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LITTLEWOOD–PALEY OPERATORS ON THE GENERALIZED LIPSCHITZ SPACES

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Abstract. Littlewood–Paley operators defined on a new kind of generalized Lipschitz spaces $\mathcal{E}_0^{\alpha,p}$ are studied. It is proved that the image of a function under the action of these operators is either equal to infinity almost everywhere or is in $\mathcal{E}_0^{\alpha,p}$, where $-n < \alpha < 1$ and $1 < p < \infty$.

1. Introduction

For $x \in \mathbb{R}^n$, $y > 0$, the Poisson kernel is $P(x,y) = c_n y(y^2 + |x|^2)^{-(n+1)/2}$. Denote the Poisson integral of f by

$$f(x,y) = \int_{\mathbb{R}^n} f(z)P(x-z,y)\,dz.$$

We have (see [1])

$$|\nabla f(x,y)| \le c_n \int_{\mathbb{R}^n} |f(z)|\,(y + |x-z|)^{-(n+1)} dz. \tag{1}$$

Let us now consider the following two kinds of Littlewood–Paley functions:

$$S(f)(x) = \Big(\iint_{\Gamma(x)} y^{1-n} |\nabla f(z,y)|^2 dz\,dy \Big)^{1/2}$$

and

$$g_\lambda^*(f)(x) = \Big\{ \iint_{\mathbb{R}^{n+1}_+} \Big(\frac{y}{y+|x-z|} \Big)^{\lambda n} y^{1-n} |\nabla f(z,y)|^2 dz\,dy \Big\}^{1/2}.$$

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The generalized Lipschitz space $\mathcal{E}^{\alpha,p}$ consists of functions f which are locally integrable and satisfy the following condition: there exists a constant C such that for any cube Q

$$\int_Q |f(x) - f_Q|^p dx \le C|Q|^{1+\frac{\alpha p}{n}}, \tag{2}$$

where $f_Q = \frac{1}{Q}\int_Q f(x)dx$. Denote the norm of f in $\mathcal{E}^{\alpha,p}$ by

$$\|f\|_{\alpha,p} = \inf\left\{C^{1/p} : C \text{ satisfies } (2)\right\}.$$

Recently, Qiu [2] has obtained the following result.

Theorem A. *Let $1 < p < \infty$, $-n/p \le \alpha < 1/2$, $\alpha \ne 0$, and $\lambda > \max(1, 2/p)$. If $f \in \mathcal{E}^{\alpha,p}$ and Tf is $S(f)$ or $g_\lambda^*(f)$, then either $Tf(x) = \infty$ a.e. or $Tf(x) < \infty$ a.e., and there exists a constant C independent of f such that*

$$\|Tf\|_{\alpha,p} \le C\|f\|_{\alpha,p}.$$

We notice that the range of α in Theorem A seems somewhat rough. It is natural to consider whether the conclusion of the above theorem holds for $-n < \alpha < 1$. The last named author of this paper proved that the conclusion of Theorem A holds for $-n/p < \alpha < 1$ (see [3]). In this paper, with the aid of the idea in [4], we shall introduce a variant of $\mathcal{E}^{\alpha,p}$, $\mathcal{E}_0^{\alpha,p}$, and prove that the conclusion of Theorem A holds for $\mathcal{E}_0^{\alpha,p}$ with $-n < \alpha < 1$. Let us first definie $\mathcal{E}_0^{\alpha,p}$.

Definition. A locally integrable function f is called a generalized Lipschitz function of central type if there exists a constant C such that (2) holds for any cube Q centered at the origin. Moreover, the space consisting of all generalized Lipschitz functions of central type is denoted by $\mathcal{E}_0^{\alpha,p}$. We call $\mathcal{E}_0^{\alpha,p}$ the generalized Lipschitz space of central type.

It is easy to see that $\mathcal{E}^{\alpha,p} \subset \mathcal{E}_0^{\alpha,p}$ and $\mathcal{E}_0^{\alpha,p}$ is just the bounded mean oscillation space of central type, BMO_0 in [4]. Let us now formulate our results.

