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THE SKOROKHOD OBLIQUE REFLECTION PROBLEM IN A CONVEX POLYHEDRON

M. SHASHIASHVILI

ABSTRACT. The Skorokhod oblique reflection problem is studied in the case of n-dimensional convex polyhedral domains. The natural sufficient condition on the reflection directions is found, which together with the Lipschitz condition on the coefficients gives the existence and uniqueness of the solution. The continuity of the corresponding solution mapping is established. This property enables one to construct in a direct way the reflected (in a convex polyhedral domain) diffusion processes possessing the nice properties.

1. INTRODUCTION.

This paper is concerned with the Skorokhod oblique reflection problems that have applications in queuing and storage theory ([1], [2],[3]). The Skorokhod problem was used for constructing the reflected diffusion processes in an n-dimensional domain G by Tanaka [4], Lions and Sznitman [5], and Saisho [6]. These authors obtained solutions to the Skorokhod problem under the following assumptions: Tanaka [4] – G convex and a normal direction of reflection; Lions and Sznitman [5] – ∂G smooth save at "convex corners" and a normal direction of reflection, or ∂G smooth and smoothly varying direction of reflection together with the admissibility condition on ∂G; Saisho [6] – as in [5], but without the admissibility condition and only for the case of normal direction of reflection.

The case of "convex corners" with oblique reflection has received a good deal of attention lately.

The case of $G = \mathbb{R}^n_+$ (i.e., of the nonnegative orthant) has been considered by Harrison and Reiman [7]. Under some restrictions on possible directions of reflection they have obtained the existence, uniqueness, and Lipschitz continuity of the solution mapping.

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The Lipschitz continuity of the solution mapping to the Skorokhod problem with oblique reflection has been extensively investigated recently by Dupuis and Ishii [8] for the case where G is a convex polyhedral domain. Dupius and Ishii [8], as well as Harrison and Reiman [7], supposed that the reflection directions are constant for each boundary hyperplane forming the faces of the polyhedron.

The aim of the present paper is to investigate and solve the Skorokhod oblique reflection problem for a convex polyhedral domain when the reflection directions are the functions of state and control, where the control strategy can depend on the whole past trajectory.

2. Statement of the Skorokhod and the Modified Skorokhod problems. Principal Assumption

Let G denote the bounded convex polyhedral domain in $\mathbb{R}^n$ defined by

$$G = \{x \in \mathbb{R}^n : n_i \cdot x > c_i,\ i = 1, \dots, N\}, \tag{1}$$

where G is assumed to be non-empty, and each of the faces F_i of the polyhedron $\overline{G}$ has dimension $n-1$. Here n_i is a unit vector normal to the hyperplane $H_i = \{x \in \mathbb{R}^n : n_i \cdot x = c_i\}$, $i = 1, \dots, N$, F_i is a part of H_i, $F_i = \{x \in \overline{G} : n_i \cdot x = c_i\}$, $i = 1, \dots, N$, $n_i \cdot x$ denotes the inner product of the vectors n_i and x.

Let a bounded closed subset U of the space $\mathbb{R}^m$ be given, which plays the role of a set of possible controls, and for each pair (x, u), $x \in H_i$, $u \in U$, let the vector $q_i(x, u)$ be defined. By this vector is meant the reflection direction at the point x, $x \in H_i$, and for the control u, $u \in U$. Naturally, each of the vector-valued function $q_i(x, u)$, $i = 1, \dots, N$, is supposed to be the continuous function of the pair (x, u). The following condition of normalization of the length of the vector $q_i(x, u)$ will considerably simplify the presentation of the main results

$$n_i \cdot q_i(x, u) = 1, \quad x \in H_i, \quad u \in U, \quad i = 1, \dots, N. \tag{2}$$

In the sequel we shall need to extend vector-valued functions $q_i(x, u)$ for arbitrary values $x \in \mathbb{R}^n$. Trivially, we define $q_i(x, u) = q_i(\mathrm{pr}_{H_i} x, u)$, $x \in \mathbb{R}^n$, $u \in U$. Suppose also that we are given a vector-valued function $b(t, x, u)$, $t \geq 0$, $x \in \overline{G}$, $u \in U$ (with values in $\mathbb{R}^n$) which is assumed to be continuous as a function of the triple (t, x, u). Since the set $\overline{G} \times U$ is a compact subset of $\mathbb{R}^n \times \mathbb{R}^m$, it is not a serious restriction to suppose that the function $b(t, x, u)$ is bounded by a constant C,

$$|b(t, x, u)| \leq C, \quad t \geq 0, \quad x \in \overline{G}, \quad u \in U. \tag{3}$$

Of course, the functions $q_i(x,u)$, $i=1,\dots,N$ are bounded in the set $\overline{G}\times U$. Let they be bounded by the same constant C,

$$|q_i(x,u)| \le C, \quad x\in\overline{G}, \quad u\in U, \quad i=1,\dots,N. \tag{4}$$

We will need one more nondecreasing continuous real-valued function A_t with $A_0=0$ which has the meaning of an integrator function.

Consider now the space $C(\mathbb{R}_+,\mathbb{R}^n)$, i.e., the space of all continuous n-dimensional functions $x=x(\cdot)=x_t$, $t\ge 0$, defined on the time interval $[0,\infty)$, and then define the control strategy $u=u_t=u_t(x(\cdot))=u_t(x[0,t])$, where for each $x=x_t$, $t\ge 0$, the corresponding function $u=u_t(x[0,t])$ is supposed to be continuous in t and takes its values in the set U of controls. The control strategy is naturally assumed to depend continuously on the trajectory $x=x_t$, $t\ge 0$, in the following sense: if for some time interval $[0,t]$

$$\sup_{0\le s\le t} |x_s^n - x_s| \underset{n\to\infty}{\to} 0$$

then necessarily

$$\sup_{0\le s\le t} |u_s(x^n[0,s]) - u_s(x[0,s])| \underset{n\to\infty}{\to} 0.$$

Now we are ready to formulate the Skorokhod oblique reflection problem in a convex polyhedron $\overline{G}$. Given $x=x_t\in C(\mathbb{R}_+,\mathbb{R}^n)$ with the initial condition $x_0\in\overline{G}$, find a pair $(z,y)=(z_t,y_t)_{t\ge 0}$ of the continuous functions $z=z_t$ and $y=y_t$ with values in $\mathbb{R}^n$ and $\mathbb{R}^N$, respectively, which jointly satisfy the following conditions:

$$\begin{aligned}&(1)\ z_t\in\overline{G}, \quad t\ge 0,\\ &(2)\ z_t = x_t + \int_0^t b(s,z_s,u_s)\,dA_s + \sum_{k=1}^N \int_0^t q_k(z_s,u_s)\,dy_s^k, \quad t\ge 0,\end{aligned} \tag{5}$$

where $u=u_s=u_s(z[0,s])$;

(3) every component-function y_t^k, $k=1,\dots,N$, is nondecreasing with $y_0^k=0$, $k=1,\dots,N$, and satisfies the requirement

$$\int_0^t I(n_i\cdot z_s > c_i)\,dy_s^i = 0, \quad i=1,\dots,N, \quad t\ge 0, \tag{6}$$

i.e., y_t^i increases only at those times t, when $n_i\cdot z_t = c_i$, i.e., when the function $z=z_t$, $t\ge 0$, is at the face F_i. For the solution of this problem we shall follow the method of Hiroshi Tanaka in [4]. Namely, we generalize this problem for functions $x=x_t$ from the space $D(\mathbb{R}_+,\mathbb{R}^n)$ (the space of all n-dimensional right continuous functions with left-hand limits on the time interval $[0,\infty)$), prove the existence of the solution of this problem for a certain subspace of $D(\mathbb{R}_+, \mathbb{R}^n)$, and then pass to the solution of the initial problem. To this end we have to assume:

(a) there is a given nondecreasing right continuous real-valued function $A = A_t$ with $A_0 = 0$ (the integration function);

(b) the control strategy $u = u_t = u_t(x[0,t])$ can be extended continuously for the functions $x = x_t \in D(\mathbb{R}_+, \mathbb{R}^n)$ in the following sense:

for each $x = x_t \in D(\mathbb{R}_+, \mathbb{R}^n)$ the function $u_t(x[0,t])$ should be right continuous with left-hand limits (with values in U), and if

$$\sup_{0 \le s \le t} |x_s^n - x_s| \underset{n\to\infty}{\to} 0,$$

where $x^n = x_t^n \in D(\mathbb{R}_+, \mathbb{R}^n)$, $x = x_t \in C(\mathbb{R}_+, \mathbb{R}^n)$, then necessarily

$$\sup_{0 \le s \le t} |u_s(x^n[0,s]) - u_s(x[0,s])| \underset{n\to\infty}{\to} 0 .$$

Thus the modified Skorokhod oblique reflection problem can be formulated as follows:

