GEORGIAN MATHEMATICAL JOURNAL: Vol. 3, No. 2, 1996, 133-140

ON THE NON-COMMUTATIVE NEUTRIX PRODUCT $(x_+^r \ln x_+) \circ x_-^{-s}$

ADEM KILIÇMAN AND BRIAN FISHER

ABSTRACT. The non-commutative neutrix product of the distributions $x_+^r \ln x_+$ and x_-^{-s} is evaluated for $r = 0, 1, 2, \dots$ and $s = 1, 2, \dots$. Further neutrix products are then deduced.

In the following, we let N be the neutrix (see van der Corput [1]) having domain $N' = \{1, 2, \dots, n, \dots\}$ and range the real numbers, with negligible functions finite linear sums of the functions $n^\lambda \ln^{r-1} n$, $\ln^r n$, $\lambda > 0$, $r = 1, 2, \dots$, and all functions which converge to zero in the normal sense as n tends to infinity.

We now let $\rho(x)$ be any infinitely differentiable function having the following properties:

(i) $\rho(x) = 0$ for $|x| \geq 1$,
(ii) $\rho(x) \geq 0$,
(iii) $\rho(x) = \rho(-x)$,
(iv) $\int_{-1}^{1} \rho(x)\, dx = 1$.

Putting $\delta_n(x) = n\rho(nx)$ for $n = 1, 2, \dots$, it follows that $\{\delta_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the Dirac delta-function $\delta(x)$.

Now let $\mathcal{D}$ be the space of infinitely differentiable functions with compact support and let $\mathcal{D}'$ be the space of distributions defined on $\mathcal{D}$. Then if f is an arbitrary distribution in $\mathcal{D}'$, we define $f_n(x) = (f * \delta_n)(x) = \langle f(t), \delta_n(x-t)\rangle$ for $n = 1, 2, \dots$. It follows that $\{f_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the distribution $f(x)$.

A first extension of the product of a distribution and an infinitely differentiable function is the following (see for example [2] or [3]).

1991 *Mathematics Subject Classification.* 46F10.
Key words and phrases. Distribution, delta-function, neutrix, neutrix limit, neutrix product.

Definition 1. Let f and g be distributions in $\mathcal{D}'$ for which on the interval (a,b), f is the kth derivative of a locally summable function F in $L^p(a,b)$ and $g^{(k)}$ is a locally summable function in $L^q(a,b)$ with $1/p+1/q=1$. Then the product $fg=gf$ of f and g is defined on the interval (a,b) by

$$fg=\sum_{i=0}^{k}\binom{k}{i}(-1)^i[Fg^{(i)}]^{(k-i)}.$$

The following definition for the neutrix product of two distributions was given in [4] and generalizes Definition 1.

Definition 2. Let f and g be distributions in $\mathcal{D}'$ and let $g_n(x)=(g*\delta_n)(x)$. We say that the neutrix product $f\circ g$ of f and g exists and is equal to the distribution h on the interval (a,b) if

$$\underset{n\to\infty}{N\text{-}\lim}\langle f(x)g_n(x),\phi(x)\rangle=\langle h(x),\phi(x)\rangle^1$$

for all functions ϕ in $\mathcal{D}$ with support contained in the interval (a,b).

Note that if

$$\lim_{n\to\infty}\langle f(x)g_n(x),\phi(x)\rangle=\langle h(x),\phi(x)\rangle,$$

we simply say that the *product* $f.g$ exists and equals h (see [4]).

It is obvious that if the product $f.g$ exists then the neutrix product $f\circ g$ exists and $f.g=f\circ g$. Further, it was proved in [4] that if the product fg exists by Definition 1 then the product $f.g$ exists by Definition 2 and $fg=f.g$. Note also that although the product defined in Definition 1 is always commutative, the product and neutrix product defined in Definition 2 is in general non-commutative.

The following theorem holds (see [5]).

