i) $\forall x_0 \in K$, $\exists x(.)$ *solution of* (10) *starting at* x_0, *such that* $\forall T' \in (0,T]$, $\exists t \in (0, T']$ *with* $x(t) \in X$;

(ii) $\forall x_0 \in X \setminus K, \exists x(.)$ *solution of* (10) *starting at* x_0, *such that* $\forall T' \in (0,T]$, $\exists t \in (0,T']$ *with* $x(t) \in \mathrm{co}(\{x_0\} \cup C)$;

Then Φ has an equilibrium in X.

Remarks. (**1**) Conditions (i)–(ii) state that if a trajectory of (10) starts in K then it must first enter X, and if it starts in $X \setminus K$ then it is first attracted by C in the sense that the trajectory must first enter the drop with vertex at the initial point and base C.

(**2**) When X is a compact convex viability domain of Φ, $R^n = R^m$ and $\Lambda = Id_{R^n}$, then (i) and (ii) are obviously satisfied with $K = C = X$ and we retrieve the equilibrium theorem of [4].

4. Proof of the Main Theorem

The starting point is the following generalization of the Browder–Ky Fan fixed point theorem for convex-valued multifunctions with open fibers defined on noncompact domains.

Theorem 4.1 ([14]). *Let X be a convex subset of a topological vector space and $\Phi : X \longrightarrow \mathcal{P}(X)$ be a multifunction satisfying the following properties:*

(i) $\forall y \in Y, \Phi^{-1}(y)$ *is open in* X;

(ii) $\forall x \in X, \Phi(x)$ *is convex nonempty;*

(iii) *there exists a compact subset K of X such that for any finite subset N of X there exists a compact convex subset C_N of X containing N such that $\Phi(x) \cap C_N \neq \varnothing$, $\forall x \in C_N \setminus K$.*

Then, $\exists x_0 \in X$ with $x_0 \in \Phi(x_0)$.

Proof. Step 1. Assuming for simplicity that X is compact (this correponds to the Browder–Ky Fan theorem), we can take $K = \varnothing$ and $C_N = X$ for any finite subset N of X and show that Φ has a fixed point as in [15]. First observe that the multifunction Φ has a so-called *Kuratowski selection*, that is: there exist a subset $N = \{y_1, ..., y_n\} \subset X$ and a single-valued continuous mapping $s : X \to \mathrm{co}(N)$ such that $s(x) \in \Phi(x)$, $\forall x \in X$. Indeed, observe that the family $\{\Phi^{-1}(y) : y \in X\}$ forms an open cover of X. Since X is compact, there exists a finite subset $N = \{y_1, ..., y_n\}$ of X such that $X = \bigcup_{i=1}^n \Phi^{-1}(y_i)$. Let $\{\lambda_i : i = 1, ..., n\}$ be a continuous partition of unity subordinated to the open cover $\{\Phi^{-1}(y_i); i = 1, ..., n\}$ of X. Define the continuous mapping $s : X \longrightarrow \mathrm{co}(N)$ by putting

$$s(x) := \sum_{i=1}^{n} \lambda_i(x) y_i,\ x \in X.$$

Let $x \in X$ be arbitrary. If $\lambda_i(x) \neq 0$, then $x \in \Phi^{-1}(y_i)$, hence $y_i \in \Phi(x)$. Since $\Phi(x)$ is convex, $s(x) \in \Phi(x)$. (Note here that the paracompactness of X is sufficient for the existence of a continuous selection for Φ; without compactness, however, the range of this selection is not necessarily finite-dimensional.)

By the Brouwer fixed point theorem, the mapping s restricted to $\mathrm{co}(N)$, $s : \mathrm{co}(N) \longrightarrow \mathrm{co}(N)$ has a fixed point which is also a fixed point for Φ.

