

## ON STRONG DIFFERENTIABILITY OF INTEGRALS ALONG DIFFERENT DIRECTIONS

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ABSTRACT. Theorems are proved as regards strong differentiability of integrals in different directions.

### § 1. INTRODUCTION

The well-known negative result in the theory of strong differentiability of integrals reads: there exists a summable function whose integral is differentiated in a strong sense in none of the directions.

Below we shall prove the theorems which in particular imply: for each pair of directions  $\gamma_1$  and  $\gamma_2$  differing from each other there exists a non-negative summable function whose integral is strongly differentiated in the directions  $\gamma_1$  and is not strongly differentiated in the direction  $\gamma_2$ .

### § 2. DEFINITIONS AND FORMULATION OF THE PROBLEM

Let  $B(x)$  be a differentiation basis at the point  $x \in \mathbb{R}^n$ , i.e., a family of bounded measurable sets with positive measure containing  $x$  and such that there is at least a sequence  $\{B_k\} \subset B(x)$  with  $\text{diam}(B_k) \rightarrow 0$  as  $k \rightarrow \infty$ , (see, [1, Ch. II, Section 2]). A collection  $B = \{B(x) : x \in \mathbb{R}^n\}$  is called a differentiation basis in  $\mathbb{R}^n$ .

For  $f \in L_{loc}(\mathbb{R}^n)$  and  $x \in \mathbb{R}^n$  by  $M_B f(x)$ ,  $\overline{D}_B(\int f, x)$ , and  $\underline{D}_B(\int f, x)$  are denoted, respectively, the maximal Hardy–Littlewood function with respect to  $B$ , and the upper and the lower derivatives of the integral  $\int f$  with respect to  $B$  at  $x$ . If  $\overline{D}_B(\int f, x) = \underline{D}_B(\int f, x)$ , then this number will be denoted by  $D_B(\int f, x)$ ; the basis  $B$  will be said to differentiate  $\int f$  if the equality  $D_B(\int f, x) = f(x)$  holds for almost all  $x \in \mathbb{R}^n$ . If the differentiation basis differentiates the integrals of all functions from some class  $M$ , then it is said to differentiate  $M$  [1, Ch. II and Ch. III].

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1991 *Mathematics Subject Classification.* 28A15.

*Key words and phrases.* Strong differentiability of an integral, Zygmund problem on differentiation with respect to rectangles, field of directions, strong maximal function.

Let  $\gamma$  denote the set of  $n$  mutually orthogonal straight lines in  $\mathbb{R}^n$  which intersect at the origin. The union of such sets will be denoted by  $\Gamma(\mathbb{R}^n)$ . Elements of  $\Gamma(\mathbb{R}^n)$  will be called directions. If  $\gamma \in \Gamma(\mathbb{R}^n)$ , then  $B_2^\gamma$  will denote the differentiation basis consisting of all  $n$ -dimensional rectangles whose sides are parallel to straight lines from  $\gamma$ .

The standard direction in  $\mathbb{R}^n$  will be denoted by  $\gamma^0$  and, for simplicity, the basis  $B_2^{\gamma^0}$  will be written as  $B_2$ .

We shall denote by  $B_3$  the differentiation basis in  $\mathbb{R}^n$  ( $n \geq 2$ ) consisting of all  $n$ -dimensional rectangles and by  $B_1$  the differentiation basis in  $\mathbb{R}^n$  consisting of all  $n$ -dimensional cubic intervals whose sides are parallel to the coordinate axes.

The fact that  $B_2$  differentiates  $L(1 + \log^+ L)^{n-1}(\mathbb{R}^n)$  is well known [2], but in a class wider than  $L(1 + \log^+ L)^{n-1}(\mathbb{R}^n)$  there exists a function whose integral is not differentiated almost everywhere by the basis  $B_2$  ([3], [4]). On the other hand, the basis  $B_3$  with "freely" rotating constituent rectangles does not differentiate  $L^\infty(\mathbb{R}^n)$  [1, Ch. V, Section 2]. The dependence of differentiation properties on the orientation of the sides of rectangles leads to the problem proposed by A. Zygmund ([1, Ch. IV, Section 2]): given a function  $f \in L(\mathbb{R}^2)$ , is it possible to choose a direction of  $\gamma \in \Gamma(\mathbb{R}^2)$  such that  $B_2^\gamma$  would differentiate  $\int f$ ?

A negative answer to A. Zygmund's question was given by D. Marstrand [5] who constructed an example of an integrable function whose integral is not differentiated almost everywhere by the basis  $B_2^\gamma$  for any fixed direction  $\gamma$ . Some generalizations of this result were later given in [6] and [7]. Hence we face the question whether the following hypothesis holds: if  $f \in L(\mathbb{R}^2)$  and  $B_2$  does not differentiate  $\int f$ , then  $B_2^\gamma$  will not differentiate  $\int f$  either whatever  $\gamma$  is.

We shall show that there exists a function  $f \in L(\mathbb{R}^2)$  for which this hypothesis does not hold.

In connection with this we have to answer the following questions: what is a set of those directions from  $\Gamma(\mathbb{R}^2)$  that differentiate  $\int f$ ? What are optimal conditions for such functions being integrable?

### § 3. STATEMENT OF THE MAIN RESULTS

Note that  $\Gamma(\mathbb{R}^2)$  corresponds in a one-to-one manner to the interval  $[0, \frac{\pi}{2})$ . To each direction  $\gamma$  we put into correspondence a number  $\alpha(\gamma)$ ,  $0 \leq \alpha(\gamma) < \frac{\pi}{2}$ , which is defined as the angle between the positive direction of the axis  $ox$  and the straight direction from  $\gamma$  lying in the first quadrant of the plane. Elements of the set  $\Gamma(\mathbb{R}^2)$  will be identified with points from  $[0, \frac{\pi}{2})$ . By the neighborhood of the point 0 will be meant the union of intervals  $[0, \varepsilon) \cup (\frac{\pi}{2} - \varepsilon, \frac{\pi}{2})$ ,  $0 < \varepsilon < \frac{\pi}{4}$ .