For the function $x = x_t \in D(\mathbb{R}_+, \mathbb{R}^n)$ with the initial condition $x_0 \in \overline{G}$ we seek a pair $(z, y) = (z_t, y_t)_{t \ge 0}$ of functions belonging to the spaces $D(\mathbb{R}_+, \mathbb{R}^n)$ and $D(\mathbb{R}_+, \mathbb{R}^N)$ such that they jointly satisfy the following conditions:

$$\begin{aligned} &(1)\ z_t \in \overline{G}, \ \ t \ge 0, \\ &(2)\ z_t = x_t + \int_0^t b(s, z_{s-}, u_{s-}) dA_s + \sum_{k=1}^N \int_0^t q_k(z_{s-}, u_{s-}) dy_s^k, \ \ t \ge 0, \end{aligned} \tag{7}$$

(3) every component-function y_t^k is nondecreasing with $y_0^k = 0$ and satisfies

$$\int_0^t I(n_i \cdot z_s > c_i) dy_s^i = 0, \ \ i = 1, \dots, N, \ \ t \ge 0, \tag{8}$$

Now we shall give a condition which turns out to be sufficient for the existence of the solution of the Skorokhod and the modified Skorokhod problem (for a certain subclass of functions). We may call this condition the principal assumption, since it is proved to be crucial in all constructions given in this paper. For each $x \in \partial G$ denote by $I(x)$ the set of those indices i_k for which $x \in F_{i_k}$. The principal assumption consists in the following:

For each $x \in \partial G$ with $I(x) = (i_1, \dots, i_m)$ we require the existence of positive numbers $a_1, \dots, a_m, \lambda$ with $0 < \lambda < 1$ such that for an arbitrary control $u \in U$ we have

$$\sum_{k=1, k \ne l}^m a_k |q_{i_l}(x, u) \cdot n_{i_k}| < \lambda a_l, \ \ l = 1, \dots, m. \tag{9}$$

From the fact that the set $\overline{G} \times U$ is compact and the functions $q_i(x,u)$, $i = 1,\dots,N$, are continuous it is easy to see that the principal assumption (9) provides (for fixed $x \in \partial G$) the existence of $r > 0$ such that

$$\sum_{k=1,k\neq l}^{m} a_k|q_{i_l}(y,u)\cdot n_{i_k}| < \lambda a_l,\ \ l = 1,\dots,m,$$

when $y \in B(x,r) = \{y : |y-x| < r\}$.

On the other hand, from the definition of $I(x)$ we have

$$n_{i_l}\cdot x - c_{i_l} = 0,\ \ l = 1,\dots,m,$$
$$n_j\cdot x - c_j > 0,\ \ j \neq i_1,\dots,i_m.$$

Obviously, by reducing $r > 0$ we have

$$\begin{gathered} n_j\cdot x - c_j > 3r,\ \ j \neq i_1,\dots,i_m, \\ \sum_{k=1,k\neq l}^{m} a_k|q_{i_l}(y,u)\cdot n_{i_k}| < \lambda a_l,\ \ l = 1,\dots,m, \end{gathered} \tag{10}$$

for all $y \in B(x,2r)$, $u \in U$.

Thus, for every fixed $x \in \partial G$ there do exist positive numbers $a_1,\dots,a_m,\lambda$, $0<\lambda<1$, and $r>0$ such that for all $y \in B(x,2r)$ and $u \in U$ the condition (10) does hold.

Take now the open covering $B(x,r)$ of the compact set ∂G. Then there exists its finite covering $B(x_p,r_p)$, $p = 1,\dots,M$, i.e., $\partial G \subseteq \bigcup_{p=1}^{M} B(x_p,r_p)$, where each point x_p (with the corresponding r_p) possesses the property (10). It is interesting (and useful for verification in practical problems) that this condition is, in fact, equivalent to the condition (9), i.e., our principal assumption (9), which is the assumption on the uncountable number of points of the boundary ∂G, is actually the assumption on a finite number of points x_p, $p = 1,\dots,M$, of the boundary: we require the existence of a finite covering $B(x_p,r_p)$, $p = 1,\dots,M$ of the boundary ∂G, where $x_p \in \partial G$, such that for each point x_p, $p = 1,\dots,M$, with $I(x_p) = (i_1,\dots,i_m)$ there do exist positive numbers $a_1,\dots,a_m,\lambda$, $0<\lambda<1$, such that for all $y \in B(x_p,2r_p)$ and $u \in U$ the condition

$$\begin{gathered} n_j\cdot x_p - c_j > 3r_p,\ \ j \neq i_1,\dots,i_m, \\ \sum_{k=1,k\neq l}^{m} a_k|q_{i_l}(y,u)\cdot n_{i_k}| < \lambda a_l,\ \ l = 1,\dots,m, \end{gathered} \tag{11}$$

holds. Indeed, let $y \in \partial G$; then necessarily $y \in B(x_p, r_p)$ for some $p = 1, \dots, M$. Suppose $I(x_p) = (i_1, \dots, i_m)$. Then by (11)

$$\sum_{k=1, k\neq l}^{m} a_k |q_{i_l}(y,u) \cdot n_{i_k}| < \lambda a_l, \quad l = 1, \dots, m,$$

for all controls $u \in U$.

If we now show that $I(y) \subseteq I(x_p)$, then, obviously, the corresponding subset of $a_1, \dots, a_m$ will suffice for a boundary point $y \in \partial G$, i.e., the requirement (9) will be true. Take $j \in I(y)$. We have to show that $j \in I(x_p)$. Let us have on the contrary $j \notin I(x_p)$. Then $n_j \cdot x_p - c_j > 3r_p$; hence $n_j \cdot y - c_j = n_j \cdot (y - x_p) + n_j \cdot x_p - c_j > -r_p + 3r_p = 2r_p > 0$. Thus $j \notin I(y)$, which is a contradiction.

Denote further $B = \bigcup\limits_{p=1}^{M} B(x_p, r_p)$. Then $\overline{G} = \overline{G} \cap B + (\overline{G} \setminus B) \subseteq B + (\overline{G} \setminus B)$. $\overline{G} \setminus B$ is a closed set, and $\partial G \subseteq B$. Therefore $\overline{G} \setminus B \subseteq G$. Thus a closed set is a subset of an open set; hence there exists its open neighborhood $O_\varepsilon(\overline{G} \setminus B)$ such that $\overline{O}_\varepsilon(\overline{G} \setminus B) \subseteq G$. Let us denote $\delta > 0$ as follows:

$$\delta = \min \left(\frac{\min(r_1, \dots, r_M)}{C \cdot \max\limits_{p=1,\dots,M} \frac{\sum_{k=1}^{m(p)} a_k(p)}{(1-\lambda(p)) \min\limits_{k=1,\dots,m(p)} a_k} + 1}, \frac{\varepsilon}{2} \right) \tag{12}$$

where $m(p)$, $a_k(p)$, $\lambda(p)$ are the numbers from the condition (11) for a boundary point x_p.

3. Existence of a Solution of a Modified Skorokhod Problem for Step-Functions with Small Jumps

The purpose of this paragraph is to give an algorithm for the construction of the solution of the modified Skorokhod problem for the data $x = x_t$, $A = A_t$ being the step-functions. We recall that in this case there exists a time sequence $0 = t_0 < t_1 < \cdots < t_r < \cdots$ tending to infinity, such that $(x_t, A_t) = (x_{t_r}, A_{t_r})$ if $t_r \le t < t_{r+1}$, $r = 0, 1, \dots$.

Theorem 1. *Let the condition* (11) *hold. Then for any pair* $(x, A) = (x_t, A_t)_{t \ge 0}$ *of the step-functions with the initial condition* $x_0 \in \overline{G}$, $A_0 = 0$ *and sufficiently small jumps* $|\Delta x_t| + C\Delta A_t < \delta$, $t \ge 0$ (*where* $\delta > 0$ *has been defined in* (12)), *there exists a solution of the modified Skorokhod problem.*

Proof. Since the function (x_t, A_t) is constant on the time interval $[t_r, t_{r+1})$, we can define $(z_t, y_t) = (z_{t_r}, y_{t_r})$ on the same time interval. Therefore the main problem consists in defining the function (z_t, y_t) at times t_r, $r =$

$0, 1, \ldots$. At the initial moment $t = 0$ we have $(z_0, y_0) = (x_0, 0)$. At the other moments t_r we have to find pairs $(z_{t_r}, \Delta y_{t_r})$ such that:

$$
\begin{aligned}
&(1)\ z_{t_r} \in \overline{G}, \quad \Delta y_{t_r} \geq 0,\\
&(2)\ z_{t_r} = z_{t_{r-1}} + \Delta x_{t_r} + b(t_r, z_{t_{r-1}}, u_{t_r-})\Delta A_{t_r} +\\
&\qquad + \sum_{k=1}^{N} q_k(z_{t_{r-1}}, u_{t_r-})\Delta y^k_{t_r},\\
&(3)\ I(n_i \cdot z_{t_r} > c_i)\Delta y^i_{t_r} = 0, \quad i = 1, \ldots, N.
\end{aligned}
\tag{13}
$$

