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ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF A SECOND ORDER NONLINEAR DIFFERENTIAL EQUATION

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ABSTRACT. Asymptotic properties of proper solutions of a certain class of essentially nonlinear binomial differential equations of the second order are investigated.

INTRODUCTION

Let us consider a nonlinear differential equation of the second order

$$y'' = \alpha_0 p(t) \exp(\sigma y)|y'|^{\lambda}, \tag{0.1}$$

where $\alpha_0 \in \{-1; 1\}$; $\sigma, \lambda \in \mathbb{R}$, $\sigma \neq 0$, $\lambda \neq 1$, $\lambda \neq 2$; $p : [a, \omega[\to]0, +\infty[$ $(-\infty < a < \omega \leq +\infty)$ is a continuously differentiable function. Opposite to the well-studied Emden–Fowler equation of the type

$$y'' = \alpha_0 p(t)|y|^{\sigma}|y'|^{\lambda} \operatorname{sign} y, \tag{0.2}$$

the above binomial equation has nonlinearity of another type. The main results about the behavior of the solutions of (0.2) when $\lambda = 0$ are given in the monograph [1]. Asymptotic behavior of monotonic solutions of (0.2) when $\lambda \neq 0$ is investigated in [2]–[6].

Equation of type (0.1) as well as of (0.2) are derived while describing different physical processes. In particular, the equation $\frac{1}{r}\frac{d}{dr}\Big(r\frac{d\varphi}{dr}\Big) = A\exp(\nu\varphi) + B\exp(-\nu\varphi)$ from electrodynamics and the equation $u'' = u\exp(\alpha x - u)/2$ from combustion theory reduce to the equation of type (0.1) with the help of some transformations [7].

In this work asymptotic representations of all proper solutions of (0.1) and their first derivatives are obtained when certain conditions on the function p are satisfied.

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§ 1. Formulation of Basic Results

A real solution y of equation (0.1) is said to be proper if it is defined in the left neighborhood of ω, and for certain t_0 from this neighborhood $y'(t) \neq 0$ for $t \in [t_0; \omega[$.

Let us introduce the auxiliary notation

$$\Gamma(t) = \frac{\alpha_0 \sigma}{\lambda - 2}\Big[\frac{1}{2-\lambda} p^{\frac{\lambda-3}{2-\lambda}}(t) p'(t) \int\limits_{\gamma_0}^{t} p^{\frac{1}{2-\lambda}}(s)ds - 1\Big],$$

$$V(t) = \Big|\frac{\sigma}{\lambda - 2} \int\limits_{\gamma_0}^{t} p^{\frac{1}{2-\lambda}}(s)ds\Big|^{\frac{\lambda-2}{\sigma}};$$

$$\gamma_0 = \begin{cases} a, \ \text{if} \ \int_a^\omega p^{\frac{1}{2-\lambda}}(s)ds = +\infty \\ \omega \ \text{if} \ \int_a^\omega p^{\frac{1}{2-\lambda}}(s)ds < +\infty \end{cases}; \quad \beta_0 = \begin{cases} -1, \ \text{if} \ \lim_{t\uparrow\omega} V(t) = 0 \\ 1, \ \text{if} \ \lim_{t\uparrow\omega} V(t) = +\infty \end{cases}.$$

When the conditions

$$\lim_{t\uparrow\omega} \Gamma(t) = \Gamma_0, \quad 0 < |\Gamma_0| < +\infty \tag{1.1}$$

are fulfilled, the following statements hold.

Theorem 1.1. *Let $\omega \leq +\infty$. If $\Gamma_0 < 0$, then each proper solution y of equation* (0.1) *admits one of the representations*

$$y(t) = c + o(1), \ \ t \uparrow \omega \ \ \textit{for} \ \ \omega < +\infty, \tag{1.2$_1$}$$

$$y(t) = c_1 t + o(1), \ \ t \to +\infty \ \ \textit{for} \ \ \omega = +\infty, \tag{1.2$_2$}$$

where $c \in \mathbb{R}$, $c_1\sigma \leq 0$.

If $\Gamma_0 > 0$ and $\alpha_0\sigma > 0$, then each proper solution y of (0.1) *admits one of the representations* (1.2_i) $(i \in \{1; 2\})$ *or*

$$y(t) = \ln[|\Gamma_0|^{\frac{1}{\sigma}} V(t)] + o(1), \ \ t \uparrow \omega. \tag{1.3}$$

If $\Gamma_0 > 0$ and $\alpha_0\sigma < 0$, then each proper solution y of (0.1) *either admits one of the representations* (1.2_i) $(i \in \{1; 2\})$, (1.3) *or there exists a sequence $\{t_k\} \uparrow \omega$, $k \to \infty$, such that $y(t_k) = \frac{1}{\sigma}\ln[V^\sigma(t_k)\Gamma(t_k)]$, $k = 1, 2, \dots$.*

Theorem 1.2. *Let $\omega < \infty$. The derivative of each proper solution y of the type* (1.2_1) *of equation* (0.1) *satisfies one of the asymptotic representations*

$$y'(t) = c_0 + o(1), \ \ t \uparrow \omega, \tag{1.4}$$

or

$$y'(t) = \left|(1-\lambda)\exp(\sigma c)\int\limits_{\rho}^{t} p(s)ds\right|^{\frac{1}{1-\lambda}}[\nu + o(1)], \quad t \uparrow \omega, \tag{1.5}$$

where $\rho = \omega$, $\nu = -\operatorname{sign}(1-\lambda)$ *if* $\int_a^\omega p(t)dt < iy$, *and* $\rho = a$, $\nu = \alpha_0 \operatorname{sign}(1-\lambda)$ *otherwise.*

For a proper solution y of equation (0.1) *admitting one of the representations* (1.2_1), (1.4) ((1.2_1), (1.5)) *to exist, it is necessary and sufficient that* $\int\limits_a^\omega p(t)dt < \infty$ $(\sigma\beta_0(1-\lambda) > 0)$.

Theorem 1.3. *Let $\omega = +\infty$. For a proper solution of equation* (0.1)*, y of the type* (1.2_2)*, where $c_1 = 0$, to exist, it is necessary and sufficient that $\sigma\beta_0(1-\lambda) > 0$. The derivative of each of such solutions satisfies* (1.5).

Theorem 1.4. *Let $\omega = +\infty$. For arbitrary c_1 satisfying the inequality $\sigma c_1 < 0$ and $c \in \mathbb{R}$, equation* (0.1) *possesses a proper solution y admitting representation* (1.2_2)*. The derivative of each of such solutions is represented in the form*

$$y'(t) = c_1 + o(1), \quad t \to +\infty.$$

Theorem 1.5. *Let $\omega \le +\infty$. For a proper solution y of equation* (0.1) *of the type* (1.3) *to exist, it is necessary and sufficient that $\Gamma_0 > 0$. The derivative of each of such solutions satisfies the relation*

$$y'(t) = \frac{V'(t)}{V(t)}[1 + o(1)], \quad t \uparrow \omega.$$

§ 2. Some Auxiliary Statements

Let us consider the system of differential equations

$$\begin{cases} u_1' = f_1(\tau) + a_{11}(\tau)u_1 + a_{12}(\tau)u_2 + g_1(\tau)X_1(\tau, u_1, u_2) \\ u_2' = f_2(\tau) + a_{21}(\tau)u_1 + a_{22}(\tau)u_2 + g_2(\tau)X_2(\tau, u_1, u_2) \end{cases}, \tag{2.1}$$

where the functions $f_1, g_1 : [T, +\infty[\to \mathbb{R}$ $(i = 1, 2)$, $a_{ij} : [T, +\infty[\to \mathbb{R}$ $(i, j = 1, 2)$ are continuous and the functions $X_i : \Omega \to \mathbb{R}$ $(i = 1, 2)$ are continuous in r, u_1, u_2 in the domain

$$\Omega = [T, +\infty[\times D, \quad D = \{(u_1, u_2) : |u_1| \le \delta, \ |u_2| \le \delta, \ \delta > 0\}. \tag{2.2}$$