Denote  $S = [0, 1]^2$ , and  $[T]$  the closure of a set  $T$ .

**Theorem 1.** Let  $\Phi(t)$  be a nondecreasing continuous function on the interval  $[0, \infty)$  and  $\Phi(t) = o(t \log^+ t)$  for  $t \rightarrow \infty$ . Then there exists a nonnegative summable function  $f \in \Phi(L)(S)$  such that

- (a)  $\overline{D}_{B_2}(\int f, x) = +\infty$  a.e. on  $S$ ;
- (b)  $D_{B_2^\gamma}(\int f, x) = f(x)$  a.e. on  $S$  for each  $\gamma$  from  $\Gamma(\mathbb{R}^2) \setminus \gamma^0$  and, moreover,

$$\sup_{\gamma: \varepsilon < \alpha(\gamma) < \frac{\pi}{2} - \varepsilon} \left\{ M_{B_2^\gamma} f(x) \right\} < \infty \text{ a.e. on } S$$

for each number  $\varepsilon$ ,  $0 < \varepsilon < \frac{\pi}{2}$ .

**Theorem 2.** Let  $\Phi(t)$  be a nondecreasing continuous function on the interval  $[0, \infty)$  and  $\Phi(t) = o(t \log^+ t)$  as  $t \rightarrow \infty$ . Let, moreover, a sequence  $(\gamma_n)_{n=1}^\infty \subset \Gamma(\mathbb{R}^2)$  be given. Then there exists a nonnegative summable function  $f \in \Phi(L)(S)$  such that

- (a) for every  $n = 1, 2, \dots$

$$\overline{D}_{B_2^{\gamma_n}}(\int f, x) = +\infty \text{ a.e. on } S; \tag{1}$$

- (b) for almost every  $\gamma \in \Gamma(\mathbb{R}^2)$

$$D_{B_2^\gamma}(\int f, x) = f(x) \text{ a.e. on } S \tag{2}$$

and, moreover, for every set  $T$  for which  $[T] \subset \Gamma(\mathbb{R}^2) \setminus (\gamma_n)_{n=1}^\infty$  a function  $f$  can be chosen such that in addition to (1) and (2) the following relation will be fulfilled:

$$\sup_{\gamma: \gamma \in T} \left\{ M_{B_2^\gamma} f(x) \right\} < \infty \text{ a.e. on } S.$$

§ 4. AUXILIARY STATEMENTS

To prove Theorem 1 we shall make use of the following two lemmas in which  $I = [0, l_1] \times [0, l_2]$ ;  $\chi_A$  and  $|A|$  will stand below for the characteristic function and Lebesgue measure of a set  $A$ , respectively.

**Lemma 1.** Let  $\gamma \in \Gamma(\mathbb{R}^2) \setminus \gamma^0$ . There exists a constant  $c(\gamma)$ ,  $1 < c(\gamma) < \infty$ , such that the inequality

$$\left| \left\{ x \in \mathbb{R}^2 : M_{B_2^\gamma}(\chi_I)(x) > \lambda \right\} \right| < 9c(\gamma)\lambda^{-1}|I|$$

holds for any  $\lambda$ ,  $0 < \lambda < 1$ , satisfying the condition

$$c(\gamma)\lambda^{-1}l_1 \leq l_2.$$

**Lemma 2.** The inequality

$$\left| \left\{ x \in \mathbb{R}^2 : M_{B_2}(\chi_I)(x) > \lambda \right\} \right| > \frac{1}{\lambda} \log \left( \frac{1}{\lambda} \right) |I|$$

holds for any number  $\lambda$ ,  $0 < \lambda < 1$ .

One can easily verify Lemma 2.

*Proof of Lemma 1.* Let  $x \notin I$ . It is easy to show that the maximal function  $M_{B_2^\gamma}(\chi_I)$  at the point  $x$  can be estimated from above as follows:

$$M_{B_2^\gamma}(\chi_I)(x) \leq c(\gamma) \frac{l_1}{\rho(x, I)}, \quad (3)$$

where

$$c(\gamma) = 2 \max \left\{ \frac{1}{\cos(\alpha(\gamma))}; \frac{1}{\sin(\alpha(\gamma))} \right\} \quad (4)$$

and  $\rho(x, I)$  denotes the distance between  $x$  and the interval  $I$ .

Hence it is obvious that

$$\begin{aligned} & \left\{ x \in \mathbb{R}^2 : M_{B_2^\gamma}(\chi_I)(x) > \lambda \right\} \subset \\ & \subset \left\{ x \in \mathbb{R}^2 : c(\gamma) \frac{l_1}{\rho(x, I)} > \lambda \right\} \subset Q(I, \lambda^{-1}, \gamma), \end{aligned} \quad (5)$$

where

$$Q(I, \lambda^{-1}, \gamma) = [-c(\gamma)\lambda^{-1}l_1, 2c(\gamma)\lambda^{-1}l_1] \times [-l_2, 2l_2]. \quad (6)$$

Clearly,

$$|Q(I, \lambda^{-1}, \gamma)| = 9c(\gamma)\lambda^{-1}|I|.$$

We eventually obtain

$$\left| \left\{ x \in \mathbb{R}^2 : M_{B_2^\gamma}(\chi_I)(x) > \lambda \right\} \right| \leq |Q(I, \lambda^{-1}, \gamma)| = 9c(\gamma)\lambda^{-1}|I|. \quad \square$$