Theorem 1. *Let f and g be distributions in $\mathcal{D}'$ and suppose that the neutrix products $f\circ g^{(i)}$ (or $f^{(i)}\circ g$) exist on the interval (a,b) for $i=0,1,2,\dots,r$. Then the neutrix products $f^{(k)}\circ g$ (or $f\circ g^{(k)}$) exist on the interval (a,b) for $k=1,2,\dots,r$ and*

$$f^{(k)}\circ g=\sum_{i=0}^{k}\binom{k}{i}(-1)^i[f\circ g^{(i)}]^{(k-i)} \tag{1}$$

or

$$f\circ g^{(k)}=\sum_{i=0}^{k}\binom{k}{i}(-1)^i[f^{(i)}\circ g]^{(k-i)} \tag{2}$$

on the interval (a,b) for $k=1,2,\dots,r$.

---

[1]See [1] or [4] for the definition of N-lim.

In the next two theorems, which were proved in [5] and [6] respectively, the distributions x_+^{-r} and x_-^{-r} are defined by

$$x_+^{-r} = \frac{(-1)^{r-1}}{(r-1)!}(\ln x_+)^{(r)}, \quad x_-^{-r} = -\frac{1}{(r-1)!}(\ln x_-)^{(r)},$$

for $r = 1, 2, \dots$ and not as in the book of Gel'fand and Shilov [7].

Theorem 2. *The neutrix products $x_+^r \circ x_-^{-s}$ and $x_-^{-s} \circ x_+^r$ exist and*

$$x_+^r \circ x_-^{-s} = x_+^r x_-^{-s} = 0, \tag{3}$$
$$x_-^{-s} \circ x_+^r = x_-^{-s} x_+^r = 0 \tag{4}$$

for $r = s, s+1, \dots$ and $s = 1, 2, \dots$ and

$$x_+^r \circ x_-^{-s} = \sum_{i=r+1}^{s} \binom{s}{i} \frac{(-1)^{i-1} r!}{(s-1)!} c_1(\rho) \delta^{(s-r-1)}(x), \tag{5}$$
$$x_-^{-s} \circ x_+^r = \sum_{i=r+1}^{s} \binom{s}{i} \frac{(-1)^{i-1} r!}{(s-1)!} [c_1(\rho) + \tfrac{1}{2}\psi(i-r-1)] \delta^{(s-r-1)}(x) \tag{6}$$

for $r = 0, 1, \dots, s-1$ and $s = 1, 2, \dots$, where

$$c_1(\rho) = \int_0^1 \ln t \rho(t)\, dt, \quad \psi(r) = \begin{cases} 0, & r = 0, \\ \sum_{i=1}^r i^{-1}, & r \geq 1. \end{cases}$$

Theorem 3. *The neutrix products $x_+^{-r} \circ x_-^{-s}$ and $x_-^{-s} \circ x_+^{-r}$ exist and*

$$x_+^{-r} \circ x_-^{-s} = \frac{(-1)^r c_1(\rho)}{(r+s-1)!} \delta^{(r+s-1)}(x), \tag{7}$$
$$x_-^{-s} \circ x_+^{-r} = \frac{(-1)^{r-1} c_1(\rho)}{(r+s-1)!} \delta^{(r+s-1)}(x) \quad \text{for } r, s = 1, 2, \dots. \tag{8}$$

It was shown in [8] that with suitable choice of the function ρ, $c_1(\rho)$ can take any negative value.

We now prove the following theorem.

Theorem 4. *The neutrix products $(x_+^r \ln x_+) \circ x_-^{-s}$ and $x_-^{-s} \circ (x_+^r \ln x_+)$ exist and*

$$(x_+^r \ln x_+) \circ x_-^{-s} = (x_+^r \ln x_+) x_-^{-s} = 0, \tag{9}$$
$$x_-^{-s} \circ (x_+^r \ln x_+) = x_-^{-s}(x_+^r \ln x_+) = 0 \tag{10}$$

for $r = s, s+1, s+2\dots$ *and* $s = 1, 2, \dots$ *and*

$$(x_+^r \ln x_+) \circ x_-^{-s} = \frac{(-1)^r}{(s-r-1)!}\Big(c_2 - \frac{\pi^2}{12}\Big)\delta^{(s-r-1)}(x) - \sum_{i=r+1}^{s-1} \frac{(-1)^i r! c_1}{(s-i-1)!i!(i-r)}\delta^{(s-r-1)}(x) - \psi(r)\sum_{i=r+1}^{s} \frac{(-1)^i s r! c_1}{i!(s-i)!}\delta^{(s-r-1)}(x), \tag{11}$$

