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# COMMUTATIVITY FOR A CERTAIN CLASS OF RINGS

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Abstract. We discuss the commutativity of certain rings with unity 1 and one-sided $s$-unital rings under each of the following conditions: $x^r[x^s,y]=\pm[x,y^t]x^n$, $x^r[x^s,y]=\pm x^n[x,y^t]$, $x^r[x^s,y]=\pm[x,y^t]y^m$, and $x^r[x^s,y]=\pm y^m[x,y^t]$, where $r$, $n$, and $m$ are non-negative integers and $t>1$, $s$ are positive integers such that either $s$, $t$ are relatively prime or $s[x,y]=0$ implies $[x,y]=0$. Further, we improve the result of [6, Theorem 3] and reprove several recent results.

Throughout the paper $R$ will represent an associative ring (with or without unity 1). Let $C(R)$ denote the commutator ideal of $R$, $Z(R)$ the center of $R$, and $H$ the heart of $R$. By $(GF(q))_2$ we mean the ring of $2\times 2$ matrices over the Galois field $GF(q)$ with $q$ elements. Set $e_{11}=\begin{pmatrix}1&0\\0&0\end{pmatrix}$, $e_{12}=\begin{pmatrix}0&1\\0&0\end{pmatrix}$, $e_{21}=\begin{pmatrix}0&0\\1&0\end{pmatrix}$, and $e_{22}=\begin{pmatrix}0&0\\0&1\end{pmatrix}$ in $(GF(p))_2$ for a prime $p$. Following [1], a ring $R$ is said to be left (resp., right) $s$-unital, if $x\in Rx$ (resp., $x\in xR$) for each element $x$ in $R$. Further, $R$ is called $s$-unital if $x\in Rx\cap xR$ (see [2] and [3]). The symbol $[x,y]$ stands for the commutator $xy-yx$ for any $x$, $y\in R$. In some particular cases several authors [1–3, 5] studied the commutativity of rings satisfying the following conditions:

($c_1$) For every $x,y\in R$ there holds $x^r[x^s,y]=\pm[x,y^t]x^n$ with integers $t>1$, $s\geq 1$, $n\geq 0$, $r\geq 0$.

($c_2$) For every $x,y\in R$ there holds $x^r[x^s,y]=\pm[x,y^t]y^m$ with integers $t>1$, $s\geq 1$, $m\geq 0$, $r\geq 0$.

($c_3$) For every $x,y\in R$ there holds $x^r[x^s,y]=\pm x^n[x,y^t]$ with integers $t>1$, $s\geq 1$, $n\geq 0$, $r\geq 0$.

($c_4$) For every $x,y\in R$ there holds $x^r[x^s,y]=\pm y^m[x,y^t]$ with integers $t>1$, $s\geq 1$, $m\geq 0$, $r\geq 0$.

To develop the commutativity of a ring $R$ satisfying one of the above conditions, we need some extra condition such as

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Q(s): For any positive integer $s$ $s[x,y] = 0$ implies $[x,y] = 0$ for all $x, y \in R$,

or

Q(s,t): $s$ and $t$ are relatively prime integers.

To prove our results we need a few preliminary lemmas. We begin with the following well-known result [6, p. 221].

**Lemma 1.** *Let $x, y \in R$ and $[[x,y],x] = 0$. Then $[x^k, y] = kx^{k-1}[x,y]$ for any positive integer $k$.*

**Lemma 2 ([7]).** *Let $R$ be a ring with unity 1, and let $f : R \to R$ be a function such that $f(x+1) = f(x)$ for every $x \in R$. If for some positive integer $n$ we have $x^n f(x) = f(x)x^n = 0$ for all $x \in R$, then necessarily $f(x) = 0$.*

**Lemma 3 ([8]).** *Let $f$ be a polynomial in the non-commuting indeterminates $x_1, x_2, \ldots, x_n$ with integer coefficients. Then the following statements are equivalent:*
(i) *$C(R)$ is a nil ideal for any ring $R$ satisfying $f = 0$.*
(ii) *$(GF(p))_2$ fails to satisfy $f = 0$ for every prime $p$.*

The main results of the paper are:

**Theorem 1.** *Let $R$ be a ring with 1 satisfying $(c_1)$ together with either $Q(s,t)$ or $Q(s)$. Then $R$ is commutative.*

