

ON STRONG MAXIMAL OPERATORS CORRESPONDING TO DIFFERENT FRAMES

G. ONIANI

ABSTRACT. The problem is posed and solved whether the conditions $f \in L(1 + \ln^+ L)^2(\mathbb{R}^2)$ and $\sup_{\theta \in [0, \pi/2)} \int_{\{M_{2,\theta}(f) > 1\}} M_{2,\theta}(f) < \infty$ are equivalent for functions $f \in L(\mathbb{R}^2)$ (where $M_{2,\theta}$ denotes the strong maximal operator corresponding to the frame $\{OX_\theta, OY_\theta\}$).

The results obtained represent a general solution of M. de Guzmán's problem that was previously studied by various authors.

1. NOTATION

Let B_1 be a family of all cubic intervals in \mathbb{R}^n . We denote by M_1 a maximal Hardy-Littlewood operator which is defined as follows:

$$M_1(f)(z) := \sup \left\{ \frac{1}{|I|} \int_I |f| : z \in I, I \in B_1 \right\}, \quad z \in \mathbb{R}^n,$$

for $f \in L_{loc}(\mathbb{R}^n)$.

Let B_2 be a family of all open rectangles in \mathbb{R}^2 whose sides are parallel to the coordinate axes; OX_θ be the straight line obtained by rotating the OX -coordinate axis through the angle θ about the point O in the positive direction (OY_θ is defined analogously); $B_{2,\theta}$ be a family of all open rectangles with the sides parallel to the straight lines OX_θ and OY_θ .

For a rectangle $I \subset \mathbb{R}^2$ we shall denote by $n(I)$ a number $\theta \in [0, \pi/2)$ for which one of the sides of I is parallel to OX_θ . The regularity factor of the rectangle I will be defined as the ratio of the length of the larger side of I to the length of the smaller side of I and will be denoted by $r(I)$.

For $\theta \in \mathbb{R}$ we shall denote by $\theta \pmod{\pi/2}$ a number such that $0 \leq \theta \pmod{\pi/2} < \pi/2$ and $\theta - \theta \pmod{\pi/2} = \pi k/2$ for some $k \in \mathbb{N}$. One can easily verify that $B_{2,\theta} = B_{2,\theta \pmod{\pi/2}}$, $\theta \in \mathbb{R}$.

The sets $\{OX_\theta, OY_\theta\}$, $\theta \in [0, \pi/2)$ will be called frames.

1991 *Mathematics Subject Classification.* 28A15.

Key words and phrases. Strong maximal operators, different frames, regularity factor of a rectangle.

For $f \in L_{loc}(\mathbb{R}^2)$ let

$$M_2(f)(z) := \sup \left\{ \frac{1}{|I|} \int_I |f| : z \in I, I \in B_2 \right\}, \quad z \in \mathbb{R}^2,$$

$$M_{2,\theta}(f)(z) := \sup \left\{ \frac{1}{|I|} \int_I |f| : z \in I, I \in B_{2,\theta} \right\}, \quad z \in \mathbb{R}^2.$$

M_2 is called the strong maximal operator, while $M_{2,\theta}$ is called the strong maximal operator corresponding to the frame $\{OX_{\theta \pmod{\pi/2}}, OY_{\theta \pmod{\pi/2}}\}$. This definition is correct because by virtue of the equality $B_{2,\theta} = B_{2,\theta \pmod{\pi/2}}$ we have $M_{2,\theta} = M_{2,\theta \pmod{\pi/2}}$. The latter equality implies that the family $\{M_{2,\theta}\}_{\theta \in [0, \pi/2]}$ exhausts the family of all operators $M_{2,\theta}$ ($\theta \in \mathbb{R}$).

2. FORMULATION OF THE QUESTION

As is known, the space $L(1 + \ln^+ L)(\mathbb{R}^n)$ can be characterized by the maximal operator M_1 as follows (see [1], [2]):

Theorem 1⁰. *Let $f \in L(\mathbb{R}^n)$. Then the following two conditions are equivalent:*

1. $f \in L(1 + \ln^+ L)(\mathbb{R}^n)$;
2. $\int_{\{M_1(f) > 1\}} M_1(f) < \infty$.

The implication $1 \Rightarrow 2$ was proved by Hardy and Littlewood [3] for $n = 1$ and by Wiener [4] for $n \geq 2$. The results of the reverse nature were obtained for the first time by Stein [5] and Tsereteli [6, 7]. Guzman and Welland [1, 2] improved the above results by formulating Theorem 1⁰.

It is known that if $f \in L(1 + \ln^+ L)^2(\mathbb{R}^2)$ then (see [1])

$$\int_{\{M_2(f) > 1\}} M_2(f) < \infty. \quad (2.1)$$

Guzman (see [1]) posed the question whether it was possible to characterize the space $L(1 + \ln^+ L)^2(\mathbb{R}^2)$ by the operator M_2 as was done for the space $L(1 + \ln^+ L)$ using the operator M_1 . Gogoladze [8, 9] and Bagby [10] answered this question in the negative. Their results give rise to

Theorem 2⁰. *For any functions $f \notin L(1 + \ln^+ L)^2(\mathbb{R}^2)$ and $f \in L(1 + \ln^+ L)(\mathbb{R}^2)$ there exists a Lebesgue measure preserving an invertible mapping*