Introduce the following notation: $a_i(\tau,t)=\exp\int_t^\tau a_{ii}(s)ds$ $(i=1,2)$;

$$A_2(\tau)=\Big|\int_{\alpha_2}^{\tau}|a_{21}(t)|a_2(\tau,t)dt\Big|;\quad A_1(\tau)=\Big|\int_{\alpha_1}^{\tau}|a_{12}(t)|A_2(t)a_1(\tau,t)dt\Big|;$$

$$F_2(\tau)=\Big|\int_{\beta_2}^{\tau}|f_2(t)|a_2(\tau,t)dt\Big|;\quad G_2(\tau)=\Big|\int_{\gamma_2}^{\tau}|g_2(t)|a_2(\tau,t)dt\Big|;$$

$$F_1(\tau)=\Big|\int_{\beta_1}^{\tau}|f_1(t)|a_1(\tau,t)dt\Big|+\Big|\int_{\beta_{12}}^{\tau}|a_{12}(t)|F_2(t)a_1(\tau,t)dt\Big|;$$

$$G_1(\tau)=\Big|\int_{\gamma_1}^{\tau}|g_1(t)|a_1(\tau,t)dt\Big|+\Big|\int_{\gamma_{12}}^{\tau}|a_{12}(t)|G_2(t)a_1(\tau,t)dt\Big|,$$

where each of the limits of integration $\alpha_i,\beta_i,\gamma_i$ $(i=1,2)$, β_{12},γ_{12} is equal either to T or to $+\infty$ and is chosen in a special way: in every integral defining the functions F_i,A_i,G_i $(i=1,2)$ and having the form

$$I(\mu,\tau)=\int_{\mu}^{\tau}|b(t)|\exp\int_{t}^{\tau}a(s)\,ds\,dt, \tag{2.3}$$

we put $\mu=+\infty$ if the integral $I(T,+\infty)$ converges, and $\mu=T$ otherwise.

Theorem 2.1. *Let the functions X_i $(i=1,2)$ have bounded partial derivatives with respect to the variables u_1,u_2 in the domain Ω and let $X_i(\tau,0,0)\equiv0$ $(i=1,2)$ for $\tau\in[T;+\infty[$. If*

$$\lim_{\tau\to+\infty}F_i(\tau)=\lim_{\tau\to+\infty}G_i(\tau)=0,\quad \lim_{\tau\to+\infty}A_i(\tau)=A_i^o<1\ \ (i=1,2),$$

then (2.1) *possesses at least one real solution $(u_1(\tau),u_2(\tau))$ tending to zero as $\tau\to+\infty$.*

Theorem 2.2. *Let $X_i(\tau,0,0)\equiv0$ $(i=1,2)$ for $\tau\ge T$, and let the functions $\frac{\partial X_i(\tau,u_1,u_2)}{\partial u_k}$ $(i,k=1,2)$ tend to zero as $|u_1|+|u_2|\to0$ uniformly with respect to $\tau\in[T,+\infty[$. If*

$$\lim_{\tau\to+\infty}F_i(\tau)=0,\quad \lim_{\tau\to+\infty}A_i(\tau)=A_i^o<1,\quad \lim_{\tau\to+\infty}G_i(\tau)=const\ \ (i=1,2),$$

then (2.1) *possesses at least one real solution $(u_1(\tau),u_2(\tau))$ tending to zero as $\tau\to+\infty$.*

Theorems 2.1 and 2.2 immediately follow from the results of Kostin's work [8].

We will use also the following statements dealing with limit properties of integrals of the type (2.3) ([2], [8]).

Lemma 2.1. *Let a function* $a : [T, +\infty[\to \mathbb{R}$ *be continuous and represented in the form* $a(t) = a_0(t) + \alpha(t)$, *where* $a_0 : [T, +\infty[\to \mathbb{R}$ *is a continuous function of constant sign* (*in particular, it can be* $a_0(t) \equiv 0$) *in a certain neighborhood of* $+\infty$; $\alpha : [T, +\infty[\to \mathbb{R}$ *is such that* $\int_T^{+\infty} \alpha(t)dt$ *converges. If* $b : [T, +\infty[\to \mathbb{R}$ *is continuous and* $\int_T^{+\infty} |b(t)|dt < \infty$, *then* $\lim_{\tau\to+\infty} I(\mu, \tau) = 0$, *where* μ *is chosen as stated above.*

Lemma 2.2. *Let the function* a *satisfy the conditions of Lemma* 2.1. *If* $\left|\int_T^{+\infty} a_0(t)dt\right| = \infty$ *and the function* $b : [T; +\infty[\to \mathbb{R}$ *is continuous and satisfies the asymptotic correlation* $|b(t)| = a_0(t)[q + o(1)]$, $t \to +\infty$ *with* $q \in \mathbb{R}$, *then* $\lim_{\tau\to+\infty} I(\tau, \mu) = 0$, *where* μ *is chosen as stated above.*

§ 3. Investigation of an Auxiliary Equation

Let us consider a second-order nonlinear differential equation

$$\Big(\frac{\xi'(\tau)}{\xi(\tau)} + \beta_0\Big)' + \beta_0 S(\tau)\Big(\frac{\xi'(\tau)}{\xi(\tau)} + \beta_0\Big) = \alpha_0 \xi^\sigma(\tau)\Big|\frac{\xi'(\tau)}{\xi(\tau)} + \beta_0\Big|^\lambda, \quad (3.1)$$

where $\alpha_0, \beta_0 \in \{-1, 1\}$; $\lambda, \sigma \in \mathbb{R}$, $\sigma \neq 0$, $\lambda \neq 1$, $\lambda \neq 2$, and the function $S : [b, +\infty[\to \mathbb{R}$ is continuous and satisfies

$$\lim_{\tau\to+\infty} S(\tau) = S_0, \quad 0 < |S_0| < \infty. \quad (3.2)$$

A real solution ξ of equation (3.1) will be said to be proper if it is defined in a certain neighborhood of $+\infty$, and for some τ_0 from this neighborhood it satisfies the inequalities $\xi(\tau) > 0$, $\xi'(\tau) + \beta_0\xi(\tau) \neq 0$ for $\tau \geq \tau_0$.

Theorem 3.1. *Each proper solution* ξ *of equation* (3.1) *either has no limit as* $\tau \to +\infty$, *and then there exists a sequence* $\{\tau_k\}_{k=1}^\infty$ *converging to* $+\infty$ *with* $\xi^\sigma(\tau_k) = \alpha_0 S(\tau_k)$, $k = 1, 2, \dots$ *or it possesses one of the properties*

$$\lim_{\tau\to+\infty} \xi(\tau) = \xi_0, \quad 0 < \xi_0 < +\infty; \quad (3.3)$$

$$\lim_{\tau\to+\infty} \xi'(\tau)/\xi(\tau) = -\beta_0; \quad (3.4)$$

$$\lim_{\tau\to+\infty} \xi'(\tau)/\xi(\tau) = \pm\infty. \quad (3.5)$$

Proof. Assume that a proper solution ξ of equation (3.1) has no limit as $\tau \to +\infty$. Then there exists a sequence $\{s_k\}_{k=1}^{\infty}$ of extremum points of this solution converging to $+\infty$. Taking into account that $\xi'(s_k) = 0$, $k = 1, 2, \dots$, equation (3.1) implies

$$\xi''(s_k) = \xi(s_k)\big[\xi^{\sigma}(s_k) - \beta_0 S(s_k)\big], \quad k = 1, 2, \dots. \tag{3.6}$$

Owing to the continuity of the functions $S(\tau)$ and $\xi^{\sigma}(\tau)$, if their graphs have no common points, then the right-hand side of equality (3.6) has the same sign when $k = 1, 2, \dots$. But this is impossible because it means that the solution ξ has only maximums or only minimums.