**Theorem 2.** *Let $R$ be a ring with 1 satisfying $(c_2)$ together with either $Q(s,t)$ or $Q(s)$. Then $R$ is commutative.*

**Theorem 3.** *Let $R$ be a ring with 1 satisfying $(c_3)$ together with either $Q(s,t)$ or $Q(s)$. Then $R$ is commutative.*

**Theorem 4.** *Let $R$ be a ring with 1 satisfying $(c_4)$ such that either $Q(s,t)$ or $Q(s)$ holds. Then $R$ is commutative.*

*Remark* 1. The well-known Grassmann algebra rules out the possibility of $t = 1$ in the above theorems. Moreover, if we drop the restriction that $R$ has unity 1 in the above theorems, then the ring $R$ may be poorly non-commutative. Indeed, the following example demonstrates this constraint: Let $D_k$ be the ring of all $k \times k$ matrices over a division ring $D$, and let $A_k = \{(a_{ij} \in D_k | a_{ij} = 0, j \geq i\}$. Then $A_k$ is a non-commutative ring for any positive integer $k > 2$. But $A_3$ satisfies $(c_1)$, $(c_2)$, $(c_3)$, and $(c_4)$ for all positive integers $s$, $t$ and non-negative integers $m$, $n$, and $r$.

According to [9], let $f$ be a polynomial in two non-commuting indeterminates with integral coefficients. Now write $f$ in the form

$$f(x,y) = \sum_{r=1}^{d} \sum_{i=0}^{r} f_{ri}(x,y),$$

where $f_{ri}$ denotes the sum of all terms of $f$ with degree $i$ in $x$ and $r-i$ in $y$. Let $s_{ri}$ denote the sum of the coefficients of $f_{ti}$. Then we note that if

$$s_{ri} = 0 \quad \text{for all} \quad r \quad \text{and} \quad i, \tag{I}$$

then all commutative rings satisfy $f = 0$. The converse is also true as proved by Kezlan [9]. For this we take a transcendental extension field of rationals and use the fact that the polynomial

$$f(X^d, X^{d+1}) = \sum_{r=1}^{d} \sum_{i=0}^{r} s_{ri} X^{(d+1)r-i}$$

in one indeterminate $X$ vanishes on it.

Thus if $f$ is to be equivalent with the commutativity it must at least satisfy (I), and so we may write

$$f(x,y) = m[x,y] + \sum_{r=3}^{d} \sum_{i=1}^{r-1} f_{ri}(x,y)$$

for some integer $m$. Moreover, if $m$ is divisible by a prime $p$, then the ring of strictly upper triangular $3 \times 3$ matrices over any field of characteristic $p$ satisfies the identity, and so we assume that

$$m = \pm 1. \tag{II}$$

Let us consider the condition

$$f_{r1} = 0 \quad \text{for all} \quad r. \tag{III}$$

In [9] Kezlan proved the following

**Theorem.** *If $f$ satisfies* (I), (II), *and* (III), *then an arbitrary ring $R$ is commutative if and only if it satisfies the identity $f = 0$.*

Also, it should be remarked that (I) in the theorem could be replaced by

$$f_{r,r-1} = 0 \quad \text{for all} \quad r. \tag{IV}$$

An example was given in [9] to show that we must assume either (III) or (IV) or some other condition concerning the terms linear in $x$ or in $y$. So (I) and (II) alone are not enough.

Further, in [9] Kezlan proved that for a polynomial $f(x,y)$ the identity $f(x,y) = 0$ is equivalent with the commutativity for all rings if $f(x,y) =$

$\pm[x,y] + \sum_{r=3}^{d} \sum_{i=1}^{r-1} f_{ri}(x,y)$, where $f_{ri}$ denotes the sum of all terms of $f(x,y)$ with degree $i$ in $x$ and $r-i$ in $y$, where $s_{si}$ denotes the sum of coefficients of $f_{ri}$. It was also shown that under certain restrictions on the terms linear in one variable or the other, the polynomial identity $f(x,y) = 0$ is indeed equivalent with the commutativity.

The following fact plays an important role in the proof of our results.