$$x_-^{-s} \circ (x_+^r \ln x_+) = \frac{(-1)^r}{(s-r-1)!}\Big(c_2 - \frac{\pi^2}{12}\Big)\delta^{(s-r-1)}(x) - \sum_{i=r+1}^{s-1} \frac{(-1)^i r! c_1}{(s-i-1)!i!(i-r)}\delta^{(s-r-1)}(x) - \psi(r)\sum_{i=r+1}^{s} \frac{(-1)^i s r!}{i!(s-i)!}[c_1 + \tfrac{1}{2}\psi(i-r-1)]\delta^{(s-r-1)}(x) \tag{12}$$

for $r = 0, 1, 2, \dots, s-1$ *and* $s = 1, 2\dots$ *, where*

$$c_2(\rho) = \int_0^1 \ln^2 t\rho(t)\,dt.$$

Proof. We first of all prove that

$$\ln x_+ \circ x_-^{-1} = \Big(c_2 - \frac{\pi^2}{12}\Big)\delta(x). \tag{13}$$

We put $(x_-^{-1})_n = x_-^{-1} * \delta_n(x)$ so that

$$(x_-^{-1})_n = -\int_x^{1/n} \ln(t-x)\delta_n'(t)\,dt$$

on the interval $[0, 1/n]$, the intersection of the supports of $\ln x_+$ and $(x_-^{-1})_n$. Then

$$\begin{aligned}\langle \ln x_+, (x_-^{-1})_n\rangle &= -\int_0^{1/n} \ln x \int_x^{1/n} \ln(t-x)\delta_n'(t)\,dt\,dx \\ &= -\int_0^{1/n} \delta_n'(t)\int_0^t \ln x \ln(t-x)\,dx\,dt \\ &= -\int_0^1 \rho'(u)\int_0^u [\ln v - \ln n][\ln(u-v) - \ln n]\,dv\,du\end{aligned}$$

on making the substitutions $nt = u$ and $nx = v$. It follows that

$$\begin{aligned} N\text{-}\lim_{n\to\infty}\langle \ln x_+, (x_-^{-1})_n\rangle &= -\int_0^1 \rho'(u)\int_0^u \ln v \ln(u-v)\,dv\,du \\ &= -\int_0^1 u\rho'(u)\int_0^1 [\ln u + \ln y][\ln u + \ln(1-y)]\,dy\,du \end{aligned} \tag{14}$$

on making the substitution $v = uy$.

Now

$$\begin{aligned} &\int_0^1 \ln y\,dy = \int_0^1 \ln(1-y)\,dy = -1, \\ &\int_0^1 \ln y \ln(1-y)\,dy = -\int_0^1 \ln(1-y)\,dy + \int_0^1 \frac{y\ln y}{1-y}\,dy \\ &\qquad = 1 + \sum_{i=1}^{\infty}\int_0^1 y^i \ln y\,dy = 1 - \sum_{i=1}^{\infty}(i+1)^{-2} = 2 - \frac{\pi^2}{6}, \\ &\int_0^1 u\rho'(u)\,du = -\int_0^1 \rho(u)\,du = -\tfrac{1}{2}, \\ &\int_0^1 u\ln u\rho'(u)\,du = -\int_0^1 (1+\ln u)\rho(u)\,du = -\tfrac{1}{2} - c_1, \\ &\int_0^1 u\ln^2 u\rho'(u)\,du = -\int (2\ln u + \ln^2 u)\rho(u)\,du = -2c_1 - c_2 \end{aligned}$$

and it follows from these equations and equation (14) that

$$N\text{-}\lim_{n\to\infty}\langle \ln x_+, (x_-^{-1})_n\rangle = c_2 - \frac{\pi^2}{12}. \tag{15}$$

Further, it follows as above that

$$\begin{aligned} \langle \ln x_+, x(x_-^{-1})_n\rangle &= -\int_0^{1/n} x\ln x\int_x^{1/n} \ln(t-x)\delta_n'(t)\,dt\,dx \\ &= -n^{-1}\int_0^1 \rho'(u)\int_0^u v[\ln v - \ln n][\ln(u-v) - \ln u]\,dv\,du \\ &= O(n^{-1}\ln n). \end{aligned}$$