Let now ξ be a proper solution of (3.1), and let $\lim_{\tau\to+\infty} \xi(\tau)$ (finite or infinite) exist.To prove the theorem it suffices to show that if this limit is equal to zero or $+\infty$, then the solution ξ has one of the properties (3.4) and (3.5). Assume that

$$\lim_{\tau\to+\infty} \xi^{\sigma}(\tau) = 0 \tag{3.7}$$

and consider the function $U_c(\tau) = -\beta_0 c S(\tau) + \alpha_0 |c|^{\lambda} \xi^{\sigma}(\tau)$ with $c \neq 0$. According to (3.2) and (3.7), the function U_c retains the sign in a certain interval $[\tau_c, +\infty[\subset [\tau_0, +\infty[$, i.e.,

$$U_c(\tau) > 0 \ \text{ or } \ U_c(\tau) < 0 \ \text{ when } \ \tau \geq \tau_c. \tag{3.8}$$

If the function $u(\tau) = \beta_0 + \xi'(\tau)/\xi(\tau)$ has no limit as $\tau \to +\infty$, then there exists a constant $c \neq 0$ such that for any $T \geq \tau_c$ there is $T_1 \geq T$ such that $u(T_1) = c$. In view of (3.1) this contradicts (3.8). It means that $\lim_{\tau\to+\infty} u(\tau)$ (finite or infinite) exists. Suppose now that

$$\lim_{\tau\to+\infty} u(\tau) = u_0. \tag{3.9}$$

Then taking into account (3.2) and (3.7), it follows from (3.1) that $\lim_{\tau\to+\infty} u'(\tau) = -\beta_0 S_0 \neq 0$, but this contradicts (3.9). Hence each proper solution ξ of (3.1) satisfying (3.7) possesses one of the properties (3.4) and (3.5).

In the case where the solution ξ instead of (3.7) satisfies the condition $\lim_{\tau\to+\infty} \xi^{\sigma}(\tau) = \infty$, the proof of the theorem is analogous. □

Corollary 3.1. *If one of the inequalities $\alpha_0 S_0 < 0$ or $\alpha_0 \sigma > 0$ is fulfilled, then each proper solution ξ of* (3.1) *possesses one of the properties* (3.3)–(3.5).

Proof. If $\alpha_0 S_0 < 0$, then the validity of the statement is obvious. Let $\alpha_0 \sigma > 0$, and let ξ be a proper solution of (3.1) for which the limit does not exist as $\tau \to +\infty$. Then, according to Theorem 3.1, there exists a sequence $\{\tau_k\}_{k=1}^{\infty}$ tending to $+\infty$ as $k \to +\infty$ such that $\xi^{\sigma}(\tau_k) = \alpha_0 S(\tau_k)$, $k = 1, 2, \dots$. Because of (3.2) it is easy to see that there will be at least one point of the local maximum m_1 or of the local minimum m_2 of the function $\xi^\sigma(\tau)$ at which the inequality $\xi^\sigma(m_1) > \alpha_0 S(m_1)$ or $\xi^\sigma(m_2) < \alpha_0 S(m_2)$ is respectively fulfilled. From (3.1) we have

$$\left[\xi^\sigma(\tau)\right]''\Big|_{\tau=m_i} = \alpha_0\sigma\xi^\sigma(m_i)\left[\xi^\sigma(m_i) - \alpha_0 S(m_i)\right], \quad i \in \{1,2\}. \quad (3.10)$$

Because $\alpha_0\sigma > 0$, it follows from (3.10) that $[\xi^\sigma(\tau)]''|_{\tau=m_1} > 0$ or $[\xi^\sigma(\tau)]''|_{\tau=m_2} < 0$. The obtained contradiction completes the proof of the Corollary. □

Thus, if a proper solution ξ of (3.1) is such that $\lim_{\tau\to+\infty}\xi(\tau)$ (finite or infinite) exists, then it possesses one of properties (3.3)–(3.5), and vice versa. Corollary 3.1 shows the conditions under which the limit exists for each proper solution ξ of (3.1). Using conditions (3.3)–(3.5), these solutions can be divided into three groups. Therefore further investigation will be performed for each group separately.

3.1. On Proper Solutions of Equation (3.1) Which Have Finite Different from Zero Limit as $\tau \to +\infty$.

Theorem 3.2. *For equation* (3.1) *to have a proper solution ξ with property* (3.3)*, it is necessary and sufficient that*

$$\alpha_0 S_0 > 0 \quad \text{and} \quad \xi_0 = |S_0|^{\frac{1}{\sigma}}. \quad (3.11)$$

Moreover, each of such solutions admits the representation

$$\xi'(\tau) + \beta_0\xi(\tau) = \beta_0\xi_0 + o(1), \quad \tau \to +\infty. \quad (3.12)$$

Proof. Let ξ be a proper solution of (3.1) with property (3.3). Since for every fixed value c which is different from the solutions of the equation $\alpha_0|c|^\lambda\xi_0^\sigma - \beta_0 c S_0 = 0$, the function $U_c(\tau) = -\beta_0 c S(\tau) + \alpha_0|c|^\lambda\xi^\sigma(\tau)$ $(c \in \mathbb{R})$ retains the sign in a certain interval $[c,+\infty[\subset[\tau_0,+\infty[$, arguing as in proof of Theorem 3.1, it is not difficult to show that $\lim_{\tau\to+\infty}\xi'(\tau)/\xi(\tau)$ (finite or infinite) exists. Then, according to (3.3), $\lim_{\tau\to+\infty}\xi'(\tau)$ also exists and equals zero. Passing to the limit as $\tau \to +\infty$ in (3.1) in which ξ is the solution in question, we obtain $S_0 = \alpha_0\xi_0^\sigma$ which proves (3.11).

Finally, because $\lim_{\tau\to+\infty}\xi'(\tau)/\xi(\tau) = 0$, the equality (3.12) is true due to (3.3).

Assume now that (3.11) holds. We shall prove that the equation (3.1) has at least one solution ξ satisfying the conditions (3.3) and (3.12).

Applying to equation (3.1) the transformation

$$\xi(\tau) = \beta_0 + u_1(\tau), \quad \xi'(\tau) + \beta_0\xi(\tau) = \xi_0\beta_0 + u_1(\tau)h + u_2(\tau), \quad (3.13)$$

where h is a constant which will be defined later on, we obtain the system

$$\begin{cases} u_1' = (h-\beta_0)u_1 + u_2 \\ u_2' = -\xi_0[S(\tau)-S_0] + a_{21}(\tau)u_1 + a_{22}(\tau)u_2 + X(u_1,u_2) \end{cases}, \quad (3.14)$$

in which

$$a_{21}(\tau) = -h^2 + h\beta_0[2 - S(\tau) + \lambda S_0] + S_0(\sigma+1-\lambda) - 1,$$
$$a_{22}(\tau) = \beta_0 - h - \beta_0[S(\tau) - \lambda S_0],$$
$$X(u_1,u_2) = (\xi_0\beta_0 + hu_1 + u_2)^2(\xi_0+u_1)^{-1} - [\xi_0 + (2\beta_0 h - 1)u_1 +$$
$$+2\beta_0 u_2] + \alpha_0\big(|\xi_0\beta_0 + hu_1 + u_2|^\lambda |\xi_0 + u_1|^{\sigma+1-\lambda} -$$
$$-\xi_0^\sigma[\xi_0 + (\sigma+1-\lambda+\beta_0 h\lambda)u_1 + \beta_0\lambda u_2]\big).$$

Define D by $[S_0(\lambda-1)/2]^2$ and consider two cases: $D \ge 0$ and $D < 0$.