Step 2. Now by the same argument above, the restriction of Φ to the compact set K has a Kuratowski selection s with values in a convex polytope $\mathrm{co}(N)$ where N is a finite subset of X. Let C_N be the compact convex subset of X provided by (iii) and containing the convex hull $\mathrm{co}(N)$ of N. Define the "compression" of Φ to C_N, $\Phi_N : C_N \longrightarrow \mathcal{P}(C_N)$, as follows:

$$\Phi_N(x) := \Phi(x) \cap C_N, x \in C_N.$$

Let us first observe that Φ_N has nonempty values. For, if $x \in C_N \cap K$, then $s(x) \in \Phi(x) \cap \mathrm{co}(N) \subset \Phi(x) \cap C_N$; and if $x \in C_N \setminus K$, then $\Phi_N(x) \neq \varnothing$ by (iii). It is clear that Φ_N has convex values and open fibers. Hence, it has a fixed point by Step 1. This fixed point is also a fixed point for Φ. $\square$

It is well known that the celebrated inf-sup inequality of Ky Fan [2] is an equivalent analytical formulation of the Browder–Ky Fan fixed point theorem. The systematic formulation of coincidence and fixed point theorems as nonlinear alternatives was presented in Ben-El-Mechaiekh, Deguire, and Granas [14] and in [16]. Theorem 4.1 has the following convenient analytical formulation.

Proposition 4.2. *Let X be a convex subset in a topological vector space, and $f : X \times X \longrightarrow R \cup \{+\infty\}$ be a function satisfying the following properties:*

(i) $\forall y \in X, x \longmapsto f(x,y)$ *is lower semicontinuous on* X*;*

(ii) $\forall x \in X$, $y \longmapsto f(x,y)$ *is quasiconcave on* X.

Assume that there exists a compact subset K of X such that for every finite subset N of X there exists a compact convex subset C_N of X such that:

(iv) $\forall x \in C_N \setminus K$, $\exists y \in C_N \cap X$ *with* $f(x,y) > 0$.

Then one of the following properties holds:

(a) $\exists x_0 \in K$ *such that* $f(x_0, y) \leq 0$, $\forall y \in X$; *or*

(b) $\exists y_0 \in X$ *such that* $f(y_0, y_0) > 0$.

Proof. Apply Theorem 4.1 to the multifunction $\Phi : X \longrightarrow \mathcal{P}(X)$ defined as

$$\Phi(x) := \{y \in X : f(x,y) > 0\}, x \in X. \quad \square$$

The next result is a more general version of a remarkable theorem of Ky Fan [12] and contains results of [17], [6], [18], and [8].

Theorem 4.3. *Let X be a convex subset in a topological vector space, Y a subset in $\{\varphi : X \longrightarrow R; \varphi$ is upper semicontinuous and quasiconcave$\}$, and $\Psi : X \longrightarrow \mathcal{P}(Y)$ be a multifunction. Assume that the following properties are satisfied:*

(i) Ψ *admits a continuous selection* s;

(ii) *there exists a compact subset K of X such that for each finite subset N of X there exists a compact convex subset C_N of X containing N such that*

$$\forall x \in C_N \setminus K, \quad \forall \varphi \in \Psi(x), \quad \varphi(x) < \max_{u \in C_N} \varphi(u).$$

Then,

$$\exists x_0 \in K, \exists \varphi_0 \in \Psi(x_0) \text{ such that } \varphi_0(x_0) = \max_{u \in X} \varphi_0(u).$$

Proof. Define $f : X \times X \longrightarrow R \cup \{+\infty\}$ by putting

$$f(x,y) := s(x)(y) - s(x)(x), \quad (x,y) \in X \times X.$$

The function f satisfies hypotheses (i)–(ii) of Proposition 4.2. Moreover, given any finite subset N of X, let $x \in C_N \setminus K$ and choose an $y \in C_N$ satisfying $s(x)(y) = \max_{u \in C_N} s(x)(u)$. By (ii), $s(x)(x) < s(x)(y)$, that is, $f(x,y) > 0$; hypothesis (iii) of Proposition 4.2 is thus satisfied. Since $f(y,y) = 0, \forall y \in X$, it follows that property (a) of Proposition 4.2 holds, that is, $f(x_0, y) \leq 0$ for some $x_0 \in X$ and all $y \in X$. The proof is complete with $\varphi_0 = s(x_0)$. $\square$