### § 5. PROOF OF THE MAIN RESULTS

*Proof of Theorem 1.* Let  $(c_n)_{n=1}^\infty$  be an increasing sequence of natural numbers and the constants  $c(\gamma)$  be defined by equality (4). Obviously,  $1 < c < \infty$ , where

$$c = \sup \left\{ c(\gamma) : \gamma \in \Gamma(\mathbb{R}^2), \varepsilon < \alpha(\gamma) < \frac{\pi}{2} - \varepsilon \right\}.$$

Let us construct a sequence of natural numbers  $(\beta_n)_{n=1}^\infty$  so as to satisfy the conditions

$$\beta_n \log(\beta_n) \geq \max \{ c_n \beta_n 2^{2n}; 2^n \Phi(\beta_n) \}. \quad (7)$$

The intervals  $I^n = [0, l_1^n] \times [0, l_2^n]$  for  $n = 1, 2, \dots$  will be constructed so as to satisfy the equality

$$c_n \beta_n 2^n l_1^n = l_2^n. \quad (8)$$

In what follows, for  $g \in L(\mathbb{R}^2)$ ,  $0 < \lambda < \infty$ ,  $\gamma \in \Gamma(\mathbb{R}^2) \setminus \gamma^0$ , we shall use the following notation:

$$H_2(g, \lambda) = \{x \in \mathbb{R}^2 : M_{B_2}g(x) \geq \lambda\},$$

$$H_2^\gamma(g, \lambda) = \{x \in \mathbb{R}^2 : M_{B_2^\gamma}g(x) > \lambda\}.$$

Using (7) and Lemma 2 we obtain

$$|H_2(\beta_n \chi_{I^n}, 1)| = |H_2(\chi_{I^n}, \beta_n^{-1})| > \beta_n \log(\beta_n) |I^n| > 2^n (c_n \beta_n 2^n |I^n|). \tag{9}$$

Consider the interval

$$Q^n = [-c_n \beta_n 2^n l_1^n, c_n \beta_n 2^{n+1} l_1^n] \times [-l_2^n, 2l_2^n].$$

Note that if  $\gamma$  satisfies the condition  $c(\gamma) < c_n$ , then (see (5), (6), (8)) then we have the inclusions

$$H_2^\gamma(\beta_n \chi_{I^n}, 2^{-n}) \subset Q(I^n, \beta_n 2^n, \gamma) \subset Q^n. \tag{10}$$

Since  $Q^n$  is the cubic interval (see (8)), it is easy to ascertain that for each direction  $\gamma$  there exists an interval  $E^{n,\gamma}$  such that  $E^{n,\gamma} \in B_2^\gamma$  and the conditions

$$Q^n \subset E^{n,\gamma} \subset 2Q^n \tag{11}$$

are fulfilled (see Figure 1).

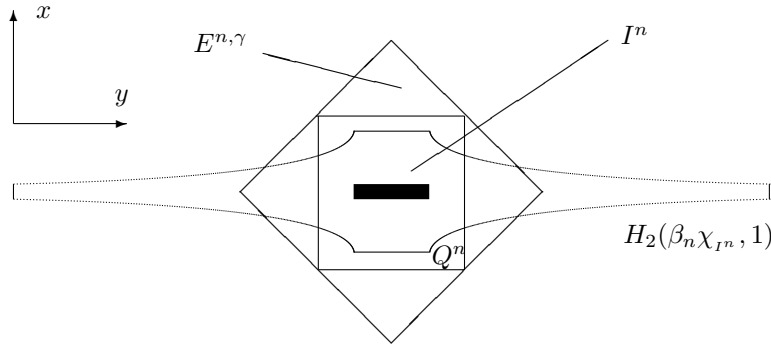


Figure 1

We have

$$|2Q^n| \leq 36c_n \beta_n 2^n |I^n|. \tag{12}$$

From (9) and (12) it follows that

$$|H_2(\beta_n \chi_{I^n}, 1)| > \frac{2^n}{36} (36c_n \beta_n 2^n |I^n|) \geq \frac{2^n}{36} |2Q^n|. \tag{13}$$

For each  $n \in \mathbb{N}$  consider the set

$$H_2^n \equiv H_2(\beta_n \chi_{I^n}, 1) \cup 2Q^n.$$

Since the set  $H_2^n$  is compact, almost the whole interval  $S$  can be represented as the union of nonintersecting sets that are homothetic to the set  $H_2^n$  and have a diameter not exceeding  $n^{-1}$  (see Lemma 1.3 from [1, Ch. III, Section 1]). Assuming that  $H_2^{n,k}$  ( $k = 1, 2, \dots$ ) are such sets, we obtain

$$\text{diam}(H_2^{n,k}) \leq n^{-1}, \quad (14)$$

$$\left| S \setminus \bigcup_{k=1}^{\infty} H_2^{n,k} \right| = 0. \quad (15)$$

Let, moreover,  $P^{n,k}$  denote a homothety transforming the set  $H_2^n$  to  $H_2^{n,k}$ . The images of the sets  $I^n$ ,  $Q^n$ ,  $E^{n,\gamma}$  for the homothety  $P^{n,k}$  will be denoted by  $I^{n,k}$ ,  $Q^{n,k}$  and  $E^{n,\gamma,k}$ .