1^0. Let $D \ge 0$. In this case we choose a constant h so that $h^2 - h\beta_0[2 + S_0(\lambda-1)] - S_0(\sigma+1-\lambda) + 1 = 0$. Note that now

$$h - \beta_0[1 + S_0(\lambda-1)] \ne 0, \quad h - \beta_0 \ne 0. \quad (3.15)$$

Consider the system (3.14) in the domain Ω (see (2.2), where $T = b$, $0 < \delta < xi_0(|h|+1)$). Partial derivatives $\frac{\partial X(u_1,u_2)}{\partial u_i}$ $(i = 1,2)$ tend to zero as $|u_1| + |u_2| \to 0$ and $X(0,0) = 0$. The functions A_i, F_i, G_i $(i = 1,2)$ defined for (3.14) in §2 are of the form

$$A_2(\tau) = \Big|\int_{\alpha_2}^{\tau} a_{21}(t)\exp\int_t^\tau a_{22}(s)ds\, dt\Big|;$$

$$A_1(\tau) = \Big|\int_{\alpha_1}^{\tau} A_2(t)\exp\int_t^\tau (h-\beta_0)ds\, dt\Big|;$$

$$F_2(\tau) = \Big|\int_{\beta_2}^{\tau} \xi_0|S(t)-S_0|\exp\int_t^\tau a_{22}(s)ds\, dt\Big|;$$

$$F_1(\tau) = \Big|\int_{\beta_1}^{\tau} F_2(t)\exp\int_t^\tau (h-\beta_0)ds\, dt\Big|;$$

$$G_2(\tau) = \Big|\int_{\gamma_2}^{\tau} \exp\int_t^\tau a_{22}(s)ds\, dt\Big|; \quad G_1(\tau) = \Big|\int_{\gamma_{12}}^{\tau} G_2(t)\exp\int_t^\tau (h-\beta_0)ds\, dt\Big|.$$

Using Lemma 2.2 and taking into account (3.2), (3.11), and (3.15), we can easily verify that $\lim_{\tau\to+\infty} A_i(\tau) = \lim_{\tau\to+\infty} F_i(\tau) = 0$ $(i = 1,2)$, $\lim_{\tau\to+\infty} G_2(\tau) = |h-\beta_0| \lim_{\tau\to+\infty} G_1(\tau) = 1/|h - \beta_0[1 + S_0(1-\lambda)]|$.

Thus system (3.14) satisfies the conditions of Theorem 2.2; hence it has at least one real solution $(u_1(\tau), u_2(\tau))$ tending to zero as $\tau \to +\infty$. Because of the transformation (3.13), this implies that there exists a solution ξ of (3.1) satisfying conditions (3.3), (3.12).

2^0. Let $D < 0$. We use the following notation: $q = \sqrt{-D}$, $p = \beta_0 S_0(\lambda - 1)/2$,

$$M(\tau)=\begin{pmatrix} \cos(q\tau) & \sin(q\tau) \\ (p+\beta_0)\cos(q\tau)-q\sin(q\tau) & q\cos(q\tau)+(p+\beta_0)\sin(q\tau) \end{pmatrix}, \quad (3.16)$$

$$\begin{pmatrix} \delta_{11}(\tau) & \delta_{12}(\tau) \\ \delta_{21}(\tau) & \delta_{22}(\tau) \end{pmatrix} = M^{-1}(\tau)\begin{pmatrix} 0 & 0 \\ 0 & -\beta_0[S(\tau)-S_0] \end{pmatrix}. \quad (3.17)$$

Putting $h = 0$ in (3.14) and applying the transformation

$$\begin{pmatrix} u_1(\tau) \\ u_2(\tau) \end{pmatrix} = M(\tau)\begin{pmatrix} z_1(\tau) \\ z_2(\tau) \end{pmatrix}, \quad (3.18)$$

we obtain a system

$$\begin{cases} z_1' = f_1(\tau) + [p+\delta_{11}(\tau)]z_1 + \delta_{12}(\tau)z_2 + X_1(\tau, z_1, z_2) \\ z_2' = f_2(\tau) + \delta_{21}(\tau)z_1 + [p+\delta_{22}(\tau)]z_2 + X_2(\tau, z_1, z_2) \end{cases}, \quad (3.19)$$

in which $f_1(\tau) = \frac{\xi_0}{q}[S(\tau) - S_0]\sin(q\tau)$, $f_2(\tau) = -\frac{\xi_0}{q}[S(\tau) - S_0]\cos(q\tau)$, $X_1(\tau, z_1, z_2) = -\frac{1}{q}\sin(q\tau)X(u_1, u_2)$, $X_2(\tau, z_1, z_2) = \frac{1}{q}\cos(q\tau)X(u_1, u_2)$.

Partial derivatives $\frac{\partial X_i(\tau,z_1,z_2)}{\partial z_k}$ $(i, k = 1, 2)$ tend to zero as $|z_1| + |z_2| \to 0$ uniformly with respect to $\tau \in [b, +\infty[$, and $X_i(\tau, 0, 0) \equiv 0$ $(i = 1, 2)$ on $[b, +\infty[$.

Consider system (3.19) in the domain Ω (see (2.2), where $T = b$, $0 < \delta < \frac{\xi_0}{2}\min\{1, 1/(|p + \beta_0| + q)\}$). The functions a_i, A_i, F_i, G_i $(i = 1, 2)$ defined for the system (3.19) in §2 are of the form

$$a_i(\tau, t) = \exp\int_t^\tau [\delta_{ii}(s) + p]ds \quad (i = 1, 2);$$

$$A_2(\tau) = \Big|\int_{\alpha_2}^\tau |\delta_{21}(t)|a_2(\tau, t)dt\Big|; \quad A_1(\tau) = \Big|\int_{\alpha_1}^\tau |\delta_{12}(t)|A_2(t)a_1(\tau, t)dt\Big|;$$

$$F_2(\tau) = \Big|\int_{\beta_2}^\tau |f_2(t)|a_1(\tau, t)dt\Big|; \quad G_2(\tau) = \Big|\int_{\gamma_2}^\tau a_2(\tau, t)dt\Big|;$$

$$F_1(\tau) = \Big|\int_{\beta_1}^\tau |f_1(t)|a_1(\tau, t)dt\Big| + \Big|\int_{\beta_{12}}^\tau |\delta_{12}(t)|F_2(t)a_1(\tau, t)dt\Big|;$$

$$G_1(\tau) = \Big|\int\limits_{\gamma_1}^{\tau} a_1(\tau,t)dt\Big| + \Big|\int\limits_{\gamma_{12}}^{\tau} |\delta_{12}(t)|G_2(t)a_1(\tau,t)dt\Big|,$$

It follows from (3.2), (3.16), and (3.17) that $\lim_{\tau\to+\infty}\delta_{ik}(\tau) = 0$ $(i,k = 1,2)$, hence $\int_b^{+\infty}[p+\delta_{ii}(\tau)]d\tau = +\infty$ $(i = 1,2)$. Then using Lemma 2.2 it is easy to make sure that $\lim_{\tau\to+\infty} A_i(\tau) = \lim_{\tau\to+\infty} F_i(\tau) = 0$; $\lim_{\tau\to+\infty} G_i(\tau) = \frac{1}{|p|}$ $(i = 1,2)$.