Theorem 1. *Let $1 < p < \infty$ and $-n < \alpha < 1$. If $f \in \mathcal{E}_0^{\alpha,p}$, then either $S(f)(x) = \infty$ a.e. or $S(f)(x) < \infty$ a.e., and there exists a constant C independent of f such that*

$$\|S(f)\|_{\alpha,p} \le C\|f\|_{\alpha,p}.$$

Theorem 2. *Let $1 < p < \infty$, $-n < \alpha < 1$, and $\lambda > \max(1, 2/p) + 2/n$. If $f \in \mathcal{E}_0^{\alpha,p}$, then either $g_\lambda^*(f)(x) = \infty$ a.e. or $g_\lambda^*(f)(x) < \infty$ a.e., and there exists a constant $C = C(n, \alpha, p, \lambda)$ such that*

$$\|g_\lambda^*(f)\|_{\alpha,p} \leq C\|f\|_{\alpha,p}.$$

2. Some Lemmas

Lemma 1. *Let $1 < p < \infty$, $-n < \alpha < 1$, $\alpha \neq 0$, and $0 < d$, $\alpha < d$. If $f \in \mathcal{E}_0^{\alpha,p}$ and Q is a cube centered at the origin with the edge length r, then there exists a constant $C = C(n, p, \alpha, d)$ such that for any $y > 0$*

$$\int\limits_{\mathbb{R}^n} \frac{|f(x) - f_Q|}{y^{n+d} + |x|^{n+d}}\, dx \leq C y^{-d}(y^\alpha + r^\alpha)\|f\|_{\alpha,p}. \tag{3}$$

See [1] and [2] for its proof.

Lemma 1′. *Let $1 < p < \infty$ and $d > 0$. If $f \in BMO_0 = \mathcal{E}_0^{0,p}$ and Q is a cube centered at the origin with the edge length r, then there exists a constant $C = C(n, p, d)$ such that for any $y > 0$,*

$$\int\limits_{\mathbb{R}^n} \frac{|f(x) - f_Q|}{y^{n+d} + |x|^{n+d}}\, dx \leq C y^{-d}\Big(1 + \Big|\log_2 \frac{y}{r}\Big|\Big)\|f\|_{0,p}.$$

Proof. By the known result in [5] we have

$$\int\limits_{\mathbb{R}^n} \frac{|f(x) - f_Q|}{r^{n+d} + |x|^{n+d}}\, dx \leq C r^{-d}\|f\|_{0,p}.$$

Let R be the cube centered at the origin with the edge length y. Then

$$\int\limits_{\mathbb{R}^n} \frac{|f(x) - f_Q|}{y^{n+d} + |x|^{n+d}}\, dx \leq \int\limits_{\mathbb{R}^n} \frac{|f(x) - f_R|}{y^{n+d} + |x|^{n+d}}\, dx$$
$$+ |f_R - f_Q| \int\limits_{\mathbb{R}^n} \frac{dx}{y^{n+d} + |x|^{n+d}}\, dx \leq C y^{-d}\|f\|_{0,p} + C y^{-d}|f_R - f_Q|.$$

Thus it remains to prove

$$|f_R - f_Q| \leq C\Big(1 + \Big|\log_2 \frac{y}{r}\Big|\Big)\|f\|_{0,p}.$$

Let $y > r$, and let k satisfy $2^k \le y < 2^{k+1}r$. Then $k \le \log_2 \frac{y}{r}$ and

$$\begin{aligned} |f_R - f_Q| &\le |f_R - f_{Q_k}| + \sum_{j=1}^{k} |f_{Q_j} - f_{Q_{j-1}}| \\ &\le 2^n \Big(\frac{1}{|R|} \int\limits_R |f(x) - f_R|^p dx \Big)^{1/p} + \sum_{j=1}^{k} 2^n \|f\|_{0,p} \\ &\le 2^n (1+k) \|f\|_{0,p} \\ &\le C\Big(1 + \log_2 \frac{y}{r}\Big) \|f\|_{0,p}, \end{aligned}$$

where Q_k is the concentric extension of Q by 2^k times.