The key observation for solving the problem (13) consists in the fact that, as it turns out, this problem is equivalent to the following one: solve first the system

$$
\begin{aligned}
\Delta y^i_{t_r} = \max\Big[0, -(n_i \cdot z_{t_{r-1}} - c_i) -&\\
-\big(n_i \cdot \Delta x_{t_r} + n_i \cdot b(t_r, z_{t_{r-1}}, u_{t_r-})\Delta A_{t_r}\big) -&\\
- \sum_{k=1,\ k\neq i}^{N} n_i \cdot q_k(z_{t_{r-1}}, u_{t_r-})\Delta y^k_{t_r}\Big],& \quad i = 1, \ldots, N,
\end{aligned}
\tag{14}
$$

at time t_r, and then define z_{t_r} from

$$
\begin{aligned}
z_{t_r} = z_{t_{r-1}} + \Delta x_{t_r} + b(t_r, z_{t_{r-1}}, u_{t_r-})\Delta A_{t_r} +&\\
+ \sum_{k=1}^{N} q_k(z_{t_{r-1}}, u_{t_r-})\Delta y^k_{t_r}.&
\end{aligned}
\tag{15}
$$

Indeed, suppose (13) holds for $(z_{t_r}, \Delta y_{t_r})$. Then, taking the inner product with vectors n_i, $i = 1, \ldots, N$, we obtain

$$
\begin{aligned}
n_i \cdot z_{t_r} - c_i = (n_i \cdot z_{t_{r-1}} - c_i) + n_i \cdot \Delta x_{t_r} + n_i \cdot b(t_r, z_{t_{r-1}}, u_{t_r-})\Delta A_{t_r} +&\\
+ n_i \cdot q_i(z_{t_{r-1}}, u_{t_r})\Delta y^i_{t_r} + \sum_{k=1,\ k\neq i}^{N} n_i \cdot q_k(z_{t_{r-1}}, u_{t_r-})\Delta y^k_{t_r},&
\end{aligned}
$$

where

$$
\begin{gathered}
n_i \cdot z_{t_r} - c_i \geq 0, \ \ \Delta y^i_{t_r} \geq 0, \ \ i = 1, \ldots, N,\\
(n_i \cdot z_{t_r} - c_i)\Delta y^i_{t_r} = 0, \ \ i = 1, \ldots, N,\\
n_i \cdot q_i(z_{t_{r-1}}, u_{t_r-}) = 1, \ \ i = 1, \ldots, N.
\end{gathered}
$$

Now, to prove that each of these systems (for $i = 1, \dots, N$) is equivalent to

$$\Delta y^i_{t_r} = \max \Big[0, -(n_i \cdot z_{t_{r-1}} - c_i) - (n_i \cdot \Delta x_{t_r} + $$
$$+ n_i \cdot b(t_r, z_{t_{r-1}}, u_{t_r-})\Delta A t_r) - \sum_{k=1,\ k\neq i}^{N} n_i \cdot q_k(z_{t_{r-1}}, u_{t_r-})\Delta y^k_{t_r}\Big],$$

it suffices only to note that for real numbers x, y, z the following two conditions are equivalent:

$$\begin{cases} z \ge 0, \ \ y \ge 0, \ \ zy = 0, \\ z = x + y, \end{cases} \qquad \begin{cases} y = \max(0, -x), \\ z = x + y, \end{cases}$$

Suppose on the contrary that (14)–(15) hold. Taking in (15) again the inner product with vectors n_i, $i = 1, \dots, N$, we get

$$n_i \cdot z_{t_r} - c_i = n_i \cdot z_{t_{r-1}} - c_i + n_i \cdot \Delta x_{t_r} + n_i \cdot b(t_r, z_{t_{r-1}}, u_{t_r-})\Delta A_{t_r} +$$
$$+ \sum_{k=1,\ k\neq i}^{N} n_i \cdot q_k(z_{t_{r-1}}, u_{t_r-})\Delta y^k_{t_r} + \Delta y^i_{t_r}, \ \ i = 1, \dots, N.$$

But as we have just seen, this, together with (14), gives

$$n_i \cdot z_{t_r} - c_i \ge 0, \ \ \Delta y^i_{t_r} \ge 0,$$
$$(n_i \cdot z_{t_r} - c_i)\Delta y^i_{t_r} = 0, \ \ i = 1, \dots, N.$$

In their turn, the last relations together with (15) are, obviously, the same as the relations (13).

Therefore we have to suggest a method of solution of the system (14)–(15). This method will be given by induction. Suppose we have defined the pair (T_p, z_{T_p}), where T_p takes its values from $t_0, t_1, \dots, t_{r-1}, t_r, \dots$. Assume that $T_p = t_{r-1}$ (for some $r = 1, \dots$). Then the algorithm will be proposed to define the next values $z_{t_r}, \dots, z_{T_{p+1}}$ until the time T_{p+1}, where the terms of this sequence and the time T_{p+1} are defined simultaneously. Define

$$j_p = \begin{cases} \min(j : 1 \le j \le M, z_{T_p} \in B(x_j, r_j) \ \text{ if such a } j \text{ exists}, \\ M + 1, \quad \text{otherwise}. \end{cases}$$

Since $z_{T_p} \in B + (\overline{G} \backslash B)$, obviously,

$$z_{T_p} \in \overline{G} \backslash B \quad \text{ if } \ j_p = M + 1,$$
$$z_{T_p} \in B(x_{j_p}, r_{j_p}) \ \text{ if } \ j_p \le M.$$

Suppose first that $j_p = M+1$; then $z_{T_p} \in \overline{G}\backslash B$. In this case it is easy to define the sequence $(z_{t_r}, \Delta y_{t_r}), \ldots, (z_{t_{r+k}}, \Delta y_{t_{r+k}})$ until $z_{t_{r+k}} \notin O_\delta(\overline{G}\backslash B)$. Indeed, we define

$$\begin{aligned}
&\Delta y_{t_r} = 0, \ldots, \Delta y_{t_{r+k}} = 0, \quad z_{t_r} = z_{t_{r-1}} + \Delta x_{t_r} + b(t_r, z_{t_{r-1}}, u_{t_r-}))\Delta A_{t_r}, \\
&\quad \ldots \\
&z_{t_{r+k}} = z_{t_{r+k-1}} + \Delta x_{t_{r+k}} + b(t_{r+k}, z_{t_{r+k-1}}, u_{t_{r+k}-})\Delta A_{t_{r+k}},
\end{aligned}$$

and have $z_{t_{r+k-1}} \in O_\delta(\overline{G}\backslash B)$, $|\Delta x_{t_{r+k}}| + C\Delta A_{t_{r+k}} < \delta$, Therefore $z_{t_{r+k}} \in O_{2\delta}(\overline{G}\backslash B) \subseteq G$. In this case T_{p+1} is defined as follows:

$$T_{p+1} = \inf\{t_r : t_r > T_p,\ z_{t_r} \notin O_\delta(\overline{G}\backslash B)\}.$$

Suppose now the second case, i.e., when $j_p \leq M$ and $z_{T_p} \in B(x_{j_p}, z_{j_p})$. Then z_{T_p} is situated near the boundary ∂G. So the construction of the next values of the solution $z_{t_r}, z_{t_{r+1}}, \ldots$ is indeed a serious problem. For simplicity, we denote the pair (x_{j_p}, r_{j_p}) by (x, r), and the corresponding ball by $B(x, r)$. Let $I(x) = (i_1, \ldots, i_m)$. By our assumption (11) we have

$$\begin{gathered}
n_j \cdot x - c_j > 3r, \quad j \neq i_1, \ldots, i_m, \\
\sum_{k=1, k\neq l}^{m} a_k |q_{i_l}(y,u) \cdot n_{i_k}| < \lambda a_l, \quad l = 1, \ldots, m,
\end{gathered} \tag{16}$$

for all $y \in B(x, 2r)$ and all $u \in U$.

We begin by considering the following auxiliary system:

$$\begin{aligned}
\Delta y_{t_r}^{i_k} = \max \Big[0, &-(n_{i_k} \cdot z_{t_{r-1}} - c_{i_k}) - \\
-\big(n_{i_k} &\cdot \Delta x_{t_r} + n_{i_k} \cdot b(t_r, z_{t_{r-1}}, u_{t_r-})\Delta A_{t_r}\big) - \\
- \sum_{l=1, l\neq k}^{m} &n_{i_k} \cdot q_{i_l}(z_{t_{r-1}}, u_{t_r-})\Delta y_{t_r}^{i_l}\Big], \quad k = 1, \ldots, m.
\end{aligned} \tag{17}$$

We shall show that this system has a unique solution. Rewrite it in a simpler way:

$$y^k = \max\Big[0, b_k - \sum_{l=1, l\neq k}^{m} n_{i_k} \cdot q_{i_l} y^l\Big], \quad k = 1, \ldots, m, \tag{18}$$

where

$$\begin{gathered}
b_k = -(n_{i_k} \cdot z_{t_{r-1}} - c_{i_k}) - \big(n_{i_k} \cdot \Delta x_{t_r} + n_{i_k} \cdot b(t_r, z_{t_{r-1}}, u_{t_r} -)\Delta A_{t_r}\big), \\
y^k = \Delta y_{t_r}^{i_k}, \quad q_{i_l} = q_{i_l}(z_{t_{r-1}}, u_{t_{r}-}).
\end{gathered}$$