Using one of the homothetic properties, from (13) we obtain

$$\left| \bigcup_{k=1}^{\infty} 2Q^{n,k} \right| \leq \frac{36}{2^n} \left| \bigcup_{k=1}^{\infty} H_2^{n,k} \right| = \frac{36}{2^n}.$$

Therefore

$$\sum_{n=1}^{\infty} \left| \bigcup_{k=1}^{\infty} 2Q^{n,k} \right| < \infty, \quad (16)$$

which implies

$$\left| \limsup_{n \rightarrow \infty} \bigcup_{k=1}^{\infty} 2Q^{n,k} \right| = 0. \quad (17)$$

The function  $f_n$  on  $S$  is defined by

$$f_n(x) = \sum_{k=1}^{\infty} \beta_n \chi_{I^{n,k}}(x), \quad n = 1, 2, \dots,$$

and  $f$  by

$$f(x) = \sup_{n \in \mathbb{N}} f_n(x).$$

Let us show that  $f \in \Phi(L)(S)$ . We have (see (15))

$$1 = |S| \geq \left| \bigcup_{k=1}^{\infty} H_2(\beta_n \chi_{I^{n,k}}, 1) \right| > \sum_{k=1}^{\infty} \beta_n \log(\beta_n) |I^{n,k}|,$$

which by virtue of (7) gives us

$$1 \geq 2^n \sum_{k=1}^{\infty} \Phi(\beta_n) |I^{n,k}|.$$

Therefore

$$\int_S \Phi(f_n) = \sum_{k=1}^{\infty} \Phi(\beta_n) |I^{n,k}| \leq 2^{-n}, \quad n = 1, 2, \dots$$

Since  $\Phi(t)$  is a continuous nondecreasing function, we obtain

$$\int_S \Phi(f) \leq \sum_{n=1}^{\infty} \int_S \Phi(f_n) \leq \sum_{n=1}^{\infty} 2^{-n} < \infty.$$

To prove the theorem we have first to show that

$$\overline{D}_{B_2}(f, x) = +\infty \text{ a.e. on } S.$$

We have (see (15))

$$\begin{aligned} 1 = |S| &= \left| \limsup_{n \rightarrow \infty} \bigcup_{k=1}^{\infty} H_2^{n,k} \right| = \left| \limsup_{n \rightarrow \infty} \bigcup_{k=1}^{\infty} (H_2(\beta_n \chi_{I^{n,k}}, 1) \cup 2Q^{n,k}) \right| \leq \\ &\leq \left| \limsup_{n \rightarrow \infty} \bigcup_{k=1}^{\infty} (H_2(\beta_n \chi_{I^{n,k}}, 1)) \right| + \left| \limsup_{n \rightarrow \infty} \bigcup_{k=1}^{\infty} 2Q^{n,k} \right|. \end{aligned}$$

Hence by virtue of (17) we conclude that

$$\left| \limsup_{n \rightarrow \infty} \bigcup_{k=1}^{\infty} (H_2(\beta_n \chi_{I^{n,k}}, 1)) \right| = 1 \tag{18}$$

which clearly implies that for almost all  $x \in S$  there exists a sequence  $(n_i, k_i)_{i=1}^{\infty}$  (depending on  $x$ ) such that

$$x \in H_2(\beta_{n_i} \chi_{I^{n_i, k_i}}, 1), \quad i = 1, 2, \dots$$

Note further that

$$\text{diam} (H_2(\beta_{n_i} \chi_{I^{n_i, k_i}}, 1)) < n_i^{-1}, \quad i = 1, 2, \dots$$

From the inclusion  $x \in H_2(\beta_{n_i} \chi_{I^{n_i, k_i}}, 1)$  and construction of sets  $H_2(\beta_n \chi_{I^{n,k}}, 1)$  it follows that there exists an interval  $R_i \in B_2(x)$  contained in the set  $H_2(\beta_i \chi_{I^{n_i, k_i}}, 1)$  and

$$\frac{1}{|R_i|} \int_{R_i} \chi_{I^{n_i, k_i}}(y) dy \geq \beta_{n_i}^{-1}, \quad i = 1, 2, \dots \tag{19}$$

We define  $f^n$  as

$$f^n(x) = \sup_{m \geq n} f_m(x).$$

By (19) the relations

$$\overline{D}_{B_2}(f^n, x) \geq \lim_{i \rightarrow \infty} \frac{1}{|R_i|} \int_{R_i} f_{n_i}(y) dy \geq \lim_{i \rightarrow \infty} \frac{\beta_{n_i}}{|R_i|} \int_{R_i} \chi_{I^{n_i, k_i}}(y) dy \geq 1$$

a.e. on  $S$ .

We have

$$|\text{supp}(f^n)| \leq \sum_{m=n}^{\infty} |\text{supp}(f_m)| \leq \sum_{m=n}^{\infty} \left| \bigcup_{k=1}^{\infty} 2Q^{m,k} \right|.$$

Hence by virtue of (16) we find that for each number  $\varepsilon$ ,  $0 < \varepsilon < 1$ , there exists a number  $n = n(\varepsilon)$  such that  $|\text{supp}(f^n)| < \varepsilon$ .

We have

$$\begin{aligned} & |\{x \in S : \overline{D}_{B_2}(ff^n, x) > f^n(x)\}| \geq \\ & \geq |\{x \in S : \overline{D}_{B_2}(ff^n, x) \geq 1, x \notin \text{supp}(f^n)\}| = \\ & = |\{x \in S : x \notin \text{supp}(f^n)\}| > 1 - \varepsilon \end{aligned}$$

which by the Besikovitch theorem (see [1, Ch. IV, Section 3]) implies

$$|\{x \in S : \overline{D}_{B_2}(ff^n, x) = +\infty\}| > 1 - \varepsilon.$$

Since  $\varepsilon$  is arbitrary and  $f(x) \geq f^n(x)$ , we have

$$|\{x \in S : \overline{D}_{B_2}(ff^n, x) = +\infty\}| = 1$$

thereby proving assertion (a) of Theorem 1.