When $y < r$, by exchanging y and r, we shall get the same estimate as above with $\log_2 \frac{r}{y} = |\log_2 \frac{y}{r}|$. □

Let $\chi_{_E}$ be the characteristic function of E. For a cube Q in $\mathbb{R}^n$ and $d > 0$ let dQ be the concentric extension of Q by d times.

Lemma 2. *Suppose that $1 < p < \infty$, $-n < \alpha < 1$, and $f \in \mathcal{E}_0^{\alpha,p}$. Let Q be a cube centered at the origin with the edge length r, and $h_Q(x) = [f(x) - f_Q]\chi_{Q^c}(x)$. If there is $x' \in dQ$ such that $S(h_Q)(x') < \infty$, where $d = (8\sqrt{n})^{-1}$, then there exists a constant $C = C(n, \alpha, p)$ such that*

$$S(h_Q)(x) < \infty, \quad \forall x \in dQ$$

and

$$|S(h_Q)(x) - S(h_Q)(x')| < Cr^\alpha \|f\|_{\alpha,p}, \quad \forall x \in dQ.$$

Proof. Let us first consider the case of $\alpha \ne 0$. Fix $x \in dQ$. Set

$$\Gamma^-(x) = \big\{(z,y) \in \Gamma(x) : y \le dr\big\}$$

and

$$\Gamma^+(x) = \big\{(z,y) \in \Gamma(x) : y > dr\big\}.$$

Then

$$S(h_Q)(x) \le S^- + S^+, \quad x \in dQ,$$

where

$$S^- = \Big(\iint\limits_{\Gamma^-(x)} y^{1-n} |\nabla h_Q(z,y)|^2 dz\, dy \Big)^{1/2}$$

and

$$S^+ = \Big(\iint\limits_{\Gamma^+(x)} y^{1-n} |\nabla h_Q(z,y)|^2 dz\, dy \Big)^{1/2}.$$

Estimating S^- as in [2], we have

$$S^- \le Cr^{\alpha}\|f\|_{\alpha,p}. \tag{4}$$

For S^+ we have

$$\begin{aligned}
S^+ &= \Big(\iint\limits_{\Gamma^+(0)} y^{1-n}|\nabla h_Q(x+z,y)|^2 dz\,dy\Big)^{1/2}\\
&\le \Big(\iint\limits_{\Gamma^+(0)} y^{1-n}|\nabla h_Q(x'+z,y)|^2 dz\,dy\Big)^{1/2}\\
&+ \Big(\iint\limits_{\Gamma^+(0)} y^{1-n}|\nabla h_Q(x+z,y)-\nabla h_Q(x'+z,y)|^2 dz\,dy\Big)^{1/2}\\
&\le S(h_Q)(x') + \Big\{\iint\limits_{\Gamma^+(0)} y^{1-n}\\
&\times \Big(\int\limits_{Q^c} |\nabla P(x+z-t,y)\nabla P(x'+z+t,y)|\,|f(t)-f_Q|dt\Big)^2 dz\,dy\Big\}^{1/2}.
\end{aligned}$$

Note that

$$|\nabla P(x,y)-\nabla P(x',y)| = \Big(\sum_{j=1}^{n+1}\Big|\frac{\partial}{\partial x_j}\,p(x,y)-\frac{\partial}{\partial x_j}\,P(x',y)\Big)^{1/2},$$

where $\frac{\partial}{\partial x_{n+1}} = \frac{\partial}{\partial y}$. By the mean value theorem we have

$$\begin{aligned}
&\Big|\frac{\partial}{\partial x_j}\,p(x,y)-\frac{\partial}{\partial x_j}\,P(x',y)\Big|\\
&= \Big|\nabla\frac{\partial}{\partial x_j}\,P(x,y)\Big|_{x+\theta_j(x-x')}|x-x'|, \quad 0<\theta_j<1,
\end{aligned}$$

where

$$\Big|\nabla\frac{\partial}{\partial x_j}\,P(x,y)\Big| \le \frac{C}{(y+|x|)^{n+2}}.$$