Corollary 4.4. *Let X be a convex subset of a normed space E, F be a normed space, $\Lambda : X \longrightarrow \Lambda(E,F)$ be a continuous mapping, and $f : X \times F^* \longrightarrow R \cup \{+\infty\}$ be two real functions satisfying the following conditions:*

(i) $\exists M > 0$ *such that* $\forall x \in X$, $\forall u \in E$ *with* $\|u\| = 1$, $\exists v \in \Lambda(x)(u)$ *such that* $\|v\| \leq M$*;*

(ii) $\forall \psi \in F^*$, $x \longmapsto f(x,\psi)$ *is upper semicontinuous on* X*;*

(iii) $\forall x \in X$, $\psi \longmapsto f(x,\psi)$ *is quasiconvex on* F^*.

Assume that there exists a compact subset K of X such that for each finite subset N of X there exists a compact convex subset C_N of X containing N such that

(iv) $\forall x \in C_N \setminus K, \forall \psi \in F^*, f(x,\psi) \geq 0$ *provided that* $\Lambda(x)^*(\psi) \cap N_{C_N}(x) \neq \varnothing$.

Then one of the following conditions is satisfied:

(1) $\exists \widehat{x} \in X$ *such that* $f(\widehat{x},\psi) \geq 0$, $\forall \psi \in F^*$; *or*

(2) $\exists (x_0, \psi_0) \in K \times F^*$ *such that* $\Lambda(x_0)^*(\psi_0) \cap N_X(x_0) \neq \varnothing$ *and* $f(x_0, \psi_0) < 0$.

Proof. Note first that being a subset of a normed space, X is paracompact. Assume that conclusion (1) fails and define the multifunction $\Theta : X \longrightarrow \mathcal{P}(F^*)$ with domain X by putting

$$\Theta(x) := \{\psi \in F^* : f(x,\psi) < 0\}, x \in X.$$

We claim that the multifunction $\Psi : X \longrightarrow \mathcal{P}(E^*)$ defined as

$$\Psi(x) := \Lambda(x)^*(\Theta(x)), x \in X,$$

admits a continuous selection.

The multifunction Ψ can be viewed as the composition product $X \stackrel{1_X \times \Theta}{\longrightarrow} X \times \mathcal{P}(F^*) \stackrel{\Omega}{\longrightarrow} \mathcal{P}(E^*)$, where $(1_X \times \Theta)(x) = \{x\} \times \Theta(x)$ and $\Omega(x, \psi) = \Lambda(x)^*(\psi)$, $x \in X, \psi \in F^*$.

Hypotheses (ii)–(iii) imply that the multifunction Θ has convex values and open fibers. Hence it admits a continuous selection t (see Step 1 in the proof of Theorem 4.1). Clearly, $1_X \times t$ is a continuous selection of $1_X \times \Theta$.

Note that $\forall x \in X, \forall \psi \in F^*$, $\Omega(x, \psi)$ is closed and convex. If we show that the multifunction Ω is lower semicontinuous, since X is paracompact and $\mathcal{P}(E^*)$ is a Banach space, then Michael's selection theorem would imply the existence of a continuous selection s of the multifunction $\Omega(1_X \times t)$. This selection s would clearly be a selection of Ψ.

In order to show that Ω is lower semicontinuous, according to Lemma 1.4 it suffices to show that the mapping $x \longmapsto \Lambda(x)^*$ is pointwise bounded and that the multifunction $x \longmapsto \operatorname{graph}(\Lambda(x)^*)$ is lower semicontinuous.