Let us now show that if  $\gamma \in \Gamma(\mathbb{R}^2) \setminus \gamma^0$  then

$$D_{B_2^\gamma}(ff, x) = f(x) \text{ a.e. on } S.$$

By virtue of the above-mentioned Besikovitch theorem it is sufficient to prove that

$$M_{B_2^\gamma} f(x) < \infty$$

at almost every point  $x \in S$ .

Fix  $\gamma \neq \gamma^0$ . Note that there exists a number  $n(\gamma)$  such that  $c(\gamma) < c_n$  for  $n > n(\gamma)$ . The inclusions (10) and (11) imply that the following inclusions are valid: if  $n > n(\gamma)$  then

$$H_2^\gamma(\beta_n \chi_{I^{n,k}}, 2^{-n}) \subset Q(I^{n,k}, \beta_n 2^n, \gamma) \subset Q^{n,k} \subset E^{n,\gamma,k} \subset 2Q^{n,k} \quad (20)$$

for all  $k = 1, 2, \dots$

By (17) we find that for almost all  $x \in S$  there exists a finite set  $p(x) \subset \mathbb{N}$  with the properties

$$x \in \bigcup_{k=1}^{\infty} 2Q^{n,k} \quad \text{for } n \in p(x)$$

and

$$x \notin \bigcup_{k=1}^{\infty} 2Q^{n,k} \quad \text{for } n \notin p(x).$$



We obtain

$$M_{B_2^\gamma} f(x) \leq \sum_{n=1}^{n(\gamma)} M_{B_2^\gamma} f_n(x) + \sum_{n \in p(x), n > n(\gamma)} M_{B_2^\gamma} f_n(x) + \sum_{n \notin p(x), n > n(\gamma)} M_{B_2^\gamma} f_n(x) = I_1(x, \gamma) + I_2(x, \gamma) + I_3(x, \gamma).$$

Let us estimate from above the values  $I_k(x, \gamma)$ ,  $k = 1, 2, 3$ . Note that

$$I_1(x, \gamma) \leq \sum_{n=1}^{n(\gamma)} \beta_n < \infty$$

for all  $x \in S$ .

Similarly,

$$I_2(x, \gamma) \leq \text{card}(p(x)) \max_{n \in p(x)} \{\beta_n\} < \infty \text{ a.e. on } S.$$

We shall now prove that

$$M_{B_2^\gamma} f_n(x) \leq 2^{-n}, \quad n \notin p(x), \quad n > n(\gamma).$$

Assume that  $R^\gamma \in B_2^\gamma(x)$  and let  $\{k_1^n, \dots, k_j^n, \dots\}$  denote a set of natural numbers for which

$$|R^\gamma \cap I^{n, k_j^n}| > 0, \quad j = 1, 2, \dots.$$

Note that  $R^\gamma \cap E^{n, \gamma, k_j^n} \in B_2^\gamma$ , and if  $n \notin p(x)$ ,  $n > n(\gamma)$ , then  $x \notin E^{n, \gamma, k_j^n}$ . Also taking into account the fact that the set  $R^\gamma \cap E^{n, \gamma, k_j^n}$  contains at least one point from  $(E^{n, \gamma, k_j^n})^c$ , we obtain (see (20))

$$|R^\gamma \cap I^{n, k_j^n}| < \beta_n^{-1} 2^{-n} |R^\gamma \cap E^{n, \gamma, k_j^n}|, \quad j = 1, 2, \dots,$$

for  $n > n(\gamma)$ ,  $n \notin p(x)$ .

Hence it follows that if  $n > n(\gamma)$ ,  $n \notin p(x)$  then

$$\begin{aligned} \frac{1}{|R^\gamma|} \int_{R^\gamma} f_n(y) dy &= \frac{\beta_n}{|R^\gamma|} \sum_{j=1}^{\infty} |R^\gamma \cap I^{n, k_j^n}| \leq \\ &\leq \frac{\beta_n}{|R^\gamma|} \sum_{j=1}^{\infty} \beta_n^{-1} 2^{-n} |R^\gamma \cap E^{n, \gamma, k_j^n}|. \end{aligned} \tag{21}$$

Since the sets  $E^{n, \gamma, k_j^n}$ ,  $j = 1, 2, \dots$ , do not intersect for  $n > n(\gamma)$  (see (20)), we have

$$|R^\gamma| \geq \left| \bigcup_{j=1}^{\infty} (R^\gamma \cap E^{n, \gamma, k_j^n}) \right| = \sum_{j=1}^{\infty} |R^\gamma \cap E^{n, \gamma, k_j^n}|. \tag{22}$$

(21)–(22) imply that if  $n \notin p(x)$ ,  $n > n(\gamma)$  then

$$M_{B_2^\gamma} f_n(x) = \sup_{R^\gamma \in B_2^\gamma(x)} \frac{1}{|R^\gamma|} \int_{R^\gamma} f_n(y) dy \leq 2^{-n}.$$

Therefore

$$I_3(x, \gamma) = \sum_{n \notin p(x), n > n(\gamma)} M_{B_2^\gamma} f_n(x) \leq \sum_{n=1}^{\infty} 2^{-n} < \infty.$$

Finally,

$$M_{B_2^\gamma} f(x) < \infty \quad (23)$$

for almost all  $x \in S$ , which proves the first part of assertion (b) of Theorem 1.