Thus

$$\begin{aligned}
&|\nabla P(x+z-t,y)-\nabla P(x'+z-t,y)|\\
&\le C|x-x'|\Big\{\sum_{j=1}^{n+1}\big(y+|x+z-t+\theta_j(x-x')|\big)^{-2(n+2)}\Big\}^{1/2}.
\end{aligned} \tag{5}$$

Since $(x,y)\in\Gamma^+(0)$, $x,x'\in dQ$, and $t\notin Q$, we have $|t|>r/2$, $|z|<y$, $|x|<r/16<|t|/8$, and $|x-x'|<r/8<|t|/4$. Thus,

$$\begin{aligned}|t|&\le|x+z-t+\theta_j(x-x')|+|x|+|z|+|x-x'|\\&\le|x+z-t+\theta_j(x-x')|+|t|/8+y+|t|/4\end{aligned}$$

and

$$\frac{5}{16}\big(y+|t|\big)\le|x+z-t+\theta_j(x-x')|+y,$$

where $1\le j\le n|+1$. Therefore

$$|\nabla P(x+z-t,y)-\nabla P(x'+z-t,y)|\le\frac{Cr}{(y+|t|)^{n+2}}.\tag{6}$$

Using (6) and (3), we obtain

$$\begin{aligned}S^+&\le S(h_Q)(x')+C\Big\{\iint\limits_{\Gamma^+(0)}y^{1-n}\Big[\int\limits_{Q^c}\frac{r|f(t)-f_Q|}{(y+|t|)^{n+2}}dt\Big]^2dzdy\Big\}^{1/2}\\&\le S(h_Q)(x')+C\Big\{\iint\limits_{\Gamma^+(0)}y^{1-n}r^2\big[y^{-2}(y^\alpha+r^\alpha)\|f\|_{\alpha,p}\big]^2dzdy\Big\}^{1/2}\\&\le S(h_Q)(x')+Cr\|f\|_{\alpha,p}\Big\{\int\limits_{dr}^{\infty}\int\limits_{|z|<y}y^{1-n}y^{-4}(y^{2\alpha}+r^{2\alpha})dzdy\Big\}^{1/2}\\&\le S(h_Q)(x')+Cr^\alpha\|f\|_{\alpha,p}.\end{aligned}\tag{7}$$

Combining (4) with (7) we have

$$S(h_Q)(x)\le S(h_Q)(x')+Cr^\alpha\|f\|_{\alpha,p}.$$

Thus $S(h_Q)(x)<\infty$. Exchanging x and x', we obtain

$$|S(h_Q)(x)-S(h_Q)(x')|\le Cr^\alpha\|f\|_{\alpha,p}.$$

Hence the proof of Lemma 2 is complete for the case of $\alpha\ne0$.

When $\alpha=0$, by using Lemma 1′ instead of Lemma 1 we obtain

$$\begin{aligned}S^+&\le S(h_Q)(x')+C\Big\{\iint\limits_{\Gamma^+(0)}y^{1-n}\Big[\int\limits_{Q^c}\frac{r|f(t)-f_Q|}{(y+|t|)^{n+2}}dt\Big]^2dzdy\Big\}^{1/2}\\&\le S(h_Q)(x')+C\Big\{\iint\limits_{\Gamma^+(0)}y^{1-n}r^2\big[y^{-2}\big(1+\big|\log_2\frac{y}{r}\big|\big)\|f\|_{0,p}\big]^2dzdy\Big\}^{1/2}\\&\le S(h_Q)(x')+Cr\|f\|_{0,p}\Big\{\int\limits_{dr}^{\infty}\int\limits_{|z|<y}y^{-3-n}\big(1+\big|\log_2\frac{y}{r}\big|\big)^2dzdy\Big\}^{1/2}\end{aligned}$$

$$\leq S(h_Q)(x') + Cr^{\alpha}\|f\|_{0,p}\Big\{\int_1^{\infty} u^{-3}\big(1+|\log_2 u|\big)^2 du\Big\}^{1/2}$$

$$\leq S(h_Q)(x') + Cr^{\alpha}\|f\|_{0,p}. \tag{8}$$

Now, it is easy to see that the conclusion of the lemma for $\alpha = 0$ follows from (8) and (4) with $\alpha = 0$. □