**Lemma 4 ([9]).** *Let $R$ be a ring satisfying the polynomial identity $f(x,y) = \pm[x,y]$, where each homogeneous component of $f(x,y)$ has integer coefficients whose sum is zero and where $f(x,y)$ has no linear terms either in $x$ or in $y$. Then $R$ is commutative.*

Now we prove

**Proposition 1.** *Let $R$ be a ring with $p[x,y] = 0$ for all $x$, $y$ in $R$, $p$ a prime, and let $s$ be a positive integer not divisible by $p$. Suppose that $R$ satisfies a polynomial identity of the form $f(x,y) = \pm m[x,y]$, where $m$ is any non-negative integer and $f(x,y)$ satisfies the same condition as in Lemma 4. Then $R$ is commutative.*

*Proof.* Let $qp + mn = 1$. By hypothesis, we have

$$f(x,y) = \pm m[x,y] \quad \text{for all} \quad x, y \in R.$$

Multiply the above identity by $n$ and use $p[x,y] = 0$ to get $nf(x,y) = \pm[x,y]$. Hence by Lemma 4 $R$ is commutative. □

**Lemma 5.** *Let $R$ be a ring with 1 satisfying $(c_1)$ or $(c_2)$ or $(c_3)$ or $(c_4)$ such that either $Q(s,t)$ or $Q(s)$ holds. Then $C(R)$ is nil.*

*Proof.* Let $x = e_{22}$ and $y = e_{21} + e_{22}$ in $(c_1)$ and $(c_2)$. Then by Lemma 3, $x$ and $y$ fail to satisfy the polynomial identities $(c_3)$ and $(c_4)$ for any prime $p$. Similarly, $x = e_{11}$ and $y = e_{21}$ fail to satisfy $(c_3)$ and $(c_4)$. Thus $C(R)$ is a nil ideal. □

*Proof of Theorem* 1. According to Lemma 5, $C(R)$ is nil. Now let $R$ satisfy $(c_1)$. Then by contradiction we assume that there exists a non-commutative ring with 1 satisfying $(c_1)$. Another step is to pass to the subdirectly irreducible case, and with $Q(s,t)$ this reduction can be obtained as in [9]. Therefore, without loss of generality, we assume that there is a ring $R$ such that

($\alpha$) $R$ is a non-commutative ring with 1, satisfies $(c_1)$, and $R$ is subdirectly irreducible with heart $H = C(R)$ with $H^2 = (0)$.

Using the condition $Q(s)$, we must slightly modify the arguments in [8] because this condition is not preserved under homomorphism. Then sssuming that we have a non-commutative ring $A$ with unity satisfying $(c_1)$ and that $Q(s)$ holds, we have $a, b \in A$ with $s[a,b] \neq 0$. By Zorn's lemma we get an ideal $M$ which is maximal with respect to the exclusion of $s[a,b]$. Then the ring $\overline{A} = A/M$ is not commutative, satisfies $(c_1)$, and is subdirectly irreducible with the heart containing $\overline{s}[\overline{a}, \overline{b}]$. Hence the ring $R = \overline{A}$ may not inherit $Q(s)$. This shows that $s$ does not annihilate all commutators of $R$. To summarize these paragraphs, if $Q(s,t)$ holds, then we have a ring $R$ satisfying $(\alpha)$. If $Q(s)$ holds, then, in addition to $(\alpha)$, we have

$(\beta)$ $s$ does not annihilate all commutators of $R$.

Now we define a mapping $F : R \to R$ for fixed $y, w \in R$ by

$$F(x) = \pm[x, (y+w)^t - y^t - w^t] \quad \text{for all} \quad x \in R. \tag{1}$$

Replace $x$ by $x+1$ in (1) to get

$$F(x+1) = \pm[x, (y+w)^t - y^t - w^t] = F(x).$$

Multiplying (1) by $x^n$ on the right, we get

$$\begin{aligned} F(x)x^n &= \pm[x, (y+w)^t - y^t - w^t]x^n = \\ &= \pm[x, (y+w)^t]x^n \mp [x, y^t]x^n \mp [x, w^t]x^n = \\ &= x^r[x^s, y+w] - \{\pm[x, y^t]x^n\} - \{\pm[x, w^t]x^n\} = \\ &= x^r[x^s, y] + x^r[x^s, w] - x^r[x^s, y] - x^r[x^s, w] = 0. \end{aligned}$$