Let us prove the second part. There exists a number  $n(\varepsilon)$  for which

$$c < c_n \quad \text{for } n > n(\varepsilon). \quad (24)$$

We can write

$$\begin{aligned} M_{B_2^\gamma} f(x) &\leq \sum_{n=1}^{n(\varepsilon)} M_{B_2^\gamma} f_n(x) + \sum_{n \in p(x), n > n(\varepsilon)} M_{B_2^\gamma} f_n(x) + \\ &+ \sum_{n \notin p(x), n > n(\varepsilon)} M_{B_2^\gamma} f_n(x) = I_1(x) + I_2(x) + I_3(x). \end{aligned}$$

We have

$$I_1(x) \leq \sum_{n=1}^{n(\varepsilon)} \beta_n < \infty$$

and

$$I_2(x) \leq \text{card}(p(x)) \max_{n \in p(x)} \{\beta_n\} < \infty \quad \text{a.e. on } S.$$

The relations (10), (11), (24) imply that if  $\varepsilon < \alpha(\gamma) < \frac{\pi}{2} - \varepsilon$ , then the inclusions

$$H_2^\gamma(\beta_n \chi_{I_n^k}, 2^{-n}) \subset Q(I_n^k, \beta_n^{-1} 2^{-n}, \gamma) \subset Q^{n,k} \subset E^{n,\gamma,k} \subset 2Q^{n,k} \quad (25)$$

hold for  $n > n(\varepsilon)$ ,  $k = 1, 2, \dots$

From (25) it follows that relations (21) and (22) are valid for any direction  $\gamma$  for which  $\varepsilon < \alpha(\gamma) < \frac{\pi}{2} - \varepsilon$  when  $n \notin p(x)$ ,  $n > n(\varepsilon)$ . This means that for such directions we have

$$M_{B_2^\gamma} f_n(x) \leq 2^{-n} \quad \text{for } n \notin p(x), n > n(\varepsilon),$$

and thus

$$M_{B_2^\gamma} f(x) \leq \sum_{n=1}^{n(\varepsilon)} \beta_n + \text{card}(p(x)) \max_{n \in p(x)} \{\beta_n\} + 1 < \infty \quad \text{a.e. on } S.$$

Since the right-hand side does not depend on  $\gamma$ , we have

$$\sup_{\gamma: \varepsilon < \alpha(\gamma) < \frac{\pi}{2} - \varepsilon} \{M_{M_2^\gamma} f(x)\} < \infty \quad \text{a.e. on } S. \quad \square$$

*Remark.* The first part of assertion (b) of Theorem 1 can be formulated as follows (see (23)):

$$M_{B_2^\gamma} f(x) < \infty \quad \text{a.e. on } S$$

for each direction  $\gamma \in \Gamma(\mathbb{R}^2) \setminus \gamma^0$ .

*Proof of Theorem 2.* Since  $\gamma_n \notin \bar{T}$ , the inclusion

$$K_n \equiv \{\gamma \in \Gamma(\mathbb{R}^2) : |\alpha(\gamma) - \alpha(\gamma_n)| > \pi \varepsilon_n^{-1}\} \supset T \tag{26}$$

holds for a sufficiently large number  $\varepsilon_n$ .

Note that if  $\gamma_n = \gamma^0$  for some  $n$ , the set  $K_n$  will have the form

$$K_n \equiv \{\gamma \in \Gamma(\mathbb{R}^2) : \pi \varepsilon_n^{-1} < \alpha(\gamma) < \frac{\pi}{2} - \pi \varepsilon_n^{-1}\}.$$

Let, moreover,

$$\sum_{n=1}^{\infty} \varepsilon_n^{-1} < \infty. \tag{27}$$

Note that by virtue of the above remark Theorem 1 implies that for any number  $n \in \mathbb{N}$  there exists a function  $f_n \in \Phi(L)(S)$ ,  $f \geq 0$ ,  $\|\Phi(f_n)\|_{n(S)} < 2^{-n}$  such that the following three conditions hold:

$$\begin{cases} \overline{D}_{B_2^{\gamma_n}}(f_n, x) = +\infty \quad \text{a.e. on } S, \\ M_{B_2^\gamma} f_n(x) < \infty, \quad \forall \gamma \in \Gamma(\mathbb{R}^2) \setminus \gamma_n \quad \text{a.e. on } S, \\ \sup_{\gamma: |\alpha(\gamma) - \alpha(\gamma_n)| > \pi \varepsilon_n^{-1}} \{M_{B_2^\gamma} f_n(x)\} \leq F_n(x), \end{cases}$$

where the function  $F_n$  is finite a.e. There exist a number  $P_n$  and a set  $E_n \subset S$  such that  $1 < P_n < \infty$ ,  $|S \setminus E_n| < 2^{-n}$ , and

$$F_n(x) \leq P_n \quad \text{for } x \in E_n, \quad n = 1, 2, \dots$$

Let

$$g_n(x) = \frac{1}{P_n 2^n} f_n(x).$$

Clearly, the following conditions hold for  $g_n$ :

$$\begin{cases} \overline{D}_{B_2^{\gamma_n}}(\int g_n, x) = \frac{1}{P_n 2^n} \overline{D}_{B_2^{\gamma_n}}(\int f, x) = +\infty \text{ a.e. on } S, \\ M_{B_2^\gamma} g_n(x) < \infty, \quad \forall \gamma \in \Gamma(\mathbb{R}^2) \setminus \gamma_n \text{ a.e. on } S, \\ \sup_{\gamma: \gamma \in K_n \supset T} \{M_{B_2^\gamma} g_n(x)\} \leq \frac{1}{2^n} \text{ on } E_n. \end{cases}$$

We defined the function  $g$  as

$$g(x) = \sup_n g_n(x).$$

Note that  $g \in \Phi(L)(S)$ . Indeed, since  $\Phi(t)$  is the nondecreasing continuous function,

$$\int_S \Phi(g) \leq \sum_{n=1}^{\infty} \int_S \Phi(g_n) \leq \sum_{n=1}^{\infty} \int_S \Phi(f_n) < \sum_{n=1}^{\infty} 2^{-n} < \infty.$$