Consider the space $\mathbb{R}^m$ with the metric

$$d(y,\widetilde{y})=\sum_{k=1}^{m}a_k|y^k-\widetilde{y}^k|$$

and introduce the mapping ψy of this space into itself,

$$(\psi y)^k=\max\Big[0,b_k-\sum_{l=1,l\neq k}^{m}n_{i_k}\cdot q_{i_l}y^l\Big],\quad k=1,\dots,m.$$

We have

$$|(\psi y)^k-(\psi\widetilde{y})^k|\leq\sum_{l=1,l\neq k}^{m}|n_{i_k}\cdot q_{i_l}|\,|y^l-\widetilde{y}^l|,\quad k=1,\dots,m.$$

Multiplying the above inequality by a_k and then taking the sum, we get (using the inequalities (16))

$$\sum_{k=1}^{m}a_k|(\psi y)^k-(\psi\widetilde{y})^k|\leq\sum_{k=1}^{m}a_k\Big(\sum_{l=1,l\neq k}^{m}|n_{i_k}\cdot q_{i_l}|\,|y^l-\widetilde{y}^l|\Big)=$$
$$=\sum_{l=1}^{m}\Big(\sum_{k=1,k\neq l}^{m}a_k|n_{i_k}\cdot q_{i_l}|\Big)|y^l-\widetilde{y}^l|<\lambda\sum_{l=1}^{m}a_l|y^l-\widetilde{y}^l|.$$

Thus $d(\psi(y),\psi(\widetilde{y}))<\lambda d(y,\widetilde{y})$, i.e., the mapping ψy is a contraction establishing thus the existence and uniqueness of the solution of the auxiliary system (17)–(18). Using the unique solution $(\Delta y_{t_r}^{i_1},\dots,\Delta y_{t_r}^{i_m})$ of this system, we can now establish that the vector $(\Delta y_{t_r}^{1},\dots,\Delta y_{t_r}^{N})$, where $\Delta y_{t_r}^{j}=0$, $j\neq i_1,\dots,i_m$, solves the system (14). In fact, we have only to verify that

$$\max\Big[0,-(n_j\cdot z_{t_{r-1}}-c_j)-(n_j\cdot\Delta x_{t_r}+n_j\cdot b(t_r,z_{t_{r-1}},u_{t_r-})\Delta A_{t_r})-$$
$$-\sum_{l=1}^{m}n_j\cdot q_{i_l}(z_{t_{r-1}},u_{t_r-})\Delta y_{t_r}^{i_l}\Big]=0,\quad j\neq i_1,\dots,i_m.$$

To this end we have to bound the sum $\sum_{l=1}^{m}\Delta y_{t_r}^{i_l}$. Obviously, $n_{i_k}\cdot z_{t_{r-1}}-c_{i_k}\geq 0$, $k=1,\dots,m$; hence from (17) we can write

$$\Delta y_{t_r}^{i_k}\leq|\Delta x_{t_r}|+C\Delta A_{t_r}+\sum_{l=1,\,l\neq k}^{m}|n_{i_k}\cdot q_{i_l}(z_{t_{r-1}},u_{t_r-})|\Delta y_{t_r}^{i_l},$$
$$k=1,\dots,m.$$

Multiplying these inequalities by a_k and then taking the sum, we obtain

$$\sum_{k=1}^{m} \Delta y_{t_r}^{i_k} \leq \frac{\sum_{k=1}^{m} a_k}{(1-\lambda) \min\limits_{k=1,\ldots,m} a_k} (|\Delta x_{t_r}| + C\Delta A_{t_r}),$$

but by the assumption of Theorem 1, $|\Delta x_t| + C\Delta A_t < \delta$, $t \geq 0$. Therefore

$$\sum_{k=1}^{m} \Delta y_{t_r}^{i_k} < \frac{\sum_{k=1}^{m} a_k}{(1-\lambda) \min a_k} \delta.$$

We have

$$n_j \cdot z_{t_{r-1}} - c_j = n_j \cdot (z_{t_{r-1}} - x) + n_j \cdot x - c_j > -2r + 3r = r,$$

hence

$$\begin{aligned} &n_j \cdot z_{t_{r-1}} - c_j + n_j \cdot \Delta x_{t_r} + n_j \cdot b(t_r, z_{t_{r-1}}, u_{t_{r}-}) \Delta A_{t_r} + \\ &+ \sum_{l=1}^{m} n_j \cdot q_{i_l}(z_{t_{r-1}}, u_{t_r -}) \Delta y_{t_r}^{i_l} > r - \delta - C \frac{\sum_{k=1}^{m} a_k}{(1-\lambda) \min a_k} \delta \geq 0 \end{aligned}$$

by the definition of δ in (12). Therefore the vector $(\Delta y_{t_r}^1, \ldots, \Delta y_{t_r}^N)$ is indeed the solution of (14). Then z_{t_r} is defined from (15).

The same procedure works recurrently and gives the values $(\Delta y_{t_{r+1}}, z_{t_{r+1}})$, $\ldots, (\Delta y_{t_{r+k}}, z_{t_{r+k}})$ until it turns out that $z_{t_{r+k}} \notin B(x_{j_p}, 2r_{j_p})$, i.e.,

$$z_{T_p} = z_{t_{r-1}} \in B(x_{j_p}, r_{j_p}), \ldots, z_{t_{r+k-1}} \in B(x_{j_p}, 2r_{j_p}),\ z_{t_{r+k}} \notin B(x_{j_p}, 2r_{j_p}).$$

We define $T_{p+1} = \inf\{t_r : t_r > T_p,\ z_{t_r} \notin B(x_{j_p}, 2r_{j_p})\}$. In general, the definition of T_{p+1} admits the form

$$T_{p+1} = \begin{cases} \inf\big(t_r : t_r > T_p, z_{t_r} \notin B(x_{j_p}, 2r_{j_p}\big) & \text{if } j_p \leq M, \\ \inf\big(t_r : t_r > T_p, z_{t_r} \notin O_\delta(\overline{G} \backslash B)\big) & \text{if } j_p = M+1. \end{cases} \tag{19}$$

Thus by induction we have given the algorithm of the solution of the system (14)–(15) which, in fact, results in the solution of the modified Skorokhod problem (7)–(8). $\square$

Let us consider now the time interval $[T_p, T_{p+1}]$ and suppose first that $j_p \leq M$. Denote for simplicity $B(x_{j_p}, r_{j_p}) = B(x, r)$; as previously $I(x) = (i_1, \ldots, i_m)$.

We have the following equation:

$$z_t - z_s = x_t - x_s + \int_s^t b(v, z_{v-}, u_{v-}) dA_v + \sum_{k=1}^{N} \int_s^t q_k(z_{v-}, u_{v-}) dy_v^k.$$

From our construction of the solution $\Delta y_t^j = 0$, $j \neq i_1, \dots, i_m$, $T_p \leq t \leq T_{p+1}$. Therefore

$$z_t - z_s = x_t - x_s + \int_s^t b(v, z_{v-}, u_{v-}) dA_v + \sum_{l=1}^m \int_s^t q_{i_l}(z_{v-}, u_{v-}) dy_v^{i_l}.$$

Multiplying this equality by the vectors n_{i_k}, $k = 1, \dots, m$,

$$n_{i_k} \cdot z_t - c_{i_k} = n_{i_k} \cdot z_s - c_{i_k} + n_{i_k} \cdot (x_t - x_s) + \\ + \int_s^t n_{i_k} \cdot b(v, z_{v-}, u_{v-}) dA_v + \sum_{l=1,\, l \neq k}^m \int_s^t n_{i_k} \cdot q_{i_l}(z_{v-}, u_{v-}) \times \\ \times dy_v^{i_l} + (y_t^{i_k} - y_s^{i_k}), \quad k = 1, \dots, m,$$

where $n_{i_k} \cdot z_s - c_{i_k} \geq 0$, $n_{i_k} \cdot z_s - c_{i_k} \geq 0$, $y_t^{i_k} - y_s^{i_k}$ are nondecreasing in t, and

$$\int_s^t I(n_{i_k} \cdot z_v - c_{i_k} > 0) d(y_v^{i_k} - y_s^{i_k}) = 0,$$

we get that the pair $(n_{i_k} \cdot z_t - c_{i_k}, y_t^{i_k} - y_s^{i_k})$, $s \leq t \leq T_{p+1}$ is the solution of the one-dimensional Skorokhod problem ([4], [10]) for the function

$$n_{i_k} \cdot z_t - c_{i_k} + n_{i_k} \cdot (x_t - x_s) + \int_s^t n_{i_k} \cdot b(v, z_{v-}, u_{v-}) dA_v + \\ + \sum_{l=1,\, l \neq k}^m \int_s^t n_{i_k} \cdot q_{i_l}(z_{v-}, u_{v-}) dy_v^{i_l}.$$