$\omega : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that

$$\int_{\{M_2(f \circ \omega) > 1\}} M_2(f \circ \omega) < \infty.$$

Now we proceed directly to formulating our problem. It is easy to verify that if $f \in L(1 + \ln^+ L)^2(\mathbb{R}^2)$ then

$$\int_{\{M_{2,\theta}(f) > 1\}} M_{2,\theta}(f) < \infty \quad \text{for any } \theta \in [0, \pi/2), \quad (2.2)$$

and, moreover,

$$\sup_{\theta \in [0, \pi/2)} \int_{\{M_{2,\theta}(f) > 1\}} M_{2,\theta}(f) < \infty. \quad (2.3)$$

Clearly, conclusion (2.3) is stronger than (2.1) and hence there is a better chance for us to improve the integral properties of the function f when (2.3) is fulfilled than in the case of fulfillment of (2.1). Having given this information, we formulate the problem:

Let $f \in L(\mathbb{R}^2)$ and $\sup_{\theta \in [0, \pi/2)} \int_{\{M_{2,\theta}(f) > 1\}} M_{2,\theta}(f) < \infty$. Is the inclusion $f \in L(1 + \ln^+ L)^2(\mathbb{R}^2)$ then valid?

We would like to note here that the functions constructed in [8–10] do not satisfy condition (2.2) and thus [8–10] do not provide a solution of the above problem. So we shall prove the following theorem which as a particular case contains the answer to the problem.

Theorem 1. *For any functions $f \notin L(1 + \ln^+ L)^2(\mathbb{R}^2)$ and $f \in L(1 + \ln^+ L)(\mathbb{R}^2)$ there exists a Lebesgue measure preserving invertible mapping $\omega : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that*

1. *the set $\{|f \circ \omega| > 1\}$ is a square interval;*

$$2. \quad \sup_{\theta \in [0, \pi/2)} \int_{\{M_{2,\theta}(f \circ \omega) > 1\}} M_{2,\theta}(f \circ \omega) < \infty.$$

3. AUXILIARY STATEMENTS

To prove Theorem 1 we shall need several lemmas. If $\Delta \subset \mathbb{R}$ is some interval and $(\text{length } \Delta) < \pi/2$, then by $M_{2,\Delta}^*$ we denote the following operator: for each $f \in L_{loc}(\mathbb{R}^2)$

$$\begin{aligned} & M_{2,\Delta}^*(f)(z) := \\ & = \sup \{M_{2,\theta}(f)(z) : \theta \in [\text{cen } \Delta - \pi/4, \text{inf } \Delta] \cup [\text{sup } \Delta, \text{cen } \Delta + \pi/4]\}, \quad z \in \mathbb{R}^2; \end{aligned}$$

where $\text{cen } \Delta = (\inf \Delta + \sup \Delta)/2$.

Lemma 1. *Let $0 < \theta < \pi/4$, $H > 0$, $0 < \lambda < H$. It is assumed that the regularity factor of rectangle I satisfies the inequality $r(I) \geq \frac{H}{\lambda \sin^2 \theta}$. Then there is a rectangle $\mathcal{I}_{I,H,\theta,\lambda}$ such that*

$$\begin{aligned} \{M_{2,\Delta}^*(H\chi_I) > \lambda\} &\subset \mathcal{I}_{I,H,\theta,\lambda}, \\ |\mathcal{I}_{I,H,\theta,\lambda}| &\leq c_1 \frac{H}{\lambda} |I|, \\ \int_{\mathcal{I}_{I,H,\theta,\lambda}} M_{2,\Delta}^*(H\chi_I) &\leq c_2 H \left(1 + \ln^+ \frac{H}{\lambda}\right) |I|, \end{aligned}$$

where $\Delta = (n(I) - \theta, n(I) + \theta)$, while c_1 and c_2 are the positive constants not depending on I , H , θ , and λ .

Proof. We begin by considering the case $\lambda = 1$, $H > 1$. Without loss of generality it will be assumed (see Fig. 1) that $n(I) = 0$, I is the rectangle $ABCD$ the sides AB and BC of which have the lengths I_1 and I_2 respectively and $I_2/I_1 \geq H/\sin^2 \theta$.

Fig. 1

The strip bounded by the straight lines containing the segments AD and BC respectively will be denoted by \tilde{I} . Using the convexity property of the rectangle, it is easy to prove the inequality

$$M_{2,\Delta}^*(H\chi_I)(z) \leq 3H \frac{I_1}{\text{dist}(z, \tilde{I})} \quad \text{for } \text{dist}(z, \tilde{I}) \geq I_1. \quad (3.1)$$

A minimal number $\alpha \geq 0$ for which the straight line l is parallel to the OX_α -axis will be denoted by $n(l)$. Let l_1 ($n(l_1) = \theta$) and l_2 ($n(l_2) = \pi - \theta$) be the straight lines passing through the points C and D , respectively. The straight lines l_1 and l_2 divide the plane into parts. We denote the right-hand

part by E_{l_1, l_2} . Due to the definition of the operator $M_{2, \theta}^*$ it is easy to show that

$$M_{2, \Delta}^*(H\chi_I)(z) \leq \frac{H|I|}{\text{dist}(z, l_1) \text{dist}(z, l_2)} \quad \text{for } z \in E_{l_1, l_2}. \quad (3.2)$$

Above and below the rectangle I let us draw the straight lines l'_1 and l'_2 which are parallel to the segment AD and situated at a distance $(3H+1)I_1$ from I . Now the rectangle $A'B'C'D'$ can be chosen so that $A'D' \subset l'_1$, $B'C' \subset l'_2$, and $\text{dist}(A'B', I) = \text{dist}(C'D', I) = 10I_2$. We shall prove that the rectangle $A'B'C'D'$ can be taken as $\mathcal{I}_{I, H, \theta, 1}$.