Note first that the definition (5) of the transpose of a closed convex process implies that

$$\forall x \in X, \forall \psi \in F^*, \sup_{\xi \in \Lambda(x)^*(\psi)} \|\xi\| \leq \|\Lambda(x)\| \, \|\psi\|. \tag{12}$$

By (i), $\|\Lambda(x)\| = \sup_{u \in B_E(0,1)} \inf_{v \in \Lambda(x)(u)} \|v\| \leq M, x \in X$; hence, (12) implies that $x \longmapsto \Lambda(x)^*$ is pointwise bounded.

In order to prove now that $x \longmapsto \operatorname{graph}(\Lambda(x)^*)$ is lower semicontinuous, let us first note that

$$\forall x, x' \in X, (\Lambda(x) - \Lambda(x'))^* = \Lambda(x)^* - \Lambda(x')^*. \tag{13}$$

Let $x \in X$ and $\epsilon > 0$ be arbitrary but fixed and let $(\psi, \varphi) \in \operatorname{graph}(\Lambda(x)^*)$. By continuity of $x \longmapsto \Lambda(x), \exists \delta > 0$ such that

$$\|\Lambda(x) - \Lambda(x')\| < \frac{\epsilon}{\|\psi\|}, \forall x' \in B_X(x, \delta).$$

By (13), given any $\varphi' \in \Lambda(x')^*(\psi)$, the linear functional $\varphi - \varphi'$ belongs to $(\Lambda(x)^* - \Lambda(x')^*)(\psi) = (\Lambda(x) - \Lambda(x'))^*(\psi)$. Hence,

$$\|\varphi - \varphi'\| \leq \sup_{\xi \in (\Lambda(x)^* - \Lambda(x')^*)(\psi)} \|\xi\| \leq \|\Lambda(x) - \Lambda(x')\| \, \|\psi\| < \epsilon.$$

Thus, $(\psi, \varphi) \in B_{F^* \times E^*}(\operatorname{graph}(\Lambda(x')^*), \epsilon)$, $\forall x' \in B_X(x, \delta)$. We have proved that, given any $x \in X$, the following containment is verified:

$$\begin{aligned} \operatorname{graph}(\Lambda(x)^*) \subset \bigcap_{\epsilon > 0} \bigcup_{\delta > 0} \bigcap_{x' \in B_X(x, \delta)} & B_{F^* \times E^*}(\operatorname{graph}(\Lambda(x')^*), \epsilon) = \\ &= \liminf_{x' \to x} \operatorname{graph}(\Lambda(x')^*), \end{aligned}$$

that is the multifunction $x \longmapsto \operatorname{graph}(\Lambda(x)^*)$ is lower semicontinuous.

All hypothesis of the Theorem 4.3 with $Y = F^*$ are now satisfied; hence, $\exists x_0 \in K, \exists \varphi_0 \in \Psi(x_0) = \Lambda(x_0)^*(\psi_0)$ for some $\psi_0 \in \Theta(x_0)$, with $\varphi_0 \in N_X(x_0)$. $\square$

Remark. If X is compact, hypothesis (i) directly follows from the continuity of Λ.

We end this section with the proof of our main theorem.

Proof of Theorem 3.1. (A) We apply Corollary 4.4 to $f(x,\psi) = \sigma(\Phi(x),\psi)$. Since Φ satisfies the condition of Ky Fan on (K, X) with respect to Λ, Lemma 2.2 implies that $\sigma(\Phi(x),\psi) \geq 0$ for all $x \in K$ and all $\psi \in F^*$ such that $\Lambda(x)^*(\psi) \cap N_X(x) \neq \varnothing$. Thus conclusion (2) of Corollary 4.4 fails. Therefore, there exists $\widehat{x} \in X$ such that $\sigma(\Phi(\widehat{x}),\psi) \geq 0$ for all $\psi \in F^*$. The fact that $\widehat{x}$ is an equilbrium for Φ follows from the characterization (4) of the closed convex hull in terms of the support function.