Thus system (3.19) satisfies all the conditions of Theorem 2.2. Therefore it has at least one real solution $(u_1(\tau), u_2(\tau))$ tending to zero as $\tau\to\infty$. Because of transformations (3.13) and (3.18) this implies that there exists a solution ξ of (3.1) satisfying (3.3) and (3.12). □

3.2. On the Proper Solutions of Equation (3.1) with the Property (3.4.). We use the following notation:

$$H(\tau) = \int\limits_b^{\tau} S(t)dt; \;\; \theta(\tau)\exp(-\delta\beta_0\tau + (1-\lambda)H(\tau)).$$

Theorem 3.3. *Each proper solution ξ of equation* (3.1) *with property* (3.4) *admits the asymptotic representation*

$$\xi(\tau) = c\exp(-\beta_0\tau)[1+o(1)], \;\; \tau\to+\infty, \tag{3.20}$$

where $c > 0$, and its derivative satisfies one of the equalities

$$\frac{\xi'(\tau)}{\xi(\tau)} + \beta_0 = c_0\exp(-H(\tau))[1+o(1)], \;\; \tau\to+\infty, \tag{3.21}$$

or

$$\frac{\xi'(\tau)}{\xi(\tau)} + \beta_0 = \nu\exp(-H(\tau))\Big|c^{\sigma}(1-\lambda)\times \times\int\limits_{\gamma}^{\tau}\theta(t)dt\Big|^{\frac{1}{1-\lambda}}[1+o(1)], \;\; \tau\to+\infty, \tag{3.22}$$

where $\nu = -\alpha_0\,\mathrm{sign}(1-\lambda)$, $\gamma = +\infty$ if $\int\limits_b^{+\infty}\theta(t)dt < \infty$ and $\nu = -\alpha_0\,\mathrm{sign}(1-\lambda)$, $\gamma = b$ otherwise, $c_0 \neq 0$.

Equation (3.1) *has a proper solution ξ with property* (3.4) *which satisfies both asymptotic equalities* (3.20), (3.21) *if and only if*

$$\beta_0 S_0 > 0, \quad \int\limits_b^{+\infty}\theta(\tau)d\tau < +\infty, \tag{3.23}$$

and equalities (3.20), (3.22) *if and only if*

$$\sigma\beta_0(1-\lambda) > 0. \tag{3.24}$$

Proof. Let ξ be a proper solution of (3.1) with property (3.4). Set

$$u(\tau) = \frac{\xi'(\tau)}{\xi(\tau)} + \beta_0, \quad \varphi(\tau) = \int\limits_{\tau_0}^{\tau} u(t)dt, \quad \Phi(\tau) = \int\limits_{r}^{\tau} \theta(t)\exp\varphi(t)dt,$$

where $r = +\infty$ if $\int_{\tau_0}^{+\infty} \theta(t)\exp\varphi(t)dt$ converges, and $r = \tau_0$ otherwise. Then

$$\lim_{\tau\to+\infty} u(\tau) = 0 \tag{3.25}$$

$$\xi(\tau) = \xi_0 \exp(-\beta_0\tau + \varphi(\tau)), \tag{3.26}$$

where ξ_0 is a certain constant. Substituting (3.26) into the right-hand side of (3.1), we find

$$|u(\tau)|^{1-\lambda} = \exp(-(1-\lambda)H(\tau))[c_1 + \alpha_0\xi_0^{\sigma}(1-\lambda)\nu_0\Phi(\tau)], \tag{3.27}$$

where $\nu_0 = \operatorname{sign} u(\tau)$, $c_1 \in \mathbb{R}$. It is clear from (3.27) that either

$$u(\tau) = c_0\exp(-H(\tau))[1+o(1)], \quad \tau\to+\infty, \tag{3.28}$$

where $c_0 \neq 0$ or

$$u(\tau) \sim \nu\exp(-H(\tau))|\xi_0^{\sigma}(1-\lambda)\Phi(\tau)|^{\frac{1}{1-\lambda}}, \quad \tau\to+\infty. \tag{3.29}$$

Moreover, (3.28) happens to be the case only if $r = \tau_0$.

Assume that the solution of (3.1) in question satisfies (3.28). This does not contradict (3.25) only if $\beta_0 S_0 > 0$. It is easy to see that if this inequality holds, then $\int_{\tau_0}^{+\infty}\theta(\tau)d\tau < \infty$, and the solution ξ satisfies asymptotic equalities (3.20), (3.21) by (3.26), (3.28).

Suppose now that the solution in question satisfies (3.29). According to (3.25), since for any $\rho \in [0, \rho^*[$, where $\rho^* = \min\left\{\left|\frac{\sigma}{1-\lambda}\right|, |S_0|\right\}$, $\varphi(\tau) = o(\tau)$, $\tau\to+\infty$, we have

$$\lim_{\tau\to+\infty} \frac{\Phi(\tau)}{\exp((1-\lambda)[H(\tau)-\rho\tau])} = \begin{cases} 0 & \text{if } \sigma\beta_0 > 0 \\ \pm\infty & \text{if } \sigma\beta_0 < 0 \end{cases}, \tag{3.30}$$

which (for $\rho = 0$) implies that (3.29) does not contradict (3.25) only if $\sigma\beta_0(1-\lambda) > 0$. Moreover, if this inequality holds, then

$$[\Phi(\tau)]^{\frac{1}{1-\lambda}} = o(\exp(H(\tau)-\rho\tau)), \quad \tau\to+\infty,$$

and therefore

$$\int_{\tau_0}^{+\infty} \exp(-H(\tau))|\Phi(\tau)|^{\frac{1}{1-\lambda}} d\tau < \infty. \tag{3.31}$$

Next, (3.26), (3.29), and (3.31) imply that the solution ξ admits representation (3.20), where $c > 0$ is a certain constant. Substituting (3.20) into the right-hand side of (3.1) and integrating the obtained equation, it is not difficult to make sure that ξ satisfies (3.22).

Let conditions (3.23) be fulfilled, and let $c > 0$, $c_0 \neq 0$ be arbitrary fixed numbers. We shall prove that there exists at least one solution ξ of equation (3.1) satisfying representations (3.20), (3.21).

Using

$$\begin{aligned} \xi(\tau) &= c\exp(-\beta_0\tau)[1+u_1(\tau)], \\ \frac{\xi'(\tau)}{\xi(\tau)} + \beta_0 &= c_0\exp(-H(\tau))[1+u_2(\tau)], \end{aligned} \tag{3.32}$$

equation (3.1) is transformed into the differential system

$$\begin{cases} u_1' = c\exp(-H(\tau))[1+u_1+u_2+X_1(u_1,u_2)] \\ u_2' = m\theta(\tau)[1+\sigma u_1+\lambda u_2+X_2(u_1,u_2)] \end{cases} \tag{3.33}$$

where $m = \alpha_0 c_0 |c_0|^{\lambda-2} c^{\sigma}$, $X_1(u_1,u_2) = u_1 u_2$, $X_2(u_1,u_2) = (1+u_1)^{\sigma} \times |1+u_2|^{\lambda} - 1 - \sigma u_1 - \lambda u_2$. Consider system (3.33) in the domain Ω (see (2.2), where $T = b$, $0 < \delta < 1$). The functions A_i, F_i, G_i $(i = 1, 2)$ defined in §2 for system (3.33) are of the form

$$A_2(\tau) = \Big| m\sigma \int_{\alpha_2}^{\tau} \theta(t)\exp(\lambda m \int_t^{\tau} \theta(s)ds\ dt \Big|;$$

$$A_1(\tau) = \Big| c_0 \int_{\alpha_1}^{\tau} A_2(t)\exp\big(-H(t) + c_0\int_t^{\tau}\exp(-H(s))ds\big)dt \Big|;$$

$$F_1(\tau) = \Big| c_0 \int_{\beta_1}^{\tau} \exp\big(-H(t) + c_0\int_t^{\tau}\exp(-H(s))ds\big)dt \Big| + \frac{1}{|\sigma|}A_1(\tau);$$

$$F_2(\tau) = G_2(\tau) = \frac{1}{|\sigma|}A_2(\tau); \quad G_1(\tau) = F_1(\tau).$$