The strip bounded by the straight lines l'_1 and l'_2 will be denoted by E ; the part of E lying on the right of $C'D'$ will be denoted by E^+ , while the part of E lying on the left of $A'B'$ will be denoted by E^- .

By (3.1) we conclude that

$$\{M_{2, \Delta}^*(H\chi_I) > 1\} \subset E. \quad (3.3)$$

By virtue of the inequality $I_2/I_1 \geq H/\sin^2 \theta$ and the definition of the rectangle $A'B'C'D'$ it is easy to show that

$$E^+ \subset E_{l_1, l_2}; \quad (3.4)$$

$$\text{dist}(D', l_1) \geq \sqrt{H|I|}; \quad (3.5)$$

$$\text{dist}(C', l_2) \geq \sqrt{H|I|}. \quad (3.6)$$

Obviously, for any $z \in E^+$

$$\text{dist}(z, l_1) \geq \text{dist}(D', l_1); \quad (3.7)$$

$$\text{dist}(z, l_2) \geq \text{dist}(C', l_2). \quad (3.8)$$

Using (3.2) and (3.4)–(3.8), we find that

$$\begin{aligned} M_{2, \Delta}^*(H\chi_I)(z) &\leq \frac{H|I|}{\text{dist}(z, l_1) \text{dist}(z, l_2)} \leq \\ &\leq \frac{H|I|}{\text{dist}(D', l_1) \text{dist}(C', l_2)} \leq \frac{H|I|}{\sqrt{H|I|} \sqrt{H|I|}} = 1 \end{aligned} \quad (3.9)$$

for any $z \in E^+$.

From (3.9) we readily obtain

$$M_{2, \Delta}^*(H\chi_I)(z) \leq 1 \quad \text{for all } z \in E^-. \quad (3.10)$$

For this it is enough to ascertain that the following equality holds:

$$M_{2, \Delta}^*(H\chi_I)(z) = M_{2, \Delta}^*(H\chi_I)(z'), \quad z \in \mathbb{R}^2,$$

where z' denotes a point symmetrical to z with respect to the center of the rectangle I .

By (3.3), (3.9), and (3.10) we have

$$\{M_{2,\Delta}^*(H\chi_I) > 1\} \subset A'B'C'D'. \quad (3.11)$$

Clearly,

$$|A'B'C'D'| \leq (6H + 3)I_1 2I_2 < 200H|I|. \quad (3.12)$$

It is easy to prove the inequality

$$\int_{E_x} M_{2,\Delta}^*(H\chi_I)(x, y) dy < 15H(1 + \ln H)I_1, \quad x \in \mathbb{R}, \quad (3.13)$$

where $E_x := \{(x, y) : y \in \mathbb{R}\} \cap E$, $x \in \mathbb{R}$.

By (3.13) we obtain

$$\begin{aligned} & \int_{A'B'C'D'} M_{2,\Delta}^*(H\chi_I)(x, y) dx dy \leq \\ & \leq 15H(1 + \ln H)I_1(\text{length } A'D') \leq 315H(1 + \ln H)|I|. \end{aligned} \quad (3.14)$$

By virtue of (3.11), (3.12), and (3.14) the lemma is proved for the case $\lambda = 1$, $H > 1$. Now it is easy to obtain the proof for the general case if we take into consideration the following obvious equalities:

$$M_{2,\Delta}^*(H\chi_I)(z) = \lambda M_{2,\Delta}^*\left(\frac{H}{\lambda}\chi_I\right)(z), \quad z \in \mathbb{R}^2; \quad (3.15)$$

$$\{M_{2,\Delta}^*(H\chi_I) > \lambda\} = \left\{M_{2,\Delta}^*\left(\frac{H}{\lambda}\chi_I\right) > 1\right\}. \quad \square \quad (3.16)$$

As is well known (see [1]), if $f \in L(1 + \ln^+ L)(\mathbb{R}^2)$, then for each $\lambda > 0$ we have the inequality

$$|\{M_2(f) > \lambda\}| \leq c_3 \int_{\mathbb{R}^2} \frac{|f|}{\lambda} \left(1 + \ln^+ \frac{|f|}{\lambda}\right), \quad (3.17)$$

where c_3 is a positive constant not depending on f and λ .

Let Γ_θ be rotation of the plane through the angle θ about the point O in the positive direction. It is easy to verify that $M_{2,\theta}(f)(z) = M_2(f \circ \Gamma_\theta)(\Gamma_\theta^{-1}(z))$. Hence we obtain $|\{M_{2,\theta}(f) > \lambda\}| = |\{M_2(f \circ \Gamma_\theta) > \lambda\}|$ which by virtue of (3.17) implies that Lemma 2 is valid.

Lemma 2. *Let $f \in L(1 + \ln^+ L)(\mathbb{R}^2)$; then the inequality*

$$|\{M_{2,\theta}(f) > \lambda\}| \leq c_3 \int_{\mathbb{R}^2} \frac{|f|}{\lambda} \left(1 + \ln^+ \frac{|f|}{\lambda}\right)$$

holds for each $\theta \in [0, \pi/2)$ and $\lambda > 0$.

By using this lemma we can easily prove

Lemma 3. *Let $f \in L(1 + \ln^+ L)^2(\mathbb{R}^2)$; then the inequality*

$$\int_{\{M_{2,\theta}(f) > 1\}} M_{2,\theta}(f) \leq c_4 \int_{\mathbb{R}^2} |f|(1 + \ln^+ |f|)^2,$$

where c_4 is the positive constant not depending on f and θ , holds for each $\theta \in [0, \pi/2)$.