Lemma 3. *Under the hypothesis of Lemma 2, if there is $x' \in dQ$ such that $g_\lambda^*(h_Q)(x') < \infty$, where $\lambda > max(1, 2/p) + 2/n$, then there exists a constant $C = C(n, \alpha, \lambda, p)$ such that $g_\lambda^*(h_Q)(x) < \infty$ and*

$$|g_\lambda^*(h_Q)(x) - g_\lambda^*(x') \leq Cr^{\alpha}\|f\|_{\alpha,p}, \quad \forall x \in dQ.$$

Proof. We only consider the case of $\alpha \neq 0$. As in Lemma 2, the proof in the case $\alpha = 0$ is similar. Let

$$J_k = \big\{(z,y) \in \mathbb{R}_+^{n+1} : |z| < 2^{k-2}r,\ 0 < y < 2^{k-2}r\big\}, \quad k \geq 0.$$

For fixed $x \in dQ$ we have

$$g_\lambda^*(h_Q)(x) \leq G^- + G^+,$$

where

$$G^- = \Big(\iint_{J_0} \Big(\frac{y}{y+|z|}\Big)^{\lambda n} y^{1-n}|\nabla h_Q(x+z,y)|^2 dz\, dy\Big)^{1/2}$$

and

$$G^+ = \Big(\iint_{\mathbb{R}_+^{n+1}\setminus J_0} \Big(\frac{y}{y+|z|}\Big)^{\lambda n} y^{1-n}|\nabla h_Q(x+z,y)|^2 dz\, dy\Big)^{1/2}.$$

Note that if $(z,y) \in J_0$, $x \in dQ$, and $t \neq Q$, then $|z| < r/4$, $|x| < r/16$, and $|t| > r/2$. Thus

$$|t| \leq |t-x-z| + |x| + |z| \leq |x+z-t| + \frac{5}{8}|t|$$

and

$$\frac{1}{16}\,(r+|t|) \leq |x+z-t| + y.$$

By (1) and Lemma 1 we get

$$G^- \leq C\Big\{\iint_{J_0} \Big(\frac{y}{y+|z|}\Big)^{\lambda n} y^{1-n}\Big[\int_{Q^c} \frac{r|f(t)-f_Q|}{(y+|x+z-t|)^{n+1}}dt\Big]^2 dzdy\Big\}^{1/2}$$

$$\leq C\Big\{\iint_{J_0} \Big(\frac{y}{y+|z|}\Big)^{\lambda n} y^{1-n}\Big[\int_{Q^c} \frac{r|f(t)-f_Q|}{(r+|t|)^{n+1}}dt\Big\}^2 dzdy\Big\}^{1/2}$$

$$\le C\Big\{\int\limits_{dr}^{\infty}\int\limits_{|z|<y}\Big(\frac{y}{y+|z|}\Big)^{\lambda n}y^{1-n}\big[r^{\alpha-1}\|f\|_{\alpha,p}\big]^2dzdy\Big\}^{1/2}$$
$$\le Cr^{\alpha-1}\|f\|_{\alpha,p}\Big(\int\limits_0^r y^{1-n}r^n dy\Big)^{1/2}$$
$$\le Cr^{\alpha}\|f\|_{\alpha,p}.$$