As is well known ([4], [10]), the solution of this problem can be written explicitly in terms of a maximal function

$$y_t^{i_k} - y_s^{i_k} = \sup_{s \leq u \leq t} \max \Big[0, -(n_{i_k} \cdot z_s - c_{i_k}) - n_{i_k} \cdot (x_u - x_s) - \\ - \int_s^u n_{i_k} \cdot b(v, z_{v-}, u_{v-})) dA_v - \sum_{l=1,\, l \neq k}^m \int_s^u n_{i_k} \cdot q_{i_l}(z_{v-}, u_{v-}) dy_v^{i_l} \Big] \leq \\ \leq \sup_{s \leq u \leq t} |x_u - x_s| + C(A_t - A_s) + \sum_{l=1,\, l \neq k}^m \int_s^t |n_{i_k} \cdot q_{i_l}(z_{v-}, u_{v-})| dy_v^{i_l}.$$

Multiplying this inequality by a_k and taking the sum, we obtain

$$\sum_{k=1}^m (y_t^{i_k} - y_s^{i_k}) \leq \frac{\sum_{k=1}^m a_k}{(1-\lambda) \min a_k} \big(\sup_{s \leq u \leq t} |x_u - x_s| + C(A_t - A_s) \big),$$

Hence

$$\sum_{k=1}^{N}(y_t^k - y_s^k) \le \frac{\sum_{k=1}^m a_k}{(1-\lambda)\min a_k}\big(\sup_{s\le u\le t}|x_u - x_s| + C(A_t - A_s)\big).$$

Using this inequality in the above-mentioned equation, we get

$$|z_t - z_s| \le \Big(1 + \frac{C\sum_{k=1}^m a_k}{(1-\lambda)\min a_k}\Big)\big(\sup_{s\le u\le t}|x_u - x_s| + C(A_t - A_s)\big).$$

From the definition of δ these bounds admit the form

$$\begin{aligned} |z_t - z_s| &\le \frac{\min(r_1,\dots,r_M)}{\delta}\big(\sup_{s\le u\le t}|x_u - x_s| + C(A_t - A_s)\big),\\ \sum_{k=1}^{N}(y_t^k - y_s^k) &\le \frac{\min(r_1,\dots,r_M)}{C\cdot\delta}\big(\sup_{s\le u\le t}|x_u - x_s| + C(A_t - A_s)\big), \end{aligned} \tag{20}$$

where $T_p \le s \le t \le T_{p+1}$, $p = 0, 1, \dots$

If $j_p = M+1$, i.e., $z_{T_p} \in \overline{G}\backslash B$, then we have $\Delta y_t^k = 0$, $k = 1, \dots N$, $T_p \le t \le T_{p+1}$. Hence in this case these bounds are also true.

4. Existence of the Solution of the Skorokhod Problem for Arbitrary Continuous Data

Theorem 2. *Suppose the assumption* (11) *holds. Then there exists the solution of the Skorokhod problem for every continuous pair of functions* $(X, A) = (X_t, A_t)_{t\ge 0}$ *with* $x_0 \in \overline{G}$, $A_0 = 0$. *For arbitrary* $0 \le s \le t \le T$ *it satisfies the inequality*

$$\begin{aligned} \sum_{i=1}^{N}(y_t^i - y_s^i) &\le \frac{\min(r_1,\dots,r_M)}{C\delta}\Big(\Big(\frac{T}{h}+1\Big)\times\\ &\quad\times\sup_{s\le u,v\le t}|x_u - x_v| + C(A_t - A_s)\Big),\\ |z_t - z_s| &\le \frac{\min(r_1,\dots,r_M)}{\delta}\Big(\Big(\frac{T}{h}+1\Big)\times\\ &\quad\times\sup_{s\le u,v\le t}|x_u - x_v| + C(A_t - A_s)\big), \end{aligned} \tag{21}$$

where $h > 0$ *is defined from the condition*

$$\begin{aligned} \Delta_T^{x,A}(h) &< \frac{\delta^2}{4\min(r_1,\dots,r_M)}, \quad \textit{where}\\ \Delta_T^{x,A}(h) &= \sup_{\substack{0\le s,t\le T\\ |t-s|\le h}}(|x_t - x_s| + C|A_t - A_s|). \end{aligned} \tag{22}$$

Proof. We define the time sequence

$$R_0^k = 0, \quad R_1^k = \inf\Big(t > 0 : |x_t - x_0| + C(A_t - A_0) > \frac{1}{k}\Big) \wedge 1, \dots$$
$$R_m^k = \inf\Big(t > R_{m-1}^k : |x_t - x_{R_{m-1}^k}| + C(A_t - A_{R_{m-1}^k}) > \frac{1}{k}\Big) \wedge m, \dots$$

for each $k = 1, 2, \dots$.

It is easy to see that $\lim_{m\to\infty} R_m^k = +\infty$, $k = 1, 2, \dots$. From the definition of these times we can write

$$\sup_{R_{m-1}^k \le t < R_m^k} (|x_t - x_{R_{m-1}^k}| + C(A_t - A_{R_{m-1}^k})) \le \frac{1}{k} \quad k = 1, 2, \dots, \quad m = 1, 2, \dots.$$

Hence, defining the step-functions $(X^k, A^k) = (X_t^k, A_t^k)_{t\ge 0}$, where

$$(X^k, A_t^k) = (X_{R_{m-1}^k}, A_{R_{m-1}^k}) \quad \text{if} \quad R_{m-1}^k \le t < R_m^k,$$
$$k = 1, 2, \dots, \quad m = 1, 2, \dots$$

we see that this sequence of step-functions (x^k, A^k) converges uniformly to the continuous pair (x, A)

$$\sup_{t\ge 0}(|x_t^k - x_t| + C(A_t^k - A_t)) \le \frac{1}{k}$$

and from the continuity of the function (x, A) we have

$$|x_{R_m^k} - x_{R_{m-1}^k}| + C(A_{R_{m^k}} - A_{R_{m-1}^k}) \le \frac{1}{k}, \quad k = 1, 2, \dots, \quad m = 1, 2, \dots.$$

Therefore $|\Delta x_t^k| + C\Delta A_t^k \le \frac{1}{k}$, $t \ge 0$, $k = 1, 2, \dots$. If we take k so large that $\frac{1}{k} \le \delta$, then Theorem 1 ensures the existence of a solution of the modified Skorokhod problem for the step-functions $(x^k, A^k) = (x_t^k, A_t^k)$. For this case denote this solution by $(z^k, y^k) = (z_t^k, y_t^k)_{t\ge 0}$ and rewrite the inequalities (20) as

$$\begin{gathered}\sum_{i=1}^N (y_t^{k,i} - y_s^{k,i}) \le \frac{\min(r_1, \dots, r_m)}{C\delta}\Big(\sup_{s\le u\le t} |x_u^k - x_s^k| + C(A_t^k - A_s^k)\Big), \\ |z_t^k - z_s^k| \le \frac{\min(r_1, \dots, r_M)}{\delta}\big(\sup_{s\le u\le t} |x_u^k - x_s^k| + C(A_t^k - A_s^k)\big),\end{gathered} \tag{23}$$

where $T_p^k \le s \le t \le T_{p+1}^k$, $p = 0, 1, \dots$. Here $z_{T_p^k}^k \in B(x_{j_p}, r_{j_p})$ if $j_p \le M$, $z_{T_p^k}^k \in \overline{G}\backslash B$ if $j_p = M + 1$ and

$$T_{p+1}^k = \begin{cases} \inf\big(t : t > T_p^k, z_t^k \notin B(x_{j_p}, 2r_{j_p}\big) & \text{if } j_p \le M, \\ \inf\big(t : t > T_p^k, z_t^k \notin O_\delta(\overline{G}\backslash B)\big) & \text{if } j_p = M + 1. \end{cases}$$

From the definition of times T_P^k we get

$$|z^k_{T^k_{p+1}} - z^k_{T^k_p}| \geq \min(r_1, \ldots, r_M) \text{ if } j_p \leq M,$$
$$|z^k_{T^k_{p+1}} - z^k_{T^k_p}| \geq \delta \text{ if } j_p = M+1.$$

In any case we have $|z^k_{T^k_{p+1}} - z^k_{T^k_p}| \geq \delta$, $k = 1, 2, \ldots$, $p = 0, 1, \ldots$.

Hence from the inequality (23) we obtain

$$\delta^2 \leq \min(r_1, \ldots, r_m)\big(\sup_{T^k_p \leq u \leq T^k_{p+1}} |x^k_u - x^k_{T^k_p}| + C(A^k_{T^k_{p+1}} - A^k_{T^k_p})\big).$$

It is easy to verify that

$$\sup_{T^k_p \leq u \leq T^k_{p+1}} |x^k_u - x^k_{T^k_p}| + C(A^k_{T^k_{p+1}} - A^k_{T^k_p}) \leq \frac{4}{k} +$$
$$+ \sup_{T^k_p \leq u \leq T^k_{p+1}} |x_u - x_{T^k_p}| + C(A_{T^k_{p+1}} - A_{T^k_p}).$$

Therefore we get

$$\frac{\delta^2}{\min(r_1, \ldots, r_M)} \leq \frac{4}{k} + \sup_{T^k_p \leq u \leq T^k_{p+1}} |x_u - x_{T^k_p}| + C(A_{T^k_{p+1}} - A_{T^k_p}).$$

From this, starting from sufficiently large k with $k \geq \dfrac{8\min(r_1, \ldots, r_M)}{\delta^2}$ we have

$$\frac{\delta^2}{4\min(r_1, \ldots, r_M)} \leq \sup_{T^k_p \leq s,t \leq T^k_{p+1}} (|x_t - x_s| + C|A_t - A_s|).$$

Fix $T > 0$ arbitrarily large and introduce the modulus of continuity of $(x, A) = (x_t, A_t)_{t \geq 0}$ on the time interval $[0, T]$:

$$\Delta^{x,A}_T(h) = \sup_{\substack{0 \leq s,t \leq T \\ |t-s| \leq h}} (|x_t - x_s| + C|A_t - A_s|);$$

then if $h > 0$ is so small that

$$\Delta^{x,A}_T(h) < \frac{\delta^2}{4\min(r_1, \ldots, r_M)}$$

then $T^k_{p+1} \leq T$ implies $T^k_{p+1} - T^k_p > h$. Thus on the time interval $[0, T]$ there may lie at most $\frac{T}{h}$ intervals $[T^k_p, T^k_{p+1}]$, $p = 0, 1, \ldots$.