(B) Given any $x_0 \in X$, define the multifunction $\Psi : X \to \mathcal{P}(F)$ by putting

$$\Psi(x) := \Phi(x) + \Lambda(x)(x - x_0), x \in X.$$

The multifunction Ψ is upper hemicontinuous with closed convex values. Given any finite subset N of X, Ψ satisfies the condition of Ky Fan on $(C_N \setminus K, \mathrm{co}(\{x_0\} \cup C_N))$ and on (K, X) with respect to Λ.

Let us prove that for any $x \in C_N \setminus K, \overline{\Lambda(x)(-S_{\mathrm{co}(\{x_0\}\cup C_N)}(x)} \cap \Psi(x) \neq \varnothing$. By hypothesis, there exists a net $\{y_i\}_{i\in I}$ in $\Lambda(-S_{C_N}(x))$ converging to some $y \in \Phi(x)$. For each $i \in I, y_i \in \Lambda(x)(-\frac{c_i - x}{t_i})$ for some $c_i \in C_N, t_i > 0$. Let y' be any element in $\Lambda(x)(x - x_0)$. For each $i \in I$, the following containments are satisfied:

$$\begin{aligned} y_i + y' &\in \Lambda(x)\Big(-\frac{c_i - x}{t_i}\Big) + \Lambda(x)(x - x_0) \subseteq \\ &\subseteq \Lambda(x)\Big(-\Big(\frac{c_i - x}{t_i} - x + x_0\Big)\Big) = \\ &= \Lambda(x)\Big(-\Big(\frac{1 + t_i}{t_i}\Big[\frac{c_i}{1 + t_i} + \frac{t_i}{1 + t_i}x_0 - x\Big]\Big)\Big) \subseteq \\ &\subseteq \Lambda(x)\big(-S_{\mathrm{co}(\{x_0\}\cup C_N)}(x)\big). \end{aligned}$$

Thus $y + y' \in \overline{\Lambda(x)(-S_{\mathrm{co}(\{x_0\}\cup C_N)}(x)} \cap \Psi(x)$. The proof of the fact that Ψ satisfies the condition of Ky Fan on (K, X) with respect to Λ is similar.

The conclusion follows from part (A) applied to the multifunction Ψ. $\square$

5. Some Related Results

An immediate consequence of Corollary 4.4 is the following coincidence property that generalizes a result of Ky Fan [12]:

Proposition 5.1. *Let X be a convex subset of a normed space E, F be a normed space, $\Phi, \Psi : X \longrightarrow \mathcal{P}(F)$ be two upper hemicontinuous multifunction with closed convex values, and $\Lambda : X \longrightarrow \Lambda(E, F)$ be a continuous mapping satisfying the boundedness condition:*

(i) *$\exists M > 0$ such that $\forall x \in X$, $\forall u \in E$ with $\|u\| = 1, \exists v \in \Lambda(x)(u)$ such that $\|v\| \leq M$.*

Assume that there exist a compact subset K of X such that for each finite subset N of X, there exists a compact convex subset C_N of X containing N such that:

(ii) *$\forall x \in C_N \setminus K$, $\forall \psi \in F^*, \sigma(\Phi(x), \psi) \geq \inf_{y \in \Psi(x)} \langle \psi, y \rangle$ provided that $\Lambda(x)^*(\psi) \cap N_{C_N}(x) \neq \varnothing$;*

(iii) *$\forall x \in K$, $\forall \psi \in F^*, \sigma(\Phi(x), \psi) \geq \inf_{y \in \Psi(x)} \langle \psi, y \rangle$ provided that $\Lambda(x)^*(\psi) \cap N_X(x) \neq \varnothing$;*