It follows from Lemma 2.1 and (3.23) that $\lim_{\tau\to+\infty} A_i(\tau) = \lim_{\tau\to+\infty} F_i(\tau) = \lim_{\tau\to+\infty} G_i(\tau) = 0$ $(i = 1, 2)$. Furthermore, partial derivatives of X_i $(i = 1, 2)$ with respect to u_1, u_2 are bounded in the domain Ω, and $X_i(0,0) = 0$ $(i = 1, 2)$. Thus the system (3.33) satisfies the conditions of Theorem 2.1 and has at least one real solution $(u_1(\tau), u_2(\tau))$ tending to zero as $\tau \to +\infty$ to which, due to the transformation (3.32), there corresponds a proper solution ξ of (3.1) satisfying asymptotic equalities (3.20), (3.21).

Let now inequality (3.24) hold, and let $c > 0$ be an arbitrary fixed number. We shall prove that equation (3.1) has at least one solution ξ satisfying representations (3.20), (3.22).

Applying the transformation

$$\begin{gathered} \xi(\tau) = c\exp(-\beta_0\tau)[1 + u_1(\tau)], \\ \frac{\xi'(\tau)}{\xi(\tau)} + \beta_0 = N(\tau)[1 + hu_1(\tau) + u_2(\tau)], \end{gathered} \tag{3.34}$$

where $N(\tau) = \nu\exp(-H(\tau))\Big|c^\sigma(1-\lambda)\int_\gamma^\tau \theta(t)dt\Big|^{\frac{1}{1-\lambda}}$, $h = \sigma/(1-\lambda)$, we get the system

$$\begin{cases} u_1' = N(\tau)[1 + (h+1)u_1 + u_2 + X_1(u_1, u_2) \\ u_2' = -hN(\tau) - h(h+1)N(\tau)u_1 - [hN(\tau) + \\ \quad +(1-\lambda)Q(\tau)]u_2 + Q(\tau)X_2(\tau, u_1, u_2), \end{cases} \tag{3.35}$$

where $Q(\tau) = \theta(\tau)\Big[(1-\lambda)\int_\gamma^\tau \theta(t)dt\Big]^{-1}$ (ν, γ are the same as in (3.22)) $X_1(u_1, u_2) = hu_1^2 + u_1u_2$, $X_2(\tau, u_1, u_2) = |1 + hu_1 + u_2|^\lambda(1 + u_1)^\sigma - 1 - (h\lambda + \sigma)u_1 - \lambda u_2 - hN(\tau)Q^{-1}(\tau)X_1(u_1, u_2)$.

Consider system (3.35) in the domain Ω (see (2.2), where $T = b$ $0 < \delta < 1/(|h|+1)$). The functions a_i, A_i, F_i, G_i $(i = 1, 2)$ defined in §2 for system (3.35) are of the form

$$a_2(\tau, t) = \exp\int_t^\tau [-hN(s) - (1-\lambda)Q(s)]ds;$$

$$a_1(\tau, t) = \exp\Big((h+1)\int_t^\tau N(s)ds\Big);$$

$$A_2(\tau) = \Big|h(h+1)\int_{\alpha_2}^\tau N(t)a_2(\tau, t)dt\Big|; \quad F_2(\tau) = \frac{1}{|h+1|}A_2(\tau);$$

$$G_2(\tau) = \Big|\int_{\gamma_2}^\tau Q(t)a_2(\tau, t)dt\Big|; \quad A_1(\tau) = \Big|\int_{\alpha_1}^\tau N(t)A_2(t)a_1(\tau, t)dt\Big|;$$

$$F_1(\tau) = \Big| \int\limits_{\beta_1}^{\tau} N(t)[1 + F_2(t)] a_1(\tau, t) dt \Big|;$$

$$G_1(\tau) = \Big| \int\limits_{\gamma_1}^{\tau} N(t)[1 + G_2(t)] a_1(\tau, t) dt \Big|.$$

Since (3.30) is fulfilled for any function $\varphi(\tau) = o(\tau)$, $\tau \to +\infty$, we have

$$N(\tau) = o(\exp(\rho_0 \tau)), \quad \tau \to +\infty \tag{3.36}$$

for arbitrary $\rho_0 \in]0, \rho^*[$. Using L'Hospital's rule it is easy to make sure that $\lim\limits_{\tau \to +\infty} \frac{\int_b^\tau \theta(t) dt}{\theta(\tau) \exp(\rho_0 \tau)} = 0$. Therefore, taking into consideration (3.36), we have

$$\lim_{\tau \to +\infty} N(\tau) Q^{-1}(\tau) = 0. \tag{3.37}$$

It follows from Lemmas 2.1, 2.2 and (3.36), (3.37) that $\lim_{\tau \to +\infty} A_i(\tau) = \lim_{\tau \to +\infty} F_i(\tau) = \lim_{\tau \to +\infty} G_1(\tau) = 0$ $(i = 1, 2)$,

$$\lim_{\tau \to +\infty} G_2(\tau) = \begin{cases} 0 \\ \frac{1}{|1-\lambda|} \end{cases} \quad \text{if} \quad \int\limits_b^{+\infty} \theta(\tau) d\tau \begin{array}{l} < +\infty \\ = +\infty \end{array}.$$

Partial derivatives $\frac{\partial X_i}{\partial u_k}$ $(i, k = 1, 2)$ tend to zero as $|u_1| + |u_2| \to 0$ uniformly with respect to $\tau \in [T, +\infty[$. Furthermore, $X_2(\tau, 0, 0) \equiv 0$ for $\tau \geq T$, $X_1(0, 0) = 0$.

Thus by Theorem 2.2 system (3.35) has at least one real solution $(u_1(\tau), u_2(\tau))$ tending to zero as $\tau \to +\infty$ to which, due to transformation (3.34), there corresponds a proper solution ξ of (3.1) satisfying (3.20), (3.22). □

3.3. On the Proper Solutions of Equation (3.1) with the Property (3.5). Below we shall use the following simple statement whose validity can be easily verified.

Lemma 3.1. *Let $f : [T, +\infty[\to \mathbb{R}$ be a continuously differentiable function such that $\lim_{t \to +\infty} |f(t)| = +\infty$. If for some $\varepsilon > 0$ there exists $\lim_{t \to +\infty} f'(t)/|f(t)|^{1+\varepsilon}$, then this limit equals zero.*

Consider first the solutions of (3.1) for which

$$\lim_{\tau \to +\infty} \xi'(\tau)/\xi(\tau) = +\infty. \tag{3.38}$$

Lemma 3.2. *Let ξ be a proper solution of* (3.1) *with the property* (3.33). *Then*

$$\lim_{\tau\to+\infty} \xi^{\sigma}(\tau)u^{\lambda-2}(\tau) = +\infty \quad \text{when } \sigma > 0. \tag{3.39}$$

and

$$\lim_{\tau\to+\infty} \xi^{\sigma}(\tau)u^{\lambda-1}(\tau) = 0 \quad \text{when } \sigma < 0. \tag{3.40}$$

Proof. Let ξ be a proper solution of (3.1) with property (3.38). Obviously,

$$\lim_{\tau\to+\infty} \xi(\tau) = +\infty. \tag{3.41}$$

First we shall show that for any $\varepsilon > 0$ the function $z(\tau) = u(\tau)\xi^{-\varepsilon}(\tau)$ has the limit as $\tau \to +\infty$, and this limit equals zero.