We have

$$\overline{D}_{B_2^{\gamma_n}}(\int g, x) \geq \overline{D}_{B_2^{\gamma_n}}(\int g_n, x) = +\infty \text{ a.e. on } S.$$

Relations (26) and (27) imply that

$$\sum_{n=1}^{\infty} |K_n^c| \leq 2\pi \sum_{n=1}^{\infty} \varepsilon_n^{-1} < \infty.$$

Therefore

$$\left| \limsup_{n \rightarrow \infty} K_n^c \right| = 0. \quad (28)$$

Similarly, since

$$\sum_{n=1}^{\infty} |E_n^c| < \sum_{n=1}^{\infty} 2^{-n} < \infty,$$

we have

$$\left| \limsup_{n \rightarrow \infty} E_n^c \right| = 0.$$

Define the sets

$$Z_n \equiv \{x \in S : F_n(x) = +\infty\}, \quad n = 1, 2, \dots,$$

$$K \equiv \liminf_{n \rightarrow \infty} K_n,$$

$$E \equiv \liminf_{n \rightarrow \infty} E_n \setminus \bigcup_{n=1}^{\infty} Z_n.$$

For  $\gamma \in K \setminus (\gamma_n)_{n=1}^{\infty}$  we set

$$G(\gamma) \equiv \bigcap_{n=1}^{\infty} \{x \in S : M_{B_2^\gamma} g_n(x) < \infty\}.$$

Clearly,

$$|E| = |G(\gamma)| = 1, \quad |K| = \frac{\pi}{2}.$$

Let  $\gamma \in K$ . It follows from (28) that there exists a finite set  $t(\gamma) \subset \mathbb{N}$  for which

$$\gamma \in K_n^c \quad \text{for } n \in t(\gamma)$$

and

$$\gamma \in K_n \quad \text{for } n \notin t(\gamma).$$

Similarly, if  $x \in E$  then there exists a finite set  $p(x) \subset \mathbb{N}$  for which

$$x \in E_n^c \quad \text{for } n \in p(x),$$

$$x \in E_n \quad \text{for } n \notin p(x).$$

Let us show that

$$M_{B_2^\gamma} g(x) < \infty$$

if  $x \in G(\gamma) \cap E$  and  $\gamma \in K \setminus (\gamma_n)_{n=1}^\infty$ .

We can write

$$M_{B_2^\gamma} g(x) \leq \sum_{n \in p(x) \cup t(\gamma)} M_{B_2^\gamma} g_n(x) + \sum_{n \notin p(x), n \notin t(\gamma)} M_{B_2^\gamma} g_n(x) = I_1(x, \gamma) + I_2(x, \gamma).$$

For  $x \in G(\gamma) \cap E$  we have

$$I_1(x, \gamma) \leq \text{card}(p(x) \cup t(\gamma)) \max_{n \in p(x) \cup t(\gamma)} \{M_{B_2^\gamma} g_n(x)\} < \infty.$$

On the other hand,

$$I_2(x, \gamma) = \sum_{n: x \in E_n, \gamma \in K_n} M_{B_2^\gamma} g_n(x) \leq \sum_{n=1}^\infty 2^{-n} < \infty.$$

Therefore for  $\gamma \in K \setminus (\gamma_n)_{n=1}^\infty$  we obtain  $(|\sigma(\gamma) \cap E| = 1)$

$$M_{B_2^\gamma} g(x) < \infty \quad \text{a.e. on } S,$$

which by the Besikovitch theorem implies

$$D_{B_2^\gamma}(\int g, x) = g(x) \quad \text{a.e. on } S.$$

Now let us prove the second part of condition (b) of Theorem 2. Since  $K_n \supset T$ , we have  $K_n^c \cap T = \emptyset$ ,  $n = 1, 2, \dots$ , and therefore

$$T \cap K_n^c = \emptyset, \quad n \in \mathbb{N}.$$

Thus if  $\gamma \in T$ , then  $t(\gamma) = \emptyset$ . We have

$$M_{B_2^\gamma} g(x) \leq \sum_{n \in p(x)} M_{B_2^\gamma} g_n(x) + \sum_{n \notin p(x)} M_{B_2^\gamma} g_n(x) = I_1(x) + I_2(x).$$

Note that if  $\gamma \in T$ ,  $x \in E$ , then

$$I_1(x) \leq \text{card}(p(x)) \max_{n \in p(x)} \{F_n(x)\} < \infty$$

and

$$I_2(x) \leq \sum_{n: x \in E_n} F_n(x) \leq \sum_{n=1}^{\infty} 2^{-n} < 1.$$

Therefore if  $\gamma \in T$  then we obtain ( $|E| = 1$ )

$$M_{B_2^\gamma} g(x) \leq \text{card}(p(x)) \max_{n \in p(x)} \{F_n(x)\} + 1 \text{ a.e. on } S,$$

and since the right-hand side of this inequality is independent of the direction  $\gamma$ , we have

$$\sup_{\gamma: \gamma \in T} \{M_{B_2^\gamma} g(x)\} < \infty \text{ a.e. on } S. \quad \square$$

## § 6. COROLLARIES

Theorem 2 implies

**Corollary 1.** *There exists a nonnegative function  $f \in L(S)$  such that the following conditions are fulfilled:*

(a) *if there is a direction  $\gamma$  such that  $\alpha(\gamma)$  is a rational number, then*

$$\overline{D}_{B_2^\gamma}(\int f, x) = +\infty \text{ a.e. on } S;$$

(b) *for almost all directions  $\gamma$*

$$D_{B_2^\gamma}(\int f, x) = f(x) \text{ a.e. on } S.$$

**Definition.** Assume that we are given a sequence of directions  $(\gamma_n)_{n=1}^{\infty}$  and let  $\gamma_n \nearrow \gamma$ ,  $n \nearrow \infty$ . Following [8], we shall say that the sequence of directions is exponential if there exists a constant  $c > 0$  such that

$$|\alpha(\gamma_i) - \alpha(\gamma_j)| > c|\alpha(\gamma_i) - \alpha(\gamma)|, \quad i \neq j.$$