Using Lemma 2, we obtain $F(x) = 0$ for all $x$, $y$ in $R$. Hence

$$\begin{aligned} &\pm [x, (y+w)^t - y^t - w^t]x^n = 0, \\ &\pm [x, (y+w)^t] \mp [x, y^t] \mp [x, w^t] = 0 \end{aligned}$$

and

$$\pm[x, (y+w)^t] = \pm[x, y^t] \pm [x, w^t] = 0. \tag{2}$$

Substituting $w = 1$ in (2), we get

$$\pm\Big(t[x,y] + \sum_{k=2}^{t-1} \binom{t}{k} [x, y^k]\Big) = 0. \tag{3}$$

By (3) we can write

$$t[x,y] + \sum_{k=2}^{t-1} \binom{t}{k} [x, y^k] = 0. \tag{3'}$$

Replacing $y$ in turn by $z_1 y, z_2 y, \dots, z_{t-1} y$ in $(3')$, we get

$$\begin{array}{l}
z_1 t[x,y] + \binom{t}{2} z_1^2 [x,y^2] + \binom{t}{3} z_1^3 [x,y^3] + \cdots + \binom{t}{t-1} z_1^{t-1} [x, y^{t-1}] = 0, \\
z_2 t[x,y] + \binom{t}{2} z_2^2 [x,y^2] + \binom{t}{3} z_2^3 [x,y^3] + \cdots + \binom{t}{t-1} z_2^{t-1} [x, y^{t-1}] = 0, \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
z_{t-1} t[x,y] + \binom{t}{2} z_{t-1}^2 [x,y^2] + \binom{t}{3} z_{t-1}^3 [x,y^3] + \cdots + \binom{t}{t-1} z_{t-1}^{t-1} [x, y^{t-1}] = 0.
\end{array}$$

The above identities can be written in the matrix form:

$$A_{(t-1)\times(t-1)} W_{(t-1)\times 1} =$$
$$= \begin{pmatrix} z_1 & z_1^2 & z_1^3 & \cdots & z_1^{t-1} \\ z_2 & z_2^2 & z_2^3 & \cdots & z_2^{t-1} \\ z_3 & z_3^2 & z_3^3 & \cdots & z_3^{t-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ z_{t-1} & z_{t-1}^2 & z_{t-1}^3 & \cdots & z_{t-1}^{t-1} \end{pmatrix} \begin{pmatrix} t[x,y] \\ \binom{t}{2}[x,y^2] \\ \binom{t}{3}[x,y^3] \\ \vdots \\ \binom{t}{t-1}[x,y^{t-1}] \end{pmatrix} = 0.$$

Multiplying the above by $\operatorname{adj}(A)$, we have $\det(A)$. In particular, $\det(A) t[x,y] = 0$ for all $x, y \in R$.

Let $z_i^j$ be the $(i,j)$th entry in the matrix $A$. Since factoring $z_i$ out of the $i$th row of $A$ gives a Vandermonde matrix, $\langle R, + \rangle$ is torsion free. Therefore $[x,y] = 0$ for all $x, y \in R$, and $R$ is commutative. (In short, if $\langle R, + \rangle$ is torsion free, then the standard Vandermonde determinant argument shows that the homogeneous components must vanish on $R$ (see [9]). So $[x,y] = 0$, since $R$ is torsion free. Thus $R$ is commutative.)

As in [9], the subdirect irreducibility gives a unique prime $p$ such that $R$ has elements of additive order $p$, from which it follows that $pH = (0)$ (as a special case $p$ annihilates all commutators).

Let $p$ not divide $t$. Then (3) yields a polynomial identity of the type in Proposition 1 and hence $R$ is commutative. Thus $p$ must divide $t$. Therefore $p$ cannot divide $s$, which is obvious if $Q(s,t)$ holds, and is also true if $Q(s)$ holds, since $s$ does not annihilate all commutators as $p$ does. Hence in either of the cases $Q(s,t)$ and $Q(s)$ we get

$$p \quad \text{divides} \quad t \quad \text{but does not divide} \quad s. \tag{4}$$

By interchanging the roles of $x$ and $y$ in (2) we obtain

$$\pm[y, (x+w)^t] = \pm[y, x^t] \pm [y, w^t]. \tag{5}$$

Replace $x$ by $x+1$ in $(c_1)$ to get the identities

$$(x+1)^r [(x+1)^s, y] = \pm[x, y^t](x+1)^n$$

and

$$\sum_{j=0}^{r}\sum_{k=1}^{s}\binom{r}{j}\binom{s}{k}x^j[x^k,y] = \pm\sum_{j=1}^{n}\binom{n}{j}[x,y^t]x^s. \tag{6}$$