Let $\Omega \subset [0, \pi/2)$, $B = \cup_{\theta \in \Omega} B_{2,\theta}$ and for each $f \in L_{loc}(\mathbb{R}^2)$

$$M_B(f)(z) := \sup \left\{ \frac{1}{|I|} \int_I |f| : z \in I, I \in B \right\}, \quad z \in \mathbb{R}^2.$$

We have

Lemma 4. *Let I_k be a rectangle, $H_k > 1$, $k \in \mathbb{N}$, and $\sum_{k=1}^{\infty} H_k |I_k| < \infty$. It is assumed that there exists a sequence $\{\mathcal{I}_k\}$ such that $\mathcal{I}_k \cap \mathcal{I}_m = \emptyset$, $k \neq m$, $\{M_B(H_k \chi_{I_k}) > \lambda\} \subset \mathcal{I}_k$, $k \in \mathbb{N}$, where $\lambda \in (0, 1]$ is a fixed number. Then*

$$\left\{ M_B \left(\sum_{k=1}^{\infty} H_k \chi_{I_k} \right) > \lambda \right\} \subset \bigcup_{k=1}^{\infty} \mathcal{I}_k. \quad (3.18)$$

Proof. Let $f_k := H_k \chi_{I_k}$ ($k \in \mathbb{N}$) and $f := \sum_{k=1}^{\infty} H_k \chi_{I_k} = \sup_{k \in \mathbb{N}} H_k \chi_{I_k}$. Clearly, (3.18) is equivalent to the inequality

$$M_B(f)(z) \leq \lambda, \quad z \notin \cup_{k=1}^{\infty} \mathcal{I}_k. \quad (3.19)$$

Let us assume that $z \notin \cup_{k=1}^{\infty} \mathcal{I}_k$, $R \ni z$, $R \in B$ and prove the inequality

$$\int_{R \cap \mathcal{I}_k} f_k \leq \lambda |R \cap \mathcal{I}_k|, \quad k \in \mathbb{N}. \quad (3.20)$$

Let $n(R) = \theta$ and $R_{x_\theta} := \{(\xi, \eta) : \xi_\theta = x_\theta, \eta_\theta \in \mathbb{R}\} \cap R$, for each $x_\theta \in \mathbb{R}$ (where $(\xi_\theta, \eta_\theta)$ denotes the coordinates of the point (ξ, η) in the $X_\theta O Y_\theta$ -system).

We consider an arbitrarily fixed number $k \in \mathbb{N}$. The following notation is introduced:

$$\begin{aligned} E_k^1 &:= \{x_\theta : R_{x_\theta} \cap \mathcal{I}_k \neq \emptyset, R_{x_\theta} \setminus \mathcal{I}_k \neq \emptyset\}; \\ E_k^2 &:= \{x_\theta : R_{x_\theta} \subset \mathcal{I}_k\}; \\ R_k^1 &:= \bigcup_{x_\theta \in E_k^1} (R_{x_\theta} \cap \mathcal{I}_k); \\ R_k^2 &:= \bigcup_{x_\theta \in E_k^2} (R_{x_\theta} \cap \mathcal{I}_k). \end{aligned}$$

Clearly,

$$R_k^1 \cap R_k^2 = \emptyset \quad \text{and} \quad R \cap \mathcal{I}_k = R_k^1 \cup R_k^2. \quad (3.21)$$

For our further discussion we need the inequality

$$M_B(f)(z) \leq \lambda, \quad z \in \partial \mathcal{I}_k, \quad (3.22)$$

which immediately follows from the condition of the lemma (∂ denotes the boundary of sets).

If $x_\theta \in E_k^1$, then one can easily find that one end of the segment $R_{x_\theta} \cap \mathcal{I}_k$ belongs to $\partial \mathcal{I}_k$ and hence by virtue of (3.22) one readily obtains

$$\int_{R_{x_\theta} \cap \mathcal{I}_k} f_k(x_\theta, y_\theta) dy_\theta \leq \lambda |R_{x_\theta} \cap \mathcal{I}_k|_1,$$

where $|\cdot|_1$ denotes the Lebesgue measure on the straight line.

The above inequality implies

$$\begin{aligned} \int_{R_k^1} f_k &= \int_{E_k^1} dx_\theta \int_{R_{x_\theta} \cap \mathcal{I}_k} f_k(x_\theta, y_\theta) dy_\theta \leq \\ &\leq \int_{E_k^1} \lambda |R_{x_\theta} \cap \mathcal{I}_k|_1 = \lambda |R_k^1|. \end{aligned} \quad (3.23)$$

It is easy to prove the following facts:

$$R_k^2 \text{ is a rectangle included in } B; \quad (3.24)$$

$$R_k^2 \text{ has a vertex belonging to } \partial \mathcal{I}_k. \quad (3.25)$$

From (3.22), (3.24), (3.25) we obtain the inequality

$$\int_{R_k^2} f_k \leq \lambda |R_k^2|. \quad (3.26)$$

Taking into account (3.12), (3.23), and (3.26), we have

$$\int_{R \cap \mathcal{I}_k} f_k = \int_{R_k^1} f_k + \int_{R_k^2} f_k \leq \lambda |R_k^1| + \lambda |R_k^2| = \lambda |R \cap \mathcal{I}_k|. \quad (3.27)$$

Since $k \in \mathbb{N}$ has been chosen arbitrarily, (3.27) implies the estimate

$$\int_R f = \sum_{k=1}^{\infty} \int_{R \cap I_k} f_k = \sum_{k=1}^{\infty} \int_{R \cap \mathcal{I}_k} f_k \leq \sum_{k=1}^{\infty} \lambda |R \cap \mathcal{I}_k| \leq \lambda |R|;$$