Now let ϕ be an arbitrary function in $\mathcal{D}$. Then $\phi(x) = \phi(0) + x\phi'(\xi x)$, where $0 < \xi < 1$. It follows that

$$\begin{aligned} \langle \ln x_+ (x_-^{-1})_n, \phi(x)\rangle - \phi(0)\langle \ln x_+, (x_-^{-1})_n\rangle &= \langle \ln x_+, x(x_-^{-1})_n\phi'(\xi x)\rangle \\ &= O(n^{-1}\ln n) \end{aligned} \tag{16}$$

since $\langle \ln x_+, x(x_-^{-1})_n \rangle = O(n^{-1}\ln n)$. Thus

$$\operatorname*{N-lim}_{n\to\infty}\langle \ln x_+(x_-^{-1})_n, \phi(x)\rangle = \operatorname*{N-lim}_{n\to\infty}\phi(0)\langle \ln x_+, (x_-^{-1})_n\rangle = \Big(c_2 - \frac{\pi^2}{12}\Big)\phi(0)$$

on using equations (15) and (16). Equation (9) follows.

We now define the function $f(x_+, r)$ by

$$f(x_+, r) = \frac{x_+^r \ln x_+ - \psi(r)x_+^r}{r!}$$

and it follows easily by induction that $f^{(i)}(x_+, r) = f(x_+, r-i)$, for $i = 0, 1, \ldots, r$. In particular, $f^{(r)}(x_+, r) = \ln x_+$, so that

$$f^{(i)}(x_+, r) = (-1)^{i-r-1}(i-r-1)!x_+^{-i+r},$$

for $i = r+1, r+2, \ldots$. Now the product of the functions x_+^i and $x_+^i \ln x_+$ and the distribution x_-^{-1} exists by Definition 1 and it is easily seen that

$$x_+^i x_-^{-1} = (x_+^i \ln x_+)x_-^{-1} = 0, \tag{17}$$

for $i = 1, 2, \ldots, r$. Using equation (13) we have

$$f^{(r)}(x_+, r) \circ x_-^{-1} = \Big(c_2 - \frac{\pi^2}{12}\Big)\delta(x) \tag{18}$$

and using equation (7) we have

$$f^{(i)}(x_+, r) \circ x_-^{-1} = -\frac{c_1}{i-r}\delta^{(i-r)}(x) \tag{19}$$

for $i = r+1, r+2, \ldots$.

Using equations (2) and (17) we now have

$$\begin{aligned}(s-1)!f(x_+, r)x_-^{-s} &= \sum_{i=0}^{s-1}\binom{s-1}{i}(-1)^i[f^{(i)}(x_+, r)x_-^{-1}]^{(s-i-1)}\\ &= \frac{(s-1)!}{r!}[x_+^r \ln x_+ - \psi(r)x_+^r]x_-^{-s} = 0,\end{aligned}$$

for $r = s, s+1, s+2, \ldots$ and $s = 1, 2, \ldots$. Equations (9) follow on using equations (3).

When $r < s$ we have

$$\begin{aligned}(s-1)!f(x_+, r) \circ x_-^{-s} &= \sum_{i=r}^{s-1}\binom{s-1}{i}(-1)^i[f^{(i)}(x_+, r) \circ x_-^{-1}]^{(s-i-1)}\\ &= \binom{s-1}{r}(-1)^r\Big(c_2 - 2 + \frac{\pi^2}{12}\Big)\delta^{(s-r-1)}(x) +\end{aligned}$$

$$-\sum_{i=r+1}^{s-1}\binom{s-1}{i}\frac{(-1)^i c_1}{i-r}\delta^{(s-r-1)}(x)$$

on using equations (2), (17), (18) and (19). It now follows that

$$(x_+^r \ln x_+)\circ x_-^{-s} = r!f(x_+,r)\circ x_-^{-s} + \psi(r)x_+^r \circ x_-^{-s}$$

and equation (11) follows on using equation (5).

We now consider the product $x_-^{-s}\circ(x_+^{-r}\ln x_+)$. The product $\ln x_- \ln x_+$ exists by Definition 1 and $\ln x_- \ln x_+ = 0$. Differentiating, we get

$$x_-^{-1}\circ \ln x_+ = \ln x_- \circ x_+^{-1} = \Big(c_2 - \frac{\pi^2}{12}\Big)\delta(x) \tag{20}$$

on replacing x by $-x$ in equation (13).