(A) *If $\forall x \in X$ one of the sets $\Phi(x)$ or $\Psi(x)$ is weakly compact, then $\exists \widehat{x} \in X$ such that $\Phi(\widehat{x}) \cap \Psi(\widehat{x}) \neq \varnothing$.*

(B) *If $\forall x \in X$ both sets $\Phi(x)$ and $\Psi(x)$ are weakly compact, then $\forall x_0 \in X$, $\exists \widehat{x} \in X$ such that $\Psi(\widehat{x}) \cap [\Phi(\hat{x}) + \Lambda(\hat{x})(\hat{x} - x_0)] \neq \varnothing$.*

Proof. We only prove (A), the proof of (B) being an immediate consequence of Theorem 3.1 (B). Since $\forall x \in X$, one of the sets $\Phi(x)$ or $\Psi(x)$ is weakly compact, then the set $\Phi(x) - \Psi(x)$ is closed and convex. Moreover, $\inf_{y \in \Psi(x)} \langle \psi, y \rangle = -\sigma(-\Psi(x), \psi)$ and by (3), $\sigma(\Phi(x), \psi) + \sigma(-\Psi(x), \psi) = \sigma(\Phi(x) - \Psi(x), \psi)$, $\forall x \in X$, $\forall \psi \in F^*$. Corollary 4.4 applied to the function $f(x, \psi) = \sigma(\Phi(x) - \Psi(x), \psi)$ implies that the multifunction $\Phi - \Psi$ has an equilibrium which is clearly a coincidence for Φ and Ψ. $\square$

Remarks. (**1**) According to Lemma 2.2, sufficient conditions for (ii) and (iii) to hold true are:

(ii)$'$ $\Phi - \Psi$ satisfies the condition of Ky Fan on $(C_N \setminus K, C_N)$ with respect to Λ, that is,

$$[\Psi(x) + \overline{\Lambda(x)(-S_{C_N}(x))}] \cap \Phi(x) \neq \varnothing, \ \forall x \in C_N \setminus K;$$

(iii)$'$ $\Phi - \Psi$ satisfies the condition of Ky Fan on (K, X) with respect to Λ, that is,

$$[\Psi(x) + \overline{\Lambda(x)(-S_X(x))}] \cap \Phi(x) \neq \varnothing, \ \forall x \in K.$$

(**2**) Again, Proposition 5.1 is true in topological spaces having sufficiently many linear functionals. It is a refinement (with tangency conditions involving a parametrized family of convex processes) of a coincidence theorem of Ky Fan [12]. Ky Fan's result corresponds to the case where $E = F$, $\Lambda(x) = Id_E$ for all $x, C_N = K$ for all N.

(**3**) When $\Lambda(x) \equiv \ell$ is a bounded linear operator, this result can be found in [6]. Particular forms of this result can be found in [3], [4], [7], [8] and others.

In the case where $E = F$ and Ψ is the inclusion $X \hookrightarrow E$, we obtain a fixed point theorem for inward or outward multifunctions that generalizes Ky Fan fixed point theorem. By way of illustration, we state this fixed point property in the case where X is compact. We will say that the multifunction $\Phi : X \longrightarrow \mathcal{P}(E)$ is *inward* with respect to a continuous family of closed convex processes $\Lambda : X \longrightarrow \Lambda(E, E)$ if the following property is verified:

$$\Phi(x) \cap [x - \overline{\Lambda(x)(-S_X(x))}] \neq \varnothing, \quad \forall x \in X.$$

The multifunction Φ is said to be *outward* with respect to Λ if

$$\Phi(x) \cap [x + \overline{\Lambda(x)(-S_X(x))}] \neq \varnothing, \ \forall x \in X.$$

Corollary 5.2. *Assume that X is a compact convex subset of a normed space E and that the multifunction $\Phi : X \longrightarrow \mathcal{P}(E)$ is upper hemicontinuous with nonempty closed convex values. If Φ is inward or outward with respect to Λ, then it has a fixed point.*

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