Because of an arbitrarily chosen R ($z \in R$, $R \in B$) we hence conclude that

$$M_B(f)(z) \leq \lambda, \quad z \notin \cup_{k=1}^{\infty} \mathcal{I}_k. \quad \square$$

For each $\theta \in [0, \pi/2)$ let $A(\theta)$ denote a set of all $f \in L(\mathbb{R}^2)$ for which $\int_{\{M_{2,\theta}(f) > 1\}} M_{2,\theta}(f) < \infty$. We have

Lemma 5. *Let f and g belong to the set $A(\theta)$. Then $f + g \in A(\theta)$ and*

$$\begin{aligned} & \int_{\{M_{2,\theta}(f+g) > 1\}} M_{2,\theta}(f+g) \leq \\ & \leq c_5 \left(\int_{\{M_{2,\theta}(f) > 1\}} M_{2,\theta}(f) + \int_{\{M_{2,\theta}(g) > 1\}} M_{2,\theta}(g) \right) + \\ & + c_6 (\|f(1 + \ln^+ |f|)\|_1 + \|g(1 + \ln^+ |g|)\|_1), \end{aligned} \quad (3.28)$$

where c_5 and c_6 are positive constants not depending on f , g , and θ .

Proof. One can easily verify the following inequalities:

$$\begin{aligned} & \int_{\{M_{2,\theta}(f+g) > 1\}} M_{2,\theta}(f+g) \leq \\ & \leq \int_{\{M_{2,\theta}(f) > 1/2\} \cup \{M_{2,\theta}(g) > 1/2\}} (M_{2,\theta}(f) + M_{2,\theta}(g)); \end{aligned} \quad (3.29)$$

$$\begin{aligned} & \int_{\{M_{2,\theta}(f) > 1/2\} \cup \{M_{2,\theta}(g) > 1/2\}} M_{2,\theta}(f) \leq \\ & \leq \int_{\{M_{2,\theta}(f) > 1/2\}} M_{2,\theta}(f) + \frac{1}{2} |\{M_{2,\theta}(g) > 1/2\}|; \end{aligned} \quad (3.30)$$

$$\int_{\{M_{2,\theta}(g) > 1/2\} \cup \{M_{2,\theta}(g) > 1/2\}} M_{2,\theta}(f) \leq$$

$$\leq \int_{\{M_{2,\theta}(g) > 1/2\}} M_{2,\theta}(g) + \frac{1}{2} |\{M_{2,\theta}(f) > 1/2\}|. \quad (3.31)$$

By virtue of (3.29), (3.30), (3.31) and Lemma 2 we conclude that (3.28) is valid. \square

Let A denote a set of all functions $f \in L(\mathbb{R}^2)$ satisfying condition (2.3). Lemma 5 immediately gives rise to

Lemma 6. *Let f and g belong to the set A . Then $f + g \in A$.*

4. PROOF OF THEOREM 1

It is assumed without loss of generality that f is positive, $|\{f > 1\}| = 1$. Let $E_k := \{k-1 \leq f < k\}$, $k \in \mathbb{N}$. Clearly, there exists $k_0 \geq 3$ for which

$$\sum_{k=k_0}^{\infty} c_1 k |E_k| < 1. \quad (4.1)$$

Choose a sequence $\{m_k\} \subset \mathbb{N}$ such that

$$\frac{k \ln^2 k |E_k|}{m_k} \leq 1, \quad k \in \mathbb{N}. \quad (4.2)$$

Let $N_0 := \{k \in \mathbb{N} : k \geq k_0, |E_k| > 0\}$ and the sequence $\{\Delta_{k,m}\}_{k \in N_0, m = \overline{1, m_k}}$ consist of pairwise nonintersecting intervals lying on the segment $[0, \pi/2)$.

For each $k \in N_0$ and $m \in [1, m_k]$ choose a rectangle $I_{k,m}$ such that

$$|I_{k,m}| = \frac{|E_k|}{m_k}; \quad (4.3)$$

$$n(I_{k,m}) = \text{cen } \Delta_{k,m}; \quad (4.4)$$

$$r(I_{k,m}) \geq \frac{k}{\sin^2 |\Delta_{k,m}|/2}. \quad (4.5)$$

By virtue of (4.5) and Lemma 1 there exists a rectangle $\mathcal{I}_{k,m}$ such that

$$\{M_{2,\Delta_{k,m}}^*(k\chi_{I_{k,m}}) > 1\} \subset \mathcal{I}_{k,m}; \quad (4.6)$$

$$|\mathcal{I}_{k,m}| \leq c_1 k |I_{k,m}|; \quad (4.7)$$

$$\int_{\mathcal{I}_{k,m}} M_{2,\Delta_{k,m}}^*(k\chi_{I_{k,m}}) \leq 2c_2 k \ln k |I_{k,m}|. \quad (4.8)$$

Let $\{Q_{k,m}\}_{k \in N_0, m = \overline{1, m_k}}$ be a sequence of pairwise rectangular intervals lying on $[0, 1]^2$ and each having height equal to 1, and

$$|Q_{k,m}| = |\mathcal{I}_{k,m}|, \quad k \in N_0, \quad m = \overline{1, m_k} \quad (4.9)$$

(such a sequence exists by virtue of (4.1), (4.3) and (4.7)).