This fact, together with the bounds (23), implies

$$
\begin{aligned}
\sum_{i=1}^{N}(y_t^{k,i}-y_s^{k,i}) &\le \frac{\min(r_1,\dots,r_M)}{C\delta}\Big(\Big(\frac{T}{h}+1\Big)\times \\
&\quad\times\sup_{s\le u,v\le t}|x_u^k-x_v^k|+C(A_t^k-A_s^k)\Big), \\
|z_t^k-z_s^k| &\le \frac{\min(r_1,\dots,r_M)}{\delta}\Big(\Big(\frac{T}{h}+1\Big)\times \\
&\quad\times\sup_{s\le u,v\le t}|x_u^k-x_v^k|+C(A_t^k-A_s^k)\Big)
\end{aligned}
\tag{24}
$$

for any $s\le t$ from the interval $[0,T]$, $0\le s\le t\le T$.

The sequence $(x^k,A^k)=(x_t^k,A_t^k)_{t\ge0}$ converges uniformly to the function $(x,A)=(x_t,A_t)_{t\ge0}$, and therefore it converges also in the Skorokhod metric. From the bounds (24) and the criterion of relative compactness in the Skorokhod metric [9] we obtain that the sequence $(z^k,y^k)=(z_t^k,y_t^k)$ is relatively compact. Let us choose its convergent subsequence, which for simplicity we denote again by (z^k,y^k). Then there exists a scaling sequence $\lambda_k(t)$ such that

$$
\sup_{0\le t\le T}|\lambda_k(t)-t|\underset{k\to\infty}{\to}0,\quad \sup_{0\le t\le T}\big(|z^k_{\lambda_k(t)}-z_t|+|y^k_{\lambda_k(t)}-y_t|\big)\underset{k\to\infty}{\to}0,
$$

where $(z_t,y_t)_{t\ge0}$ is a pair of functions which are right continuous with left-hand limits, z_t and y_t are respectively n-dimensional and y_t N-dimensional, and the component-functions y_t^i, $i=1,\dots,N$, are nondecreasing.

Performing a time change $t\to\lambda_k(t)$ in the inequalities (24) and then making k tend to infinity, we shall have

$$
\begin{aligned}
\sum_{i=1}^{N}(y_t^{i}-y_s^{i}) &\le \frac{\min(r_1,\dots,r_m)}{C\delta}\Big(\Big(\frac{T}{h}+1\Big)\times \\
&\quad\times\sup_{s\le u,v\le t}|x_u-x_v|+C(A_t-A_s)\Big) \\
|z_t-z_s| &\le \frac{\min(r_1,\dots,r_M)}{\delta}\Big(\Big(\frac{T}{h}+1\Big)\times \\
&\quad\times\sup_{s\le u,v\le t}|x_u-x_v|+C(A_t-A_s)\Big),
\end{aligned}
\tag{25}
$$

where $0\le s\le t\le T$.

In this inequality let s tend to t. Then, obviously, we have

$$
\sum_{i=1}^{N}(y_t^i-y_{t-}^i)=0,\quad |z_t-z_{t-}|=0.
$$

Hence the pair (z_t, y_t) is continuous in t.

In this case the convergence in the Skorokhod metric is equivalent to the uniform convergence (on bounded time interval), i.e., we have

$$\sup_{0\le t\le T} (|z_t^k - z_t| + |y_t^k - y_t|) \underset{k\to\infty}{\to} 0.$$

Let us show that the pair $(z_t, y_t)_{t\ge 0}$ is the solution of the Skorokhod problem for the functions (x_t, A_t). We have

(1) $z_t^k \in \overline{G}$, $t \ge 0$,

(2) $z_t^k = x_t^k + \int_0^t b(s, z_{s-}^k, u_{s-}^k) dA_s^k + \sum_{j=1}^N \int_0^t q_j(z_{s-}^k, u_{s-}^k) dy_s^{k,j}$, where $u_s^k = u_s(z^k[0,s])$,

(3) $\int_0^t I(n_i \cdot z_s^k - c_i > 0) dy_s^{k,i} = 0$, $i = 1, \dots, N$.

Passing to the limit in the first relation, we obtain $z_t \in \overline{G}$, $t \ge 0$. The third relation is obviously equivalent to the following one:

(3′) For every bounded continuous function $f(x)$ with $f(0) = 0$ we should have

$$\int_0^t f(n_i \cdot z_s^k - c_i > 0) dy_s^{k,i} = 0, \quad i = 1, \dots, N.$$

Consider the time interval $[0, T]$. The function $b(t, x, u)$ is continuous, and therefore uniformly continuous on $[0, T] \times \overline{G} \times U$. The same is true for the functions $q_j(x, u)$, $j = 1, \dots, N$. They are uniformly continuous on $\overline{G} \times U$.

From the continuity of the control strategy $u = u_s = u_s(z(0, s])$ we have

$$\sup_{0\le s\le T} |u_s^k - u_s| \underset{k\to\infty}{\to} 0, \quad \sup_{0<s\le T} |u_{s-}^k - u_s| \underset{k\to\infty}{\to} 0.$$

Using now these uniform continuity properties and Helly's theorem on passing to the limit in the Stieltjes integrals, it is not difficult to pass to the limit in the relations (2) and (3′) and to get that the continuous pair $(z_t, y_t)_{t\ge 0}$ is indeed the solution of the Skorokhod problem. □

5. Uniqueness of the Solution of the Skorokhod Problem and the Continuity of the Solution Mapping under the Natural Lipschitz Condition

Theorem 3. *Let the assumption* (11) *hold and suppose the vector-valued functions $b(t, x, u)$ and $q_j(x, u)$, $j = 1, \dots, N$, to be Lipschitz continuous in (x, u):*

for every $T > 0$ and arbitrary $t \le T$,

$$\begin{gathered} |b(t, x, u) - b(t, \widetilde{x}, \widetilde{u})| \le L_T(|x - \widetilde{x}| + |u - \widetilde{u}|), \\ |q_j(x, u) - q_j(\widetilde{x}, \widetilde{u})| \le L(|x - \widetilde{x}| + |u - \widetilde{u}|). \ j = 1, \dots, N. \end{gathered} \tag{26}$$

Suppose also that the control strategy $u_t(z[0,t])$ is Lipschitz continuous:

$$\sup_{0\le t\le T}|u_t(z[0,t])-u_t(\widetilde{z}[0,t])|\le K_T\sup_{0\le t\le T}|z_t-\widetilde{z}_t|. \tag{27}$$

Then for every pair of continuous functions $(x,A)=(x_t,A_t)_{t\ge 0}$ with $x_0\in\overline{G}$, $A_0=0$ there does exist a unique solution of the Skorokhod problem, and the solution mapping $(x,A)\to(x,A,z,y)$ is continuous in the uniform topology (on bounded time intervals).

Proof. Let $(z,y)=(z_t,y_t)$, $(\widetilde{z},\widetilde{y})=(\widetilde{z}_t,\widetilde{y}_t)$ be two solutions of the Skorokhod problem for the pair (x,A). By analogy with (19) define the following time sequences:

$$T_{p+1}=\begin{cases}\inf\big(t:t>T_p,z_t\notin B(x_{j_p},2r_{j_p}\big) & \text{if } j_p\le M,\\ \inf\big(t:t>T_p,z_t\notin O_\delta(\overline{G}\setminus B)\big) & \text{if } j_p=M+1,\quad p=0,1,\dots,\end{cases}$$

where

$$j_p=\begin{cases}\min(j:1\le j\le M,\ z_{T_p}\in B(x_j,r_j) & \text{if such a } j \text{ exists},\\ \quad M+1 & \text{otherwise},\end{cases}$$

$$\widetilde{T}_{p+1}=\begin{cases}\inf\big(t:t>\widetilde{T}_p,\widetilde{z}_t\notin B(x_{\widetilde{j}_p},2r_{\widetilde{j}_p}\big) & \text{if } \widetilde{j}_p\le M,\\ \inf\big(t:t>\widetilde{T}_p,\widetilde{z}_t\notin O_\delta(\overline{G}\setminus B)\big) & \text{if } \widetilde{j}_p=M+1,\quad p=0,1,\dots,\end{cases}$$

$$\widetilde{j}_p=\begin{cases}\min(j:1\le j\le M,\ \widetilde{z}_{\widetilde{T}_p}\in B(x_j,r_j) & \text{if such a } j \text{ exists},\\ \quad M+1 & \text{otherwise}.\end{cases}$$

Obviously, $|z_{T_{p+1}}-z_{T_p}|\ge\delta$, $|\widetilde{z}_{\widetilde{T}_{p+1}}-\widetilde{z}_{\widetilde{T}_p}|\ge\delta$, $p=0,1,\dots$.