To estimate C^+ we observe that

$$G^+\le\Big\{\iint\limits_{\mathbb{R}^{n+1}_+\setminus J_0}\Big(\frac{y}{y+|z|}\Big)^{\lambda n}y^{1-n}|\nabla h_Q(x'+z,y)|^2dzdy\Big\}^{1/2}$$
$$+\Big\{\iint\limits_{\mathbb{R}^{n+1}_+\setminus J_0}\Big(\frac{y}{y+|z|}\Big)^{\lambda n}y^{1-n}|\nabla h_Q(x+z,y)-$$
$$-\nabla h_Q(x'+z,y)|^2dzdy\Big\}^{1/2}$$
$$\le g^*_\lambda(h_Q)(x')+D,$$

where

$$D=\Big\{\iint\limits_{\mathbb{R}^{n+1}_+\setminus J_0}\Big(\frac{y}{y+|z|}\Big)^{\lambda n}y^{1-n}|\nabla h_Q(x+z,y)-$$
$$-\nabla h_Q(x'+z,y)|^2dzdy\Big\}^{1/2}$$
$$\le C\Big\{\sum_{k=1}^{\infty}(2^k r)^{-\lambda n}\iint\limits_{J_k\setminus J_{k-1}}y^{\lambda n+1-n}$$
$$\times\Big[\int\limits_{Q^c}|\nabla P(x+z-t,y)-\nabla P(x'+z-t,y)|\,|f(t)-f_Q|dt\Big]^2dzdy\Big\}^{1/2}$$
$$\le C\Big\{\sum_{k=1}^{\infty}(2^k r)^{-\lambda n}(A_k+B_k)\Big\}^{1/2}.$$

Here

$$A_k=\iint\limits_{J_k\setminus J_{k-1}}y^{\lambda n+1-n}$$
$$\times\Big[\int\limits_{Q^c_{k+1}}|\nabla P(x+z-t,y)-\nabla P(x'+z-t,y)|\,|f(t)-f_Q|dt\Big]^2dzdy,$$

$$B_k = \iint_{J_k \setminus J_{k-1}} y^{\lambda n+1-n} \times \Big[\int_{Q_{k+1}\setminus Q} |\nabla P(x+z-t,y) - \nabla P(x'+z-t,y)|\,|f(t)-f_Q|dt\Big]^2 dzdy,$$

and $Q_{k+1} = 2^{k+1}Q$. Without loss of generality we may assume that $\max(1,2/p)+2/n < \lambda < 3+2/n$. By the easy inequality (see [3])

$$|\nabla P(x,y) - \nabla P(x',y)| \le C|x-x'|\Big(\frac{1}{(y+|x|)^{n+2}} + \frac{1}{(y+|x'|)^{n+2}}\Big), \quad \forall x,x' \in \mathbb{R}^n, \quad y>0,$$

together with the Minkowski inequality for integrals, we have

$$\begin{aligned}
B_k &\le Cr^2 \iint_{J_k\setminus J_{k-1}} y^{\lambda n+1-n}\Big\{ \int_{Q_{k+1}\setminus Q} |f(t)-f_Q|\Big(\frac{1}{(y+|x+z-t|)^{n+2}} \\
&\quad + \frac{1}{(y+|x'+z-t|)^{n+2}}\Big)dt\Big\}^2 dz\,dy \\
&\le Cr^2 \int_0^\infty \int_{\mathbb{R}^n} y^{\lambda n+1-n}\Big\{ \int_{Q_{k+1}} |f(t)-f_Q|\Big(\frac{1}{(y+|x+z-t|)^{n+2}} \\
&\quad + \frac{1}{(y+|x'+z-t|)^{n+2}}\Big)dt\Big\}^2 dz\,dy \\
&\le Cr^2 \int_{\mathbb{R}^n}\Big[\int_{Q_{k+1}} |f(t)-f_Q|\Big(\int_0^\infty \frac{y^{\lambda n+1-n}}{(y+|x+z-t|)^{2(n+2)}}dy\Big)^{1/2}dt\Big]^2 dz \\
&\quad + Cr^2 \int_{\mathbb{R}^n}\Big[\int_{Q_{k+1}} |f(t)-f_Q|\Big(\int_0^\infty \frac{y^{\lambda n+1-n}}{(y+|x'+z-t|)^{2(n+2)}}dy\Big)^{1/2}dt\Big]^2 dz \\
&= Cr^2 \int_{\mathbb{R}^n}\Big[\int_{Q_{k+1}} \frac{|f(t)-f_Q|}{|z+x-t|^{(3n-\lambda n+2)/2}}\Big(\int_0^\infty \frac{y^{\lambda n+1-n}}{(1+y)^{2(n+2)}}dy\Big)^{1/2}dt\Big]^2 dz \\
&\quad + Cr^2 \int_{\mathbb{R}^n}\Big[\int_{Q_{k+1}} \frac{|f(t)-f_Q|}{|z+x'-t|^{(3n-\lambda n+2)/2}}\Big(\int_0^\infty \frac{y^{\lambda n+1-n}}{(1+y)^{2(n+2)}}dy\Big)^{1/2}dt\Big]^2 dz \\
&= Cr^2 \int_{\mathbb{R}^n}\Big(\int_{Q_{k+1}} \frac{|f(t)-f_Q|}{|u-t|^{n-[(\lambda n-1)-2]/2}}dt\Big)^2 du.
\end{aligned}$$