For each $k \in N_0$ and $m \in [1, m_k]$ we complete the rectangle $Q_{k,m}$ with pairwise nonintersecting rectangles $\{\mathcal{I}_{k,m,q}\}$ which are homothetic to the rectangle $\mathcal{I}_{k,m}$, i.e., we have

$$\begin{aligned} \mathcal{I}_{k,m,q} &= P_{k,m,q}(\mathcal{I}_{k,m}) \\ \text{where } P_{k,m,q} &\text{ is the homothety } (q \in \mathbb{N}); \end{aligned} \quad (4.10)$$

$$\mathcal{I}_{k,m,q} \subset Q_{k,m}, \quad q \in \mathbb{N}; \quad (4.11)$$

$$\mathcal{I}_{k,m,q} \cap \mathcal{I}_{k,m,q'} = \emptyset, \quad q \neq q'; \quad (4.12)$$

$$\left| Q_{k,m} \setminus \bigcup_{q \in \mathbb{N}} \mathcal{I}_{k,m,q} \right| = 0. \quad (4.13)$$

Let $I_{k,m,q} := P_{k,m,q}(I_{k,m})$, $k \in N_0$, $m = \overline{1, m_k}$, $q \in \mathbb{N}$. Clearly,

$$\sum_{q \in \mathbb{N}} |I_{k,m,q}| = |I_{k,m}| = \frac{|E_k|}{m_k}, \quad k \in N_0, \quad m \in [1, m_k]. \quad (4.14)$$

Since $P_{k,m,q}$ is a homotopy, by (4.6)–(4.8) we conclude that for each $k \in N_0$, $m \in [1, m_k]$, and $q \in \mathbb{N}$

$$\{M_{2,\Delta_{k,m}}^*(k\chi_{I_{k,m,q}}) > 1\} \subset \mathcal{I}_{k,m,q}; \quad (4.15)$$

$$|\mathcal{I}_{k,m,q}| \leq c_1 k |I_{k,m,q}|; \quad (4.16)$$

$$\int_{\mathcal{I}_{k,m,q}} M_{2,\Delta_{k,m}}^*(k\chi_{I_{k,m,q}}) \leq 2c_2 k \ln k |I_{k,m,q}|. \quad (4.17)$$

We introduce the notation $g_{k,m} := \sup_{q \in \mathbb{N}} (k\chi_{I_{k,m,q}})$, $k \in N_0$, $m_0 = \overline{1, m_k}$, and $g := \sup\{g_{k,m} : k \in N_0, m \in [1, m_k]\}$. Let us prove that $g \in A$.

It is assumed that $\theta \in [0, \pi/2)$ is an arbitrary fixed number. Two cases are possible:

(a) $\theta \in \bigcup_{k,m} \Delta_{k,m}$;

(b) $\theta \notin \bigcup_{k,m} \Delta_{k,m}$;

(a) Let $\theta \in \Delta_{k(\theta), m(\theta)}$. We introduce the notation $T := \{(k, m, q) : k \in N_0, m \in [1, m_k], q \in \mathbb{N}, (k, m) \neq (k(\theta), m(\theta))\}$.

Since $\theta \notin \Delta_{k,m}$ for $(k, m) \neq (k(\theta), m(\theta))$, by the definition of $M_{2,\Delta}^*$ and (4.15) we obtain

$$\begin{aligned} \{M_{2,\theta}(k\chi_{I_{k,m,q}}) > 1\} &\subset \{M_{2,\Delta_{k,m}}^*(k\chi_{I_{k,m,q}}) > 1\} \subset \mathcal{I}_{k,m,q}, \\ &(k, m, q) \in T. \end{aligned}$$

Hence by (4.12) and Lemma 4 we conclude that

$$\{M_{2,\theta}(g - g_{k(\theta),m(\theta)}) > 1\} \subset \bigcup_{(k,m,q) \in T} \mathcal{I}_{k,m,q}, \quad (4.18)$$

$$\begin{aligned} M_{2,\theta}(g - g_{k(\theta),m(\theta)})(z) &\leq M_{2,\theta}(k\chi_{I_{k,m,q}})(z) + 1 \\ \text{for } z \in \mathcal{I}_{k,m,q}, \quad (k,m,q) &\in T. \end{aligned} \quad (4.19)$$

On account of (4.14) and (4.16)–(4.19) we write

$$\begin{aligned} &\int_{\{M_{2,\theta}(g - g_{k(\theta),m(\theta)}) > 1\}} M_{2,\theta}(g - g_{k(\theta),m(\theta)}) \leq \\ &\leq \sum_{(k,m,q) \in T} \int_{\mathcal{I}_{k,m,q}} M_{2,\theta}(g - g_{k(\theta),m(\theta)}) \leq \\ &\leq \sum_{(k,m,q) \in T} \int_{\mathcal{I}_{k,m,q}} [M_{2,\theta}(k\chi_{I_{k,m,q}}) + 1] \leq \\ &\leq \sum_{(k,m,q) \in T} [2c_2k \ln k |I_{k,m,q}| + |\mathcal{I}_{k,m,q}|] \leq \\ &\leq 1 + \sum_{k \in N_0} 2c_2k \ln k |E_k| \leq 1 + 8c_2 \|f \ln^+ f\|_1. \end{aligned} \quad (4.20)$$

By (4.2), (4.14) and Lemma 3 we have

$$\begin{aligned} &\int_{\{M_{2,\theta}(g_{k(\theta),m(\theta)}) > 1\}} M_{2,\theta}(g_{k(\theta),m(\theta)}) \leq c_4 \int_{\mathbb{R}^2} g_{k(\theta),m(\theta)} (1 + \ln^+ g_{k(\theta),m(\theta)})^2 = \\ &= c_4 \sum_{q \in \mathbb{N}} k(\theta) (1 + \ln k(\theta))^2 |I_{k(\theta),m(\theta),q}| = \\ &= c_4 k(\theta) (1 + \ln k(\theta))^2 \frac{|E_{k(\theta)}|}{m_{k(\theta)}} \leq 4c_4. \end{aligned} \quad (4.21)$$