As above, we have

$$x_-^{-1}x_+^i = x_-^{-1}(x_+^i \ln x_+) = 0, \tag{21}$$

for $i = 0, 1, \dots, r-1$. Using equation (20) we have

$$x_-^{-1}\circ f^{(r)}(x_+,r) = \Big(c_2 - \frac{\pi^2}{12}\Big)\delta(x) \tag{22}$$

and using equation (8) we have

$$x_-^{-1}\circ f^{(i)}(x_+,r) = \frac{c_1}{i-r}\delta^{(i-r)}(x), \tag{23}$$

for $i = r+1, r+2, \dots$. Equations (10) follow as above on using equations (1) and (21) and equations (12) follow on using equations (1), (6), (20), (21), (22), and (23). □

Corollary. *The neutrix products* $(x_-^{-r}\ln x_-)\circ x_+^{-s}$ *and* $x_+^{-s}\circ(x_-^r \ln x_-)$ *exist and*

$$(x_-^r \ln x_-)\circ x_+^{-s} = (x_-^{-r}\ln x_-)x_+^{-s} = 0,$$
$$x_+^{-s}\circ(x_-^r \ln x_-) = x_+^{-s}(x_-^r \ln x_-) = 0,$$

for $r = s, s+1, s+2, \dots$ *and* $s = 1, 2, \dots$ *and*

$$\begin{aligned}(x_-^r \ln x_-)\circ x_+^{-s} = {}& \frac{(-1)^{s+1}}{(s-r-1)!}\Big(c_2 - \frac{\pi^2}{12}\Big)\delta^{(s-r-1)}(x)\\ &+ \sum_{i=r+1}^{s-1}\frac{(-1)^{s-r+i}r!c_1}{(s-i-1)!i!(i-r)}\delta^{(s-r-1)}(x)\\ &+ \psi(r)\sum_{i=r+1}^{s}\frac{(-1)^{s-r+i}sr!c_1}{i!(s-i)!}\delta^{(s-r-1)}(x),\end{aligned}$$

$$x_+^{-s} \circ (x_-^r \ln x_-) = \frac{(-1)^{s+1}}{(s-r-1)!}\Big(c_2 - \frac{\pi^2}{12}\Big)\delta^{(s-r-1)}(x) + \sum_{r+1}^{s-1} \frac{(-1)^{s-r+i} r! c_1}{(s-i-1)! i! (i-r)} \delta^{(s-r-1)}(x) + \psi(r) \sum_{i=r+1}^{s} \frac{(-1)^{s-r+i} s r!}{i!(s-i)!}[c_1 + \tfrac{1}{2}\psi(i-r-1)]\delta^{(s-r-1)}(x)$$

for $r = 0, 1, 2, \dots, s-1$ *and* $s = 1, 2, \dots$.

Proof. The results follow immediately on replacing x by $-x$ in equations (9), (10), (11), and (12). □

References

1. J. G. van der Corput, Introduction to the neutrix calculus. *J. Analyse Math.* **7**(1959–60), 291–398.
2. B. Fisher, The product of distributions. *Quart. J. Math. Oxford* (2) **22**(1971), 291–298.
3. B. Fisher, On defining the product of distributions. *Math. Nachr.* **99**(1980), 239–249.
4. B. Fisher, A non-commutative neutrix product of distributions. *Math. Nachr.* **108**(1982), 117–127.
5. B. Fisher, E. Savaş, S. Pehlivan, and E. Özçağ, Results on the non-commutative neutrix product of distributions. *Math. Balkanica* **7**(1993), 347–356.
6. B. Fisher and A. Kiliçman, The non-commutative neutrix product of the distributions x_+^{-r} and x_-^{-s}. *Math. Balkanica* (*to appear*).
7. I. M. Gel'fand and G. E. Shilov, Generalized functions. Vol. I: Properties and operations. *Academic Press, New York–London,* 1964; Russian original: Fizmatgiz, Moscow, 1958.
8. B. Fisher, Some results on the non-commutative neutrix product of distributions. *Trabajos de Matematica* **44**, *Buenos Aires,* 1983.

(Received 10.08.1994)

Authors' address:
Department of Mathematics and Computer Science
University of Leicester, Leicester
LE1 7RH, England