Hence on the time interval $[0,T]$ there are only a finite number of times

$$T_0,T_1,\dots,T_p,\dots,\quad \widetilde{T}_0,\widetilde{T}_1,\dots,\widetilde{T}_p,\dots,\quad T_0=0,\quad \widetilde{T}_0=0.$$

Fix the time interval $[0,T]$. By induction we shall prove that $T_{p+1}=\widetilde{T}_{p+1}$ and $(\widetilde{z},\widetilde{y})=(z,y)$ on the time interval $[T_p,T_{p+1}]$. For this purpose we assume $(\widetilde{z},\widetilde{y})=(z,y)$ on the time interval $[0,T_p]=[0,\widetilde{T}_p]$.

Consider the value $z_{T_p}=\widetilde{z}_{\widetilde{T}_p}\in\overline{G}$. Then

(1) $z_{T_p}\in B(x_{j_p},r_{j_p})$ if $j_p\le M$,

(2) $z_{T_p}\in\overline{G}\setminus B$ if $j_p=M+1$.

Suppose the first case, i.e., $z_{T_p}\in B(x_{j_p},r_{j_p})$. Then

$$T_{p+1}=\inf\big(t:t>T_p,z_t\notin B(x_{j_p},2r_{j_p}\big),$$
$$\widetilde{T}_{p+1}=\inf\big(t:t>T_p,\widetilde{z}_t\notin B(x_{j_p},2r_{j_p}\big).$$

Define now $T^*_{p+1} = \min(T_{p+1}, \widetilde{T}_{p+1})$. For simplicity denote again the ball $B(x_{j_p}, r_{j_p})$ by $B(x,r)$. Let $I(x) = (i_1, \dots, i_m)$. Then we have $z_{T_p} = \widetilde{z}_{\widetilde{T}_p} \in B(x,r)$,

$$\sum_{k=1, k\neq l}^{m} a_k |q_{i_l}(z_t, u_t) \cdot n_{i_k}| < \lambda a_l,$$

$$\sum_{k=1, k\neq l}^{m} a_k |q_{i_l}(\widetilde{z}_t, \widetilde{u}_t) \cdot n_{i_k}| < \lambda a_l, \quad l = 1, \dots, m,$$

when $T_p \leq t < T^*_{p+1}$, and also $n_j \cdot z_t - c_j > r > 0$, $n_j \cdot \widetilde{z}_t - c_j > r > 0$, $j \neq i_1, \dots, i_m$. Hence ${y_t}^j = {y_{T_p}}^j$, ${\widetilde{y}_t}^j = {\widetilde{y}_{T_p}}^j$, $j \neq i_1, \dots, i_m$, $T_p \leq t < T^*_{p+1}$.

Therefore we have

$$z_t = z_{T_p} + x_t - x_{T_p} + \int_{T_p}^{t} b(s, z_s, u_s) dA_s + \sum_{l=1}^{m} \int_{T_p}^{t} q_{i_l}(z_s, u_s) dy_s^{i_l},$$

$$\widetilde{z}_t = z_{T_p} + x_t - x_{T_p} + \int_{T_p}^{t} b(s, \widetilde{z}_s, \widetilde{u}_s) dA_s + \sum_{l=1}^{m} \int_{T_p}^{t} q_{i_l}(\widetilde{z}_s, \widetilde{u}_s) d{\widetilde{y}_s}^{i_l}.$$

Multiplying these equations by the vectors n_{i_k}, $k = 1, \dots, m$, we get

$$\begin{gathered}
n_{i_k} \cdot z_t - c_{i_k} = n_{i_k} \cdot z_{T_p} - c_{i_k} + \\
+ n_{i_k} \cdot (x_t - x_{T_p}) + \int_{T_p}^{t} n_{i_k} \cdot b(s, z_s, u_s) dA_s + \\
+ \sum_{l=1,\, l\neq k}^{m} \int_{T_p}^{t} n_{i_k} \cdot q_{i_l}(z_s, u_s) dy_s^{i_l} + (y_t^{i_k} - y_{T_p}^{i_k}), \\
n_{i_k} \cdot \widetilde{z}_t - c_{i_k} = n_{i_k} \cdot z_{T_p} - c_{i_k} + \\
+ n_{i_k} \cdot (x_t - x_{T_p}) + \int_{T_p}^{t} n_{i_k} \cdot b(s, \widetilde{z}_s, \widetilde{u}_s) dA_s + \\
+ \sum_{l=1,\, l\neq k}^{m} \int_{T_p}^{t} n_{i_k} \cdot q_{i_l}(\widetilde{z}_s, \widetilde{u}_s) d{\widetilde{y}_s}^{i_l} + ({\widetilde{y}_t}^{i_k} - {\widetilde{y}_{T_p}}^{i_k}), \quad k = 1, \dots, m, \\
T_p \leq t < T^*_{p+1}.
\end{gathered}$$

Now we essentially use the variational lemma of maximal functions [10] which implies that

$$\int_{T_p}^{t} \left| d(y_s^{i_k} - {\widetilde{y}_s}^{i_k}) \right| \leq \int_{T_p}^{t} f_v^k n_{i_k} \cdot (b(v, z_v, u_v) -$$

$$-b(v,\widetilde{z}_v,\widetilde{u}_v))dA_v+\sum_{l=1,l\neq k}^m\int_{T_p}^t f_v^k n_{i_k}\cdot q_{i_l}(z_v,u_v)dy_v^{i_l}-$$

$$-\sum_{l=1,l\neq k}^m\int_{T_p}^t f_v^k n_{i_k}\cdot q_{i_l}(\widetilde{z}_v,\widetilde{u}_v)d\widetilde{y}_v{}^{i_l},\quad k=1,\dots,m,$$

where f_v^k, $k=1,\dots,m$, are Borel measurable functions with two values ± 1.

Multiplying again these inequalities by a_k, taking the sum and using then the Lipschitz continuity properties (26)–(27), we obtain

$$\sum_{k=1}^m a_k\int_{T_p}^t\big|d(y_s^{i_k}-\widetilde{y}_s{}^{i_k})\big|\le\frac{\sum_{k=1}^m a_k}{(1-\lambda)}(L+L_T)(1+K_T)\times$$

$$\times\int_{T_p}^t\sup_{s\le v}|z_s-\widetilde{z}_s|d\Big(A_v+\sum_{l=1}^m y_v^{i_l}\Big).\tag{28}$$

Further we have

$$z_t-\widetilde{z}_t=\int_{T_p}^t(b(s,z_s,u_s)-b(s,\widetilde{z}_s,\widetilde{u}_s))dA_s+\sum_{l=1}^m\int_{T_p}^t(q_{i_l}(z_s,u_s)-$$

$$-q_{i_l}(\widetilde{z}_s,\widetilde{u}_s))dy_s^{i_l}+\sum_{l=1}^m\int_{T_p}^t q_{i_l}(\widetilde{z}_s,\widetilde{u}_s)d(y_s^{i_l}-\widetilde{y}_s{}^{i_l}).$$

Taking into account the previous bound (28), we get

$$|z_t-\widetilde{z}_t|\le\int_{T_p}^t\sup_{s\le v}|z_s-\widetilde{z}_s|d\overline{A}_v,\quad T_p\le t<T_{p+1}^*,\quad\text{where}$$

$$\overline{A}_t=(L+L_T)(1+k_T)\Big(1+\frac{C\sum_{k=1}^m a_k}{(1-\lambda)\min a_k}\Big)\Big(A_t+\sum_{l=1}^m y_t^{i_l}\Big).$$

Using now the Gronwall inequality, we have $\sup_{T_p\le s\le t}|z_s-\widetilde{z}_s|=0$, $T_p\le t<T_{p+1}^*$ and from the inequality (28) it follows that

$$\sum_{k=1}^m\int_{T_p}^t\big|d(y_s^{i_k}-\widetilde{y}_s{}^{i_k})\big|=0,$$

i.e., $y_t^{i_k}=\widetilde{y}_t{}^{i_k}$, $T_p\le t<T_{p+1}^*$.

Thus $(z_t,y_t)=(\widetilde{z}_t,\widetilde{y}_t)$ on $T_p\le t<T_{p+1}^*$. From the continuity we obtain $z_{T_{p+1}^*}=\widetilde{z}_{T_{p+1}^*}$, which implies $T_{p+1}=\widetilde{T}_{p+1}$. In fact, suppose on the contrary that $T_{p+1}\neq\widetilde{T}_{p+1}$ and let $T_{p+1}<\widetilde{T}_{p+1}$ (the case $\widetilde{T}_{p+1}<T_{p+1}$ can be considered similarly). Then $\widetilde{z}_{T_{p+1}}\in B(x,2r)$, i.e., $z_{T_{p+1}}\in B(x,2r)$, which

by the very definition is a contradiction. Thus $\widetilde{T}_{p+1} = T_{p+1}$, $(\widetilde{z}, \widetilde{y}) = (z, y)$ on $T_p \le t < T_{p+1}$.