Assume on the contrary that $\lim_{\tau\to+\infty} z(\tau)$ does not exist. Then there exists a constant $\overline{c}$ different from zero and $\varepsilon^{\frac{1}{\lambda-2}}$, such that the graph of the function $z = z(\tau)$ intersects the straight line $z = \overline{c}$ at $\tau = t_k$, $k = 1, 2, \dots$, and the sequence $\{t_k\}_{k=1}^{\infty}$ tends to infinity. Since by (3.1), $z'(\tau) \equiv z(\tau)\beta_0[\varepsilon - S(\tau)]$ for $\tau \geq t_0$, this implies that due to (3.2) and (3.41) the values $z'(t_k)$, $k = N, N{+}1, \dots$ for some N are of the same sign, which is impossible. Hence $\lim_{\tau\to+\infty} z(\tau)$ exists, and because of the fact that $z(\tau) \sim \xi'(\tau)/\xi^{1+\varepsilon}(\tau)$ as $\tau \to +\infty$, (3.41), and Lemma 3.1, we have

$$\lim_{\tau\to+\infty} z(\tau) = 0. \tag{3.42}$$

By virtue of (3.38) and (3.41) the validity of (3.39) and (3.40) is obvious if $\lambda > 2$ and $\lambda < 1$, respectively.

Let $\sigma > 0$ and $\lambda < 2$. Choosing ε such that $\sigma + (\lambda - 2)\varepsilon > 0$ and taking into account (3.38), (4.42), we obtain

$$\lim_{\tau\to+\infty} \xi^{\sigma}(\tau)u^{\lambda-2}(\tau) = \lim_{\tau\to+\infty} \xi^{\sigma+(\lambda-r)\varepsilon}(\tau)z^{\lambda-2}(\tau) = +\infty,$$

i.e., (3.39) holds when $\lambda < 2$.

If $\sigma < 0$ and $\lambda > 1$ we choose ε so that $\sigma + (\lambda - 1)\varepsilon < 0$. Then because of (3.38), (3.42) we have

$$\lim_{\tau\to+\infty} \xi^{\sigma}(\tau)u^{\lambda-1}(\tau) = \lim_{\tau\to+\infty} \xi^{\sigma+(\lambda-1)\varepsilon}(\tau)z^{\lambda-1}(\tau) = 0,$$

i.e., (3.40) holds when $\lambda > 1$. Thus Lemma 3.2 is proved. □

Theorem 3.4. *Equation* (3.1) *has solutions with the property* (3.38) *if and only if*

$$\sigma < 0, \quad \beta_0 S_0 < 0. \tag{3.43}$$

Furthermore, each of such solutions admits asymptotic representations

$$\xi(\tau) = c\exp\Big(-\beta_0\tau + c_0\int\limits_b^\tau \exp(-H(t))dt\Big)[1+o(1)], \quad \tau\to+\infty, \tag{3.44}$$

$$\frac{\xi'(\tau)}{\xi(\tau)} + \beta_0 = c_0\exp(-H(\tau))[1+o(\exp(-k\tau))], \quad \tau\to+\infty, \tag{3.45}$$

where $c>0$, $c_0>0$, $k>0$.

Proof. Let ξ be a proper solution of (3.1) with property (3.38) and $u(\tau) = \beta_0+\xi'(\tau)/\xi(\tau)$. When $\sigma>0$, it follows from (3.1), (3.2), (3.38), and Lemma 3.2 that

$$\lim_{\tau\to+\infty}\frac{\alpha_0 u'(\tau)}{u^2(\tau)} = \lim_{\tau\to+\infty}\big[-\alpha_0\beta_0 S(\tau)u^{-1}(\tau) + u^{\lambda-2}(\tau)\xi^\sigma(\tau)\big] = +\infty,$$

which contradicts Lemma 3.1.

When $\sigma<0$ and $\beta_0 S_0>0$, it follows from (3.1), (3.2) and Lemma 3.2 that $\lim_{\tau\to+\infty}\frac{u'(\tau)}{u(\tau)} = -\beta_0 S_0 < 0$, which contradicts (3.38).

Thus equation (3.1) can have a proper solution with property (3.38) provided only that (3.43) holds. Let inequalities (3.43) be fulfilled, and let ξ be such a solution. Put $\varepsilon(\tau) = \beta_0 S(\tau) + u'(\tau)/u(\tau)$, $\psi(\tau) = \int_{\tau_0}^\tau \varepsilon(t)dt$. Then $u(\tau) = c_1\exp(-H(\tau)+\psi(\tau))$,

$$\xi(\tau) = \xi_0\exp\Big(-\beta_0\tau + c_1\int\limits_{\tau_0}^\tau \exp(-H(t)+\psi(t))dt\Big), \tag{3.46}$$

where $\varepsilon_0>0$, $c_1>0$. It follows from (3.1), (3.2), (3.43), and Lemma 3.2 that

$$\lim_{\tau\to+\infty}\varepsilon(\tau) = 0. \tag{3.47}$$

Substituting (3.46) into the right-hand side of (3.1) and taking into account (3.38), (3.43), and (3.47), we find that

$$u(\tau) = \exp(-H(\tau))\Big[\bar c_1 + (1-\lambda)\alpha_0\xi_0^\sigma \times \\ \times\int\limits_\infty^\tau \theta(t)\exp\Big(\sigma c_1\int\limits_{\tau_0}^t \exp(-H(s)+\psi(s))ds\Big)dt\Big]^{\frac{1}{1-\lambda}}, \tag{3.48}$$

where $\bar c_1\ge 0$. Because of (3.42), (3.46)–(3.48) by using L'Hospital's rule, it is not difficult to verify that if $\bar c_1=0$, then $\lim_{\tau\to+\infty}u(\tau)=0$ when $\lambda<1$, and

$$\lim_{\tau\to+\infty}\xi^\sigma(\tau)u^{\lambda-1}(\tau) = +\infty \text{ when } \lambda>1,$$

which contradicts (3.38) and (3.39), respectively. Consequently, $\overline{c}_1 > 0$.

Note that owing to (3.43) and (3.47),

$$\int_{\tau}^{+\infty} \theta(t)\exp\Big(\sigma c_1 \int_{\tau_0}^{t} \exp(-H(s)+\psi(s))ds\Big)dt = o(\exp(-k\tau)), \quad \tau\to+\infty, \tag{3.49}$$

for any $k > 0$. Therefore, representation (3.48) can be expressed in the form (3.45) which implies that ξ satisfies (3.44) with certain constants $c > 0$, $c_0 > 0$, $k > 0$.

Next we shall prove that conditions (3.43) are sufficient for (3.1) to have a proper solution ξ satisfying (3.44), (3.45).