From the construction we easily find that

$$\begin{aligned} &\|(g - g_{k(\theta),m(\theta)})(1 + \ln^+(g - g_{k(\theta),m(\theta)}))\|_1 \leq \\ &\leq \|g(1 + \ln^+ g)\|_1 \leq 4 \|f \ln^+ f\|_1, \end{aligned} \quad (4.22)$$

Analogously,

$$\|g_{k(\theta),m(\theta)}(1 + \ln^+ g_{k(\theta),m(\theta)})\|_1 \leq 4 \|f \ln^+ f\|_1 \quad (4.23)$$

Using the representation $g = (g - g_{k(\theta),m(\theta)}) + g_{k(\theta),m(\theta)}$, by virtue of (4.20)–(4.23) and Lemma 5 we obtain

$$\begin{aligned} & \int_{\{M_{2,\theta}(g) > 1\}} M_{2,\theta}(g) \leq \\ & \leq c_5(1 + 8c_2\|f \ln^+ f\|_1 + 4c_4) + 8\|f \ln^+ f\|_1 c_6. \end{aligned} \quad (4.24)$$

In proving the case (b) we have no “dangerous” term $g_{k(\theta),m(\theta)}$ and therefore, applying the same reasoning as for (4.20), we can write

$$\begin{aligned} & \int_{\{M_{2,\theta}(g) > 1\}} M_{2,\theta}(g) \leq \\ & \leq \sum_{(k,m,q) \in T_0} \int_{\mathcal{I}_{k,m,q}} M_{2,\theta}(g) \leq \sum_{(k,m,q) \in T_0} \int [M_{2,\theta}(k\chi_{\mathcal{I}_{k,m,q}}) + 1] \leq \\ & \leq \sum_{(k,m,q) \in T_0} [2c_2 k \ln k |I_{k,m,q}| + |\mathcal{I}_{k,m,q}|] \leq \\ & \leq 1 + \sum_{k \in N_0} 2c_2 k \ln k |E_k| \leq 1 + 8c_2\|f \ln^+ f\|, \\ & \text{where } T_0 := \{(k, m, q) : k \in N_0, m = \overline{1, m_k}, q \in \mathbb{N}\}. \end{aligned} \quad (4.25)$$

Since $\theta \in [0, \pi/2)$ was chosen arbitrarily, by virtue of (4.24) and (4.25) we conclude that $g \in A$.

We shall now find the desired mapping of ω . From the construction it easily follows that

$$E_k \cap E_{k'} = \emptyset, \quad k \neq k', \quad (4.26)$$

$$\left(\bigcup_{(m,q) \in T_k} I_{k,m,q} \right) \cap \left(\bigcup_{(m,q) \in T_{k'}} I_{k',m,q} \right) = \emptyset, \quad k \neq k', \quad (4.27)$$

where $T_k := \{(m, q) : m \in [1, m_k], q \in \mathbb{N}\}$ for $k \in N_0$;

$$|E_k| = \left| \bigcup_{(m,q) \in T_k} I_{k,m,q} \right| > 0, \quad k \in N_0, \quad (4.28)$$

$$\left| \{f > 1\} \setminus \bigcup_{k \in N_0} E_k \right| = \left| (0, 1)^2 \setminus \bigcup_{(k,m,q) \in T_0} I_{k,m,q} \right| > 0. \quad (4.29)$$

By conditions (4.26)–(4.29) and one familiar result on measure-preserving transformations (see, e.g., [11, Chapter: Uniform Approximation]) we conclude that there exists a Lebesgue measure preserving invertible mapping

$\omega : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that

$$\omega\left(\bigcup_{(m,q) \in T_k} I_{k,m,q}\right) = E_k, \quad k \in N_0; \quad (4.30)$$

$$\omega\left((0,1)^2 \setminus \bigcup_{(k,m,q) \in T_0} I_{k,m,q}\right) = \{f > 1\} \setminus \bigcup_{k \in N_0} E_k; \quad (4.31)$$

$$\omega(\mathbb{R}^2 \setminus (0,1)^2) = \{f \leq 1\}. \quad (4.32)$$

(4.30) and (4.31) clearly imply

$$\{f \circ \omega > 1\} = \omega^{-1}(\{f > 1\}) = (0,1)^2. \quad (4.33)$$

From the construction and conditions (4.30)–(4.32) we obtain

$$(f \circ \omega)\chi_{\bigcup_{k \in N_0} E_k} \leq g,$$

$$(f \circ \omega)\chi_{\mathbb{R}^2 \setminus \bigcup_{k \in N_0} E_k} \in L(1 + \ln^+ L)^2(\mathbb{R}^2).$$

Hence, taking into account $f \circ \omega = (f \circ \omega)\chi_{\bigcup_{k \in N_0} E_k} + (f \circ \omega)\chi_{\mathbb{R}^2 \setminus \bigcup_{k \in N_0} E_k}$, inclusions $g \in A$, and Lemmas 3 and 6, we conclude that $f \circ \omega \in A$, i.e.,

$$\sup_{\theta \in [0, \pi/2)} \int_{\{M_{2,\theta}(f \circ \omega) > 1\}} M_{2,\theta}(f \circ \omega) < \infty. \quad (4.34)$$

By (4.33) and (4.34) ω is the desired mapping. \square

5. REMARKS

Remark 1. On overcoming certain technical difficulties, we can prove by a technique similar to that used to prove Theorem 1 the following generalization.