Now we prove that $H \subseteq Z(R)$. Let $x \in H$ and $y \in R$. Thus using $H^2 = (0)$, from (6) we have

$$s[x,y] = \pm[x,y^t] \quad \text{for all} \quad x \in H,\ \ y \in R. \tag{7}$$

Substitute $w = y$ into (5) to get

$$\pm[y, xy^{t-1} + yxy^{t-2} + \cdots + y^{t-1}x] = 0$$

and so

$$\pm[x,y^t] = 0 \quad \text{for all} \quad x \in H,\ \ y \in R. \tag{8}$$

From (7), (8) and the fact that $p$ does not divide $s$ we have $[x,y] = 0$ for $x \in H$, $y \in R$. Hence

$$H \subseteq Z(R). \tag{9}$$

Thus all commutators are central. By Lemma 5 we get

$$[x,y^t] = ty^{t-1}[x,y] = 0 \quad \text{for all} \quad x,y \in R. \tag{10}$$

This condition shows that $t$ divisible by $p$ annihilates all commutators. Hence (6) can be rewritten as

$$s[x,y] + \sum_{k=2}^{s}\binom{s}{k}[x^k,y] = \pm\sum_{j=1}^{r}\sum_{k=1}^{s}\binom{r}{j}\binom{s}{k}x^j[x^k,y]$$

which is the form of Proposition 1. Hence $R$ is commutative. □

*Proof of Theorem* 2. Let $R$ satisfy $(c_2)$. As above, it is easy to observe that the reduction to a subdirectly irreducible ring $R$ satisfying $(\alpha)$ with $Q(s,t)$ holds and so do both conditions $(\alpha)$ and $(\beta)$ with $Q(s)$. Replacing $x+1$ by $x$ and $y+1$ by $y$ in $c_2$ gives the following identities:

$$s[x,y] = \pm G(x,y),$$
$$t[x,y] = \pm H(x,y),$$

where $G$ and $H$ satisfy the conditions of Lemma 4. As in the proof of Theorem 1, we have a unique prime $p$ such that $pH = (0)$. Thus by Proposition 1, $p$ must divide both $t$ and $s$, which is impossible if $Q(s,t)$ holds. Otherwise, since $s$ is divisible by $p$, it must annihilate all commutators. Thus $(\beta)$ gives a contradiction if $Q(s)$ holds. Hence $R$ is commutative. □

Since Theorems 3 and 4 can be proved in the same way, we omit their proofs.

A recent commutativity study [10] deals with rings satisfying related conditions of the form

$$[xy - p(yx), x] = 0$$

or

$$[xy - p(yx), y] = 0.$$

This becomes possible by interchanging the roles of $x$ and $y$ as

$$[yx - p(xy), y] = 0$$

or

$$[yx - p(xy), x] = 0.$$

The following result is proved in [10].

**Theorem 5.** *Let $R$ be a ring with 1 such that for each $x, y \in R$, there exist $p(t)$, $q(t) \in t^2 Z[t]$ for which $[xy - p(yx), x] = 0$ and $[xy - q(yx), y] = 0$. Then $R$ is commutative.*

Now we generalize Theorem 5.

**Theorem 6.** *Let $R$ be a ring with unity 1 such that for each $x, y \in R$ there exists $p(x) \in x^2 Z[x]$ for which either $[yx - p(yx), x] = 0$ or $[yx - p(yx), y] = 0$. Then $R$ is commutative.*

*Proof.* Suppose that $p(x) = a_2x^2 + a_3x^3 + a_4x^4 + \cdots + a_nx^n$, where $a_1, a_2, ..., a_n$ are integers. By hypothesis, we can write for all $x, y \in R$

$$\begin{aligned} [yx, x] &= [p(yx), x], \\ [x, y]x &= [p(yx), x]. \end{aligned} \tag{1'}$$