The same is true for the second case, $j_p = M + 1$, which can be easily verified.

Now we shall prove that the solution mapping $(x, A) \to (x, A, z, y)$ is continuous. Let $(x^m, A^m) = (x_t^m, A_t^m)_{t \ge 0}$ be a sequence of continuous pairs converging to the pair $(x, A) = (x_t, A_t)_{t \ge 0}$:

$$\sup_{0 \le t \le T} (|x_t^m - x_t| + |A_t^m - A_t|) \underset{m \to \infty}{\to} 0$$

for arbitrary $T > 0$.

Let $(z^m, y^m) = (z_t^m, y_t^m)_{t \ge 0}$ be the unique solution of the corresponding Skorokhod problem. From the inequality (21) we have

$$\begin{aligned} \sum_{i=1}^{N} (y_t^{m,i} - y_s^{m,i}) &\le \frac{\min(r_1, \ldots, r_M)}{C\delta} \Big(\Big(\frac{T}{h^m} + 1 \Big) \times \\ &\quad \times \sup_{s \le u, v \le t} |x_u^m - x_v^m| + C(A_t^m - A_s^m) \Big), \\ |z_t^m - z_s^m| &\le \frac{\min(r_1, \ldots, r_M)}{\delta} \Big(\Big(\frac{T}{h^m} + 1 \Big) \times \\ &\quad \times \sup_{s \le u, v \le t} |x_u^m - x_v^m| + C(A_t^m - A_s^m) \Big), \end{aligned}$$

where $h^m > 0$ is defined from the condition $\Delta_T^{x^m, A^m}(h^m) < \frac{\delta^2}{4 \min(r_1, \ldots, r_M)}$, where

$$\Delta_T^{x^m, A^m}(h^m) = \sup_{\substack{0 \le s, t \le T \\ |t-s| \le h^m}} (|x_t^m - x_s^m| + C|A_t^m - A_s^m|).$$

Obviously,

$$|\Delta_T^{x^m, A^m}(h) - \Delta_T^{x, A}(h)| \le 2 \sup_{0 \le t \le T} (|x_t^m - x_t| + C|A_t^m - A_t|).$$

Therefore, choosing $h > 0$ from the condition

$$\Delta_T^{x, A}(h) < \frac{\delta^2}{8 \min(r_1, \ldots, r_M)}$$

and taking m so large that

$$\sup_{0 \le t \le T} (|x_t^m - x_t| + C|A_t^m - A_t|) < \frac{\delta^2}{16 \min(r_1, \ldots, r_M)},$$

we have

$$\Delta_T^{x^m, A^m}(h) < \frac{\delta^2}{4 \min(r_1, \ldots, r_M)}.$$

Thus h^m can be chosen independent of m : $h^m = h$. Therefore, by the Arzela–Ascoli theorem the sequence (x^m, A^m, z^m, y^m) is relatively compact.

Now we shall show that (z^m, y^m) converges to (z, y), where (z, y) is the solution of the Skorokhod problem for (x, A).

Choose an arbitrary subsequence m' of m. Then by the relative compactness there does exist its own subsequence m'' such that $(x^{m''}, A^{m''}, z^{m''}, y^{m''})$ is convergent to some $(x, A, \widetilde{z}, \widetilde{y})$. By the standard arguments analogous to those used in the proof of Theorem 2, it is not difficult to show that $(\widetilde{z}, \widetilde{y})$ is the solution of the Skorokhod problem for a pair (x, A). But the solution is unique, and therefore $(\widetilde{z}, \widetilde{y}) = (z, y)$. Thus, whatever the subsequence m' may be, there exists its own subsequence m'' such that $(x^{m''}, A^{m''}, z^{m''}, y^{m''})$ converges to one and the same limit (x, A, z, y). But this means that the sequence (x^m, A^m, z^m, y^m) itself converges to (x, A, z, y).

Thus we have established the continuity of the solution mapping. $\square$

6. Reflected Diffusion Processes in an n-dimensional Polyhedral Domain

Let $(\Omega, \mathcal{F}, P)$ be any probability space on which there is given an m-dimensional standard Brownian motion $B = (B_t)_{t\geq 0}$, where $B_t = (B_t^1, \dots, B_t^m)$, $B_0 = 0$. Define now an n-dimensional stochastic process $\mathrm{X} = (\mathrm{X}_t)_{t\geq 0}$, $\mathrm{X}_t = (\mathrm{X}_t^1, \dots, \mathrm{X}_t^n))$ with the initial condition $\mathrm{X}_0(w) = x$, where $x \in \overline{G}$, as follows:

$$\mathrm{X}_t(w) = x + \sum_{j=1}^{m} \sigma_j B_t^j(w),$$

where σ_j is the jth column of the matrix $\sigma = (\sigma_{kj})$ $k = 1, \dots, n$, $j = 1, \dots, m$. We seek a pair of continuous processes $(z, y) = (z_t, y_t)_{t\geq 0}$, where $z_t = (z_t^1, \dots, z_t^n)$, $y_t = (y_t^1, \dots, y_t^N)$, which jointly satisfy (P – a.s.) the following conditions:

(1) $z_t \in \overline{G}$, $t \geq 0$,

(2) $z_t = x + \sum_{j=1}^m \sigma_j B_t^j + \int_0^t b(z_s)ds + \sum_{k=1}^N \int_0^t q_k(z_s)dy_s^k$, $t \geq 0$, (29)

(3) the component functions y_t^i, $i = 1, \dots, N$, are nondecreasing processes with $y_0^i = 0$, $i = 1, \dots, N$, and with the property

$$\int_0^t I(n_i \cdot z_s > c_i)dy_s^i = 0, \;\; i = 1, \dots, N.$$

Let us introduce the σ-algebras $\mathcal{F}_t = \sigma(\mathrm{X}_s, s \leq t)$, $t \geq 0$.

Theorem 4. *Let the assumption* (11) *hold and the functions* $b(x)$, $q_k(x)$, $k = 1, \ldots, N$, *be Lipschitz continuous,*

$$|b(x) - b(\widetilde{x})| \leq L|x - \widetilde{x}|,$$
$$|q_k(x) - q_k(\widetilde{x})| \leq L|x - \widetilde{x}|, \quad k = 1, \ldots, N.$$

Then there exists (P *– a.s.*) *the unique solution of the system* (29)*, which is adapted to* $\mathcal{F}_t$, z_t *is a strong Markov process with stationary transition probabilities.*

Proof. First we consider the corresponding Skorokhod problem for an arbitrary continuous n-dimensional function $x = x_t$, $t \geq 0$, $x_0 \in \overline{G}$. As we know from Theorem 3, there exists the unique solution $(z, y) = (z_t, y_t)_{t \geq 0}$ of the problem, and if we introduce the mappings $z = \Phi(x)$, $y = \psi(x)$, then they will be continuous in the uniform topology. Further, the restrictions of z and y to $[0, t]$ depend only on the restrictions of x to $[0, t]$. From the uniqueness of the solution we obtain the following shift property of the mappings Φ and ψ:

$$z_{s+t} = \Phi(z_s + (x_{s+\cdot} - x_s))_t,$$
$$y_{s+t} = y_s + \psi(z_s + (x_{s+\cdot} - x_s))_t.$$

Let us now define the continuous processes $z_t(w) = \Phi(\mathrm{X}(w))_t$, $y_t(w) = \psi(\mathrm{X}(w))_t$, where

$$X_t(w) = x + \sum_{j=1}^{m} \sigma_j B_t^j(w).$$

Then we obtain the solution (z_t, y_t) of the system (29), which is obviously unique. From the continuity of the above-mentioned mappings we get that the processes z_t and y_t are adapted to the filtration $\mathcal{F}_t$. Let $\tau(w)$ be an arbitrary $\mathcal{F}_{t+}$-stopping time. Then we have $z_{\tau+t} = \Phi(z_\tau + (\mathrm{X}_{\tau+\cdot} - \mathrm{X}_\tau))_t$.

Now, since z_τ is $\mathcal{F}_{\tau+}$-measurable and the process $(\mathrm{X}_{\tau+u} - \mathrm{X}_\tau)$, $u \geq 0$, is independent of the σ-algebra $\mathcal{F}_{\tau+}$ (by the Markov property of the Brownian motion $B = B_t$, $t \geq 0$, with respect to the filtration $\mathcal{F}_{t+}^B$), having the probability distribution which is also independent of τ, we assert that $z = z_t$, $t \geq 0$, is indeed a strong Markov process with stationary transition probabilities. $\square$

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Author's address:
Laboratory of Stochastic Analysis
and Statistical Decisions
I. Javakhishvili Tbilisi State University
2, University St., Tbilisi 380043
Republic of Georgia