Theorem 2. *Let a function $f \notin L(1 + \ln^+ L)^2(\mathbb{R}^2)$, $f \in L(1 + \ln^+ L)(\mathbb{R}^2)$. It is assumed that a set G_1 , $|G_1| > 0$, is such that $f\chi_{\mathbb{R}^2 \setminus G_1} \in L(1 + \ln^+ L)^2(\mathbb{R}^2)$. Then for any set G_2 , $|G_2| = |G_1|$, there exists a Lebesgue measure preserving an invertible mapping $\omega : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that*

1. $\omega(G_1) = G_2$,
2. $\{z : \omega(z) \neq z\} \subset G_1 \cup G_2$,
3. $\sup_{\theta \in [0, \pi/2)} \int_{\{M_{2,\theta}(f \circ \omega) > 1\}} M_{2,\theta}(f \circ \omega) < \infty$.

Theorem 2 yields as a corollary

Theorem 3. *Let a function $f \notin L(1+\ln^+ L)^2(\mathbb{R}^2)$, $f \in L(1+\ln^+ L)(\mathbb{R}^2)$. It is assumed that a set G is such that $f\chi_{\mathbb{R}^2 \setminus G} \in L(1+\ln^+ L)^2(\mathbb{R}^2)$. Then there exists a Lebesgue measure preserving an invertible mapping $\omega : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that*

1. $\{z : \omega(z) \neq z\} \subset G$,
2. $\sup_{\theta \in [0, \pi/2)} \int_{\{M_{2,\theta}(f \circ \omega) > 1\}} M_{2,\theta}(f \circ \omega) < \infty$.

Remark 2. Let X be a set and R an equivalence relation on X . A subset $Y \subset X$ will be called an R -set if the fact that $y \in Y$ implies that $[y]_R \subset Y$, where $[y]_R$ denotes a set of all elements from X R -equivalent to y . The following problem is posed in [6]: Given an equivalence relation R on the set X , characterize the set $E \subset X$ from the standpoint of R , i.e., give in explicit terms the kernel $\underline{E}(R)$ (the greatest R -set contained in E) and the hull $\overline{E}(R)$ (the least R -set containing E).

Consider an arbitrary set $G \subset \mathbb{R}^2$ and choose $|G| > 0$, X_G , R_G and E_G in the following manner: $X_G := \{f \in L(\mathbb{R}^2), \text{supp } f \subset G\}$, $E_G := \{f \in X_G, f \in A\}$, f and $g \in R_G$ will be called R_G -equivalent if there exists a Lebesgue measure preserving an invertible mapping $\omega : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, $\{z : \omega(z) \neq z\} \subset G$, such that $g = f \circ \omega$.

Let us agree that $\varphi(L)(G)$ denotes a class of functions $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ with the following properties: $\text{supp } f \subset G$, $\int_G \varphi(|f|) < \infty$. One can easily show that $\underline{E}_G(R_G) = L(1 + \ln^+ L)^2(G)$, while by virtue of (2.3) and Theorem 3 we have the equality $\overline{E}_G(R_G) = L(1 + \ln^+ L)(G)$. Thus the next theorem is valid.

Theorem 4. *For each $G \subset \mathbb{R}^2$, $|G| > 0$, we have*

$$\underline{E}_G(R_G) = L(1 + \ln^+ L)^2(G) \quad \text{and} \quad \overline{E}_G(R_G) = L(1 + \ln^+ L)(G).$$

ACKNOWLEDGMENT

The author wishes to express his thanks to T. Zerekidze for useful hints and discussions.

REFERENCES

1. M. de Guzmán, Differentiation of integrals in \mathbb{R}^n . *Lecture Notes in Mathematics* 481, Springer, 1975.
2. M. de Guzmán and G. V. Welland, On the differentiation of integrals. *Rev. Un. Mat. Argentina* **25**(1971), 253–276.

3. G. H. Hardy and J. E. Littlewood, A maximal theorem with function-theoretic applications. *Acta Math.* **54**(1930), 81–116.
4. N. Wiener, The ergodic theorem. *Duke Math. J.* **5**(1939), 1–18.
5. E. M. Stein, Note on class $L \log L$. *Studia Math.* **31**(1969), 305–310.
6. O. D. Tsereteli, On the inversion of some Hardy and Littlewood theorems. (Russian) *Bull. Acad. Sci. Georgian SSR* **56**(1969), No. 2, 269–271.
7. O. D. Tsereteli, A metric characteristic of the set of functions whose maximal functions are summable. (Russian) *Trudy Tbilissk. Mat. Inst.* **42**(1972), 104–118.
8. L. D. Gogoladze, The Hardy-Littlewood maximal functions. (Russian) *Bull. Acad. Sci. Georgian SSR* **111**(1983), 257–259.
9. L. D. Gogoladze, On the summation of multiple trigonometric series and conjugate functions. (Russian) *Dissertation for the Doctor of Science Degree, Tbilisi*, 1984.
10. R. J. Bagby, A note on the strong maximal function. *Proc. Amer. Math. Soc.* **88**(1983), 648–650.
11. P. R. Halmos, Lectures on the ergodic theory. *Math. Soc. Japan, Tokyo*, 1956.

(Received 23.05.1994)

Author's address:
Faculty of Mechanics and Mathematics
I. Javakhishvili Tbilisi State University
2, University St., Tbilisi 380043
Republic of Georgia