Using the Hardy–Littlewood–Sobolev theorem on fractional integration with $\gamma = [(\lambda-1)n-2]/2$, $q=2$, and $1/s = 1/q + \gamma/n = \lambda/2 - 1/n$ (see [6]), we obtain

$$B_k \le Cr^2 \Big(\int_{Q_{k+1}} |f(z) - f_Q|^s dz \Big)^{2/s}.$$

Since $\lambda \ge 2/p + 2/n$, then $p \ge s$. Thus

$$\begin{aligned} B_k &\le Cr^2 \Big(\int_{Q_{k+1}} |f(z) - f_Q|^p dz \Big)^{2/p} |Q_{k+1}|^{2(1/s-1/p)} \\ &\le Cr^2 \Big\{ \Big(\int_{Q_{k+1}} |f(z) - f_Q|^p dz \Big)^{1/p} \\ &\qquad + |Q_{k+1}|^{1/p} |f_{Q_{k+1}} - f_Q| \Big\}^2 |Q_{k+1}|^{2(1/s-1/p)} \\ &\le Cr^2 \big\{ |Q_{k+1}|^{1/p+\alpha/n} \|f\|_{\alpha,p} \\ &\qquad + |Q_{k+1}|^{1/p} (2^k r)^\alpha \|f\|_{\alpha,p} \big\}^2 |Q_{k+1}|^{2(1/s+1/p)} \\ &\le Cr^2 (2^k r)^{2\alpha} (2^k r)^{\lambda n - 2} \|f\|_{\alpha,p} \\ &\le C(2^k r)^{\lambda n} (2^{2k(\alpha-1)} r^{2\alpha} \|f\|_{\alpha,p}. \end{aligned}$$

To estimate A_k we observe that if $(z,y) \in J_k \backslash J_{k-1}$ and $t \notin Q_{k+1}$, then $|t| > 2^k r$, $k \ge 1$, and $|z| < 2^{k-2} r < |t|/4$. Thus,

$$\begin{aligned} |t| &\le |x + z - t + \theta_j (x - x')| + |x| + |z| + |x - x'| \\ &\le |x + z - t + \theta (x - x')| + \frac{5}{16} |t|. \end{aligned}$$

By Lemma 1 we have

$$\begin{aligned} A_k &\le C \iint_{J_k \backslash J_{k+1}} y^{\lambda n + 1 - n} \Big[\int_{Q_{k+1}^c} \frac{r|f(t) - f_Q|}{(2^k r + |t|)^{n+2}} \, dt \Big]^2 dz \, dy \\ &\le Cr^2 \iint_{J_k \backslash J_{k+1}} y^{\lambda n + 1 - n} \big\{ (2^k r)^{-2} [(2^k r)^\alpha + r^\alpha] \|f\|_{\alpha,p} \big\}^2 dz \, dy \\ &\le Cr^2 (2^k r)^{-4+2\alpha} \|f\|_{\alpha,p} \int_0^{2^k r} \int_{|z| < 2^k r} y^{\lambda n + 1 - n} dz \, dy \\ &\le Cr^{2\alpha} (2^k r)^{\lambda n} 2^{2k(\alpha-1)} \|f\|_{\alpha,p}. \end{aligned}$$