Let c, c_0, k be arbitrary fixed numbers satisfying inequalities $c>0$, $c_0>0$,

$$k > -\beta_0 S_0. \tag{3.50}$$

Applying to (3.1) the transformation

$$\begin{gathered} \xi(\tau) = c\exp\Big(-\beta_0\tau + c_0\int_b^{\tau} \exp(-H(t))dt\Big)[1+u_1(\tau)], \\ \frac{\xi'(\tau)}{\xi(\tau)} + \beta_0 = c_0\exp(-H(\tau))[1+\exp(-k\tau)u_2(\tau)], \end{gathered} \tag{3.51}$$

we obtain the system

$$\begin{cases} u_1' = c_0\exp(-H(\tau)-k\tau)[u_2 + X_1(u_1,u_2)] \\ u_2' = ku_2 + \alpha_0 c_0^{\lambda-1}c^{\sigma}\theta(\tau)\times \\ \quad \times\exp\Big(\sigma c_1\int_b^{\tau}\exp(-H(t))dt + k\tau\Big)[1+X_2(\tau,u_1,u_2)], \end{cases} \tag{3.52}$$

where $X_1(u_1,u_2) = u_1u_2$, $X_2(\tau,u_1,u_2) = (1+u_1)^{\sigma}|1+\exp(-k\tau)u_2|^{\lambda} - 1$.

Consider system (3.52) in the domain Ω (see (2.2), where $T = b$, $0 < \delta < \min\{1, \exp(kT)\}$). Partial derivatives of X_i $(i = 1,2)$ with respect to u_i, u_2 are bounded in the domain Ω, and $X_1(0,0) = 0$, $X_2(\tau,0,0) \equiv 0$ for $\tau \geq T$. The functions A_i, F_i, G_i $(i = 1,2)$ defined in §2 for system (3.52) are of the form

$$\begin{gathered} A_2(\tau) \equiv A_1(\tau) \equiv 0; \;\; F_2(\tau) = c_0^{\lambda-1}c^{\sigma}\exp(k\tau)\Big|\int_{\beta_2}^{\tau}\theta(\tau)\times \\ \times\exp\Big(\sigma c_0\int_b^{t}\exp(-H(s))ds\Big)dt\Big|; \end{gathered}$$

$$F_1(\tau) = c_0\Big|\int\limits_{\beta_{12}}^{\tau} \exp(-H(t) - kt)F_2(t)dt\Big|; \;\; G_2(\tau) \equiv F_2(\tau);$$

$$G_1(\tau) = c_0\Big|\int\limits_{\gamma_1}^{\tau} \exp(-H(t) - kt)dt\Big| + F_1(\tau).$$

It is easily seen that asymptotic equality (3.49) under the conditions (3.43) remains true if we set $\varphi(\tau) \equiv 0$. It follows that $\lim_{\tau\to+\infty} F_2(\tau) = \lim_{\tau\to+\infty} G_2(\tau) = 0$. This implies $\lim_{\tau\to+\infty} F_1(\tau) = \lim_{\tau\to+\infty} G_1(\tau) = 0$ due to (3.51). Thus, by Theorem 2.1 system (3.52) has at least one real solution $(u_1(\tau), u_2(\tau))$ tending to zero as $\tau \to +\infty$. Taking into account transformation (3.51), we complete the proof of the theorem. □

Consider now the solutions of (3.1) satisfying

$$\lim_{\tau\to+\infty} \xi'(\tau)/\xi(\tau) = -\infty. \tag{3.53}$$

We make the substitution $1/\xi(\tau) = \mu(\tau)$ to obtain the equation

$$\Big(\frac{\mu'(\tau)}{\mu(\tau)} - \beta_0\Big)' + \beta_0 S(\tau)\Big(\frac{\mu'(\tau)}{\mu(\tau)} - \beta_0\Big) = -\alpha_0\mu^{-\sigma}(\tau)\Big|\frac{\mu'(\tau)}{\mu(\tau)} - \beta_0\Big|^{\lambda}. \tag{3.54}$$

Clearly, a proper solution ξ of (3.1) with property (3.53) corresponds to the solution μ of (3.54) with the property $\lim_{\tau\to+\infty} \mu'(\tau)/\mu(\tau) = +\infty$, and vice versa. Since equations (3.1) and (3.54) are of the same form, using the above arguments it is not difficult to see that the following statement is true.

Theorem 3.5. *Equation* (3.10) *has solutions with the property* (3.53) *if and only if* $\sigma > 0$, $\beta_0 S_0 < 0$. *Furthermore, each of such solutions* ξ *admits asymptotic representations*

$$\xi(\tau) = c\exp\Big(-\beta_0\tau - c_0\int\limits_{b}^{\tau} \exp(-H(t))dt\Big)[1 + o(1)], \;\; \tau \to +\infty,$$

$$\frac{\xi'(\tau)}{\xi(\tau)} + \beta_0 = -c_0\exp(-H(\tau))[1 + (\exp(-k\tau))], \;\; \tau \to +\infty,$$

where $c > 0$, $c_0 > 0$, $k > 0$.

§ 4. Proofs of Theorems 1.1–1.5

Applying to (0.1) the transformation

$$y(t) = \ln[V(t)\xi(\tau)], \quad \tau = \beta_0 \ln V(t), \tag{4.1}$$

we get $\tau'(t) > 0$ for $t \in [a_1, \omega[$ (a_1 is a certain number in the interval $]a, \omega[$), and $\lim_{t\uparrow\omega} \tau(t) = +\infty$. The transformation (4.1) yields equation (3.1) in which $S(\tau) = S(\tau(t)) = \alpha_0 \Gamma(t)$, $b = \beta_0 \ln V(a)$. Moreover, proper solution ξ of (3.1) corresponds to each proper solution y of (0.1), and vice versa. Taking into account transformation (4.1) and the notation introduced in §§1 and 3, it is easy to see that

$$H(\tau) = H(\tau(t)) = \ln\Big|\frac{V'(t)}{V(t)}\Big| + \ln\Big|\frac{V(a)}{V'(a)}\Big|,$$

$$\int\limits_{\gamma}^{\tau(t)} \theta(s)ds = \Big|\frac{V'(a)}{V(a)}\Big|^{1-\lambda} \int\limits_{\rho}^{t} p(s)ds,$$

where $\rho = \omega$ if $\int_a^\omega p(t)dt < +\infty$, and $\rho = a$ otherwise,

$$\int\limits_{b}^{+\infty} \exp\Big(-\beta_0 \int\limits_{b}^{\tau} S(t)dt\Big)d\tau = \Big|\frac{V'(a)}{V(a)}\Big| \int\limits_{a}^{\omega} dt. \tag{4.2}$$

Because of (1.1), the function S satisfies the condition (3.2), and $S_0 = \alpha_0 \Gamma_0$. Therefore it follows from (4.2) that if either $\alpha_0\beta_0\Gamma_0 > 0$ or $\alpha_0\beta_0\Gamma_0 < 0$, then $\omega < +\infty$ or $\omega = +\infty$, respectively.

Taking into consideration the above arguments, it is easily seen that Theorems 3.1–3.6 result in Theorems 1.1–1.5.

Remark 1. The results dealing with the asymptotic behavior of proper solutions of (0.1) in the case $\lambda = 1$ may be found in [9].

In the case $\lambda = 2$, Theorems 1.1–1.5 in which

$$\Gamma_0 = -\alpha_0 \sigma \lim_{t\uparrow\omega} \frac{p(t)p''(t)}{[p'(t)]^2}, \quad V(t) = p^{-\frac{1}{\sigma}}(t)$$

are valid under an additional assumption that p is twice continuously differentiable function satisfying one of the conditions $\lim_{t\uparrow\omega} p(t) = 0$ or $\lim_{t\uparrow\omega} p(t) = +\infty$.

Remark 2. Paper [10] contains results on the asymptotic properties of proper solutions of (0.1) in the case where $\Gamma_0 = \pm\infty$.

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