Putting $x + 1$ for $x$ in $(1')$ gives

$$\begin{aligned} [x, y](x+1) &= [p(y(x+1)), x], \\ [x, y]x + [x, y] &= \big[a_2(y(x+1))^2 + a_3(y(x+1))^3 + \cdots + a_n(y(x+1))^n, x\big], \\ [x, y]x + [x, y] &= \big[a_2y(x+1)y + a_3y(x+1)y(x+1)y + \cdots + \\ &\quad + a_ny(x+1)y(x+1)\cdots y(x+1)y, x\big](x+1) = \\ &= \big[a_2(yxy + y^2) + a_3(yxyxy + yxy^2 + y^2xy + y^3) + \cdots + \\ &\quad + a_n(yxyx\cdots y + (x+1), x\big](x+1), \\ [x, y]x + [x, y] &= \big[a_2yxy + a_3yxyxy + \cdots + a_nyxyxyx\cdots yx, x\big] + \\ &\quad + \big[a_2y^2 + a_3y^3 + \cdots + a_ny^n, x\big] + H(x, y), \end{aligned}$$

where each homogeneous component of $G$ has the sum of coefficients which is equal to zero. Thus $H$ has no terms linear in $y$, and each term of $H$ has a degree greater than 1 in $x$. Hence

$$[x,y]x + [x,y] = [p(yx),x] + [p(y),x] + H(x,y). \tag{2'}$$

Using (1′) and (2′), we obtain

$$[x,y] = [x,p(y)] + G(x,y).$$

Thus $R$ is commutative by Kezlan's theorem [11] or Lemma 4. Similarly, with the help of Lemma 4, $R$ is commutative if $R$ satisfies $[yx - p(xy), y] = 0$. □

Similarly to the proof of Theorem 6, we can reprove the next theorem using Lemma 4.

**Theorem 7 ([12]).** *Let $R$ be a ring with unity 1 satisfying*

$$[xy - p(yx), y] = 0 \quad \textit{for all} \quad x,y \in R,$$

*where $p(x) \in x^2 Z[x]$. Then $R$ is commutative.*

Now let $P$ be a ring property. If $P$ is inherited by every subring and every homomorphic image, then $P$ is called an $h$-property. More weakly, if $P$ is inherited by every finitely generated subring and every natural homomorphic image modulo the annihilator of a central element, then $P$ is called an $H$-property.

A ring property $P$ such that a ring $R$ has the property $P$ if and only if all its finitely generated subrings have $P$ is called an $F$-property.

**Proposition 2 ([13, Proposition 1]).** *Let $P$ be an $H$-property, and $P'$ be an $F$-property. If every ring $R$ with unity 1 having the property $P$ has the property $P'$, then every $s$-unital ring having $P$ has $P'$.*

Finally, Theorems 1–4, 6, and 7 are automatically generalized from a unital ring to $s$-unital ones due to Proposition 2. Indeed, we have

**Theorem 8.** *Let $R$ be a left (resp., right) $s$-unital ring satisfying* $(c_1)$ *(resp.,* $(c_2)$*). Then $R$ is commutative.*

**Theorem 9.** *Let $R$ be a left $s$-unital (resp., right) ring satisfying* $(c_3)$ *(resp.,* $(c_4)$*). Then $R$ is commutative.*

## References

1. H. A. S. Abujabal and V. Perić, Commutativity of $s$-unital rings through a Streb result. *Radovi Mat.* **7**(1991), 73–92.

2. H. Komatsu, A commutativity theorem for rings. *Math. J. Okayama Univ.* **26**(1984), 109–111.

3. M. A. Quadri and M. A. Khan, A commutativity theorem for associative rings. *Math. Japonica* **33**(1988), 275–279.

4. H. G. Moore, Generalized $n$-like rings and commutativity. *Canad. Math. Bull.* **23**(1980), 449–452.

6. N. Jacobson, Structure of rings. *Amer. Math. Soc., Colloq. Publ.,* 1964.

7. W. K. Nicholson and A. Yaqub, Commutativity theorems for rings and groups. *Canad. Math. Bull.* **22**(1979), 419–423.

8. T. P. Kezlan, A note on commutativity of semi-prime PI rings. *Math. Japonica* **27**(1982), 267–268.

9. T. P. Kezlan, On identities which are equivalent with commutativity. *Math. Japonica* **29**(1984), 135–139.

10. H. E. Bell, M. A. Quadri, and M. Ashraf, Commutativity of rings with some commutator constraints. *Radovi Mat.* **5**(1989), 223–230.

11. T. P. Kezlan, On commutativity theorems for PI ring with unity. *Tamkang J. Math.* **24**(1993), 29–36.

12. H. E. Bell, M. A. Quadri, and M. A. Khan, Two commutativity theorems for rings. *Radovi Mat.* **3**(1987), 255–260.

13. Y. Hirano, Y. Kobayashi, and H. Tominaga, Some polynomial identities and commutativity of $s$-unital rings. *Math. J. Okayama Univ.* **24**(1982), 7–13.

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