Combining the estimate of A_k with that of B_k, we obtain

$$D \le C\Big\{\sum_{k=1}^{\infty}(2^k r)^{-\lambda n}(2^k r)^{\lambda n} r^{2\alpha} 2^{2k(\alpha-1)}\|f\|_{\alpha,p}\Big\}^{1/2} \le Cr^{\alpha}\|f\|_{\alpha,p}.$$

Therefore

$$g_{\lambda}^{*}(h_Q)(x) \le g_{\lambda}^{*}(h_Q)(x') + Cr^{\alpha}\|f\|_{\alpha,p}.$$

As in the last part of the proof of Lemma 2, we have

$$|g_{\lambda}^{(}h_Q)(x) - g_{\lambda}^{*}(h_Q)(x') \le Cr^{\alpha}\|f\|_{\alpha,p}. \quad \square$$

3. The Proofs of the Theorems

Let T be one of the Littlewood–Paley functions as in Section 1. Suppose that $Tf(x) \ne \infty$ a.e. Then $|E| \triangleq |\{x : Tf(x) < \infty\}| > 0$. Thus there is a cube Q' centered at the origin such that $|Q' \cap E| > 0$. Set $Q = (1/d)Q'$ (then $Q' = dQ$). We write f as

$$\begin{aligned} f(x) &= f_Q + [f(x) - f_Q]\chi_{Q}(x) + [f(x) - f_Q]\chi_{Q^c}(x) \\ &\triangleq f_Q + g_Q(x) + h_Q(x). \end{aligned}$$

Since

$$Tf(x) \le Tg_Q(x) + Th_Q(x) \tag{9}$$

and

$$Th_Q(x) \le Tf(x) + Tg_Q(x), \tag{10}$$

it is easy to see that the inequality

$$\|g_Q\|_p = \Big(\int_Q |f(t) - f_Q|^p dt\Big)^{1/p} \le C|Q|^{1/p+\alpha/n}\|f\|_{\alpha,p} \tag{11}$$

implies that $g_Q \in L^p$. Then it follows from the L^p-boundedness of the Littlewood–Paley operator that $Tg_Q(x) < \infty$ a.e.. Since $|Q' \cap E| > 0$, there is $x' \in Q' \cap E \subset dQ$ such that $Tf(x') < \infty$ and $Tg_Q(x') < \infty$. By (10) and Lemmas 2 and 3, we have $Th_Q(x') < \infty$ and

$$Th_Q(x) < \infty, \quad \forall x \in dQ = Q'.$$

By (9) we obtain

$$Tf(x) < \infty \quad \text{a.e. } x \in Q'.$$

Finally, let the edge length of Q' tend to ∞; we have $Tf(x) < \infty$ a.e., $x \in \mathbb{R}^n$.

Let Q' be a cube centered at the origin, and $Q = (1/d)Q'$. Choose $x' \in dQ$ so that $Th_Q(x') < \infty$. Then it follows from (11) and Lemmas 2 and 3 that

$$\begin{aligned}
&\Big(\int_{Q'} |Tf(x) - Th_Q(x')|^p dx\Big)^{1/p} \\
&\le \Big(\int_{Q'} |Tg_Q(x)|^p dx\Big)^{1/p} + \Big(\int_{Q'} |Th_Q(x) - Th_Q(x')|^p dx\Big)^{1/p} \\
&\le C\|g_Q\|_p + C|Q'|^{1/p} r^\alpha \|f\|_{\alpha,p} \\
&\le C|Q'|^{1/p+\alpha/n} \|f\|_{\alpha,p}.
\end{aligned}$$

This completes the proof of the theorems. □

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