**Corollary 2.** *Assume that we are given two sequences of directions  $(\gamma_n)_{n=1}^{\infty}$  and  $(\gamma'_n)_{n=1}^{\infty}$ , the sequence  $(\gamma_n)_{n=1}^{\infty}$  being exponential, and let*

$$\gamma'_m \in \Gamma(\mathbb{R}^2) \setminus \overline{(\gamma_n)_{n=1}^{\infty}}, \quad m = 1, 2, \dots$$

*There exists  $f \in L(S)$ ,  $f \geq 0$ , such that*

$$\overline{D}_{B_2^{\gamma'_n}}(\int f, x) = +\infty \text{ a.e. on } S, \quad n = 1, 2, \dots,$$

*and*

$$D_B(\int f, x) = f(x) \text{ a.e. on } S,$$

where the differentiation basis  $B$  at the point  $x$  is defined as follows:

$$B(x) = \cup_n B_2^{\gamma_n}(x).$$

*Proof.* By virtue of Theorem 2 from [8] we find that the basis  $B$  with the exponential property differentiates the space  $L^p(\mathbb{R}^2)$ ,  $p > 2$ . Therefore the basis with this property has the property of density. Using this fact and the fact that

$$M_B f(x) < \infty \quad \text{a.e. on } S$$

(see Theorem 2,  $T = (\gamma_n)_{n=1}^\infty$ ), by virtue of the de Guzmán and Menárguez theorem (see [1, Ch. IV, Section 3]), we obtain

$$D_B(\int f, x) = f(x) \quad \text{a.e. on } S. \quad \square$$

§ 7. REMARKS

1. A set of functions described by Theorems 1 and 2 forms a first-category set in  $L(\mathbb{R}^2)$  (see Saks' theorem ([3]; [1, Ch. VII, Section 2])).

2. Let  $\gamma_1, \gamma_2 \in \Gamma(\mathbb{R}^m)$ ,  $m \geq 3$ . Denote by  $\alpha_k(\gamma_1, \gamma_2)$ ,  $k = 1, 2, \dots, m$ , the angle formed by the  $k$ th straight line of the direction  $\gamma_1$  and by the  $k$ th straight line of the direction  $\gamma_2$ . If  $\gamma \in \Gamma(\mathbb{R}^m)$  then we denote by  $\bar{\gamma}$  the following subset from  $\Gamma(\mathbb{R}^m)$ :

$$\bar{\gamma} \equiv \{ \gamma' \in \Gamma(\mathbb{R}^m) : \exists K (1 \leq k \leq m), \exists j (1 \leq j \leq 4), \alpha_k(\gamma, \gamma') = \frac{\pi}{2}(j-1) \}.$$

Without changing the essence of the proof of the main results, we can prove, for example,

**Theorem 3.** *Let  $\Phi(t)$  be a nondecreasing continuous function on the interval  $[0, \infty)$  and  $\Phi(t) = o(t(\log^+ t)^{m-1})$  for  $t \rightarrow \infty$  ( $m \geq 3$ ). For each pair of directions  $\gamma_1$  and  $\gamma_2$  for which  $\gamma_2 \notin \bar{\gamma}_1$ , there exists a nonnegative summable function  $f \in \Phi(L)([0, 1]^m)$  such that*

- (a)  $\bar{D}_{B_2^{\gamma_1}}(\int f, x) = +\infty$  a.e. on  $[0, 1]^m$ ;
- (b)  $D_{B_2^{\gamma_2}}(\int f, x) = f(x)$  a.e. on  $[0, 1]^m$ .

3. We have ascertained that for one class of functions the so-called basis rotation changes the strong differentiability property of integrals. Note that there exist functions such that the basis rotation changes the integrability property of a strong maximal function. More exactly, for any number  $\varepsilon$ ,  $0 < \varepsilon < \frac{\pi}{2}$ , there exists a function  $f \in L(U)$ ,  $U = [-1, 1]^2$ , such that

$$(a) \quad \int_{\{M_{B_2} f > 1\}} M_{B_2} f(y) dy = +\infty;$$

(b) for any direction  $\gamma$  such that  $\varepsilon < \alpha(\gamma) < \frac{\pi}{2} - \varepsilon$  we have

$$\int_{\{M_{B_2^\gamma} f > 1\}} M_{B_2^\gamma} f(y) dy < \infty.$$

Indeed, let the constant  $c$ ,  $1 < c < \infty$ , be defined by the equality

$$c = \sup \{c(\gamma) : \gamma \in \Gamma(\mathbb{R}^2), \varepsilon < \alpha(\gamma) < \frac{\pi}{2} - \varepsilon\}.$$

Consider the nonnegative function  $g$  for which the following three conditions are fulfilled:

$$g \in L \log^+ L(U) \setminus L(\log^+ L)^2(U), \quad (29)$$

$$\|g\|_1 > (2c)^{-1}, \quad \text{supp}(g) \subset U. \quad (30)$$

Assuming that  $0 \leq \lambda < \infty$  and

$$E_\lambda = \{x \in U : g(x) > \lambda\},$$

we define the interval  $I_\lambda$  by

$$I_\lambda = [-l'_\lambda, l'_\lambda] \times [-1, 1],$$

where

$$l'_\lambda = 4^{-1}|E_\lambda|.$$

The function  $f$  is defined by

$$f(x) = \int_0^\infty \chi_{I_\lambda}(x) d\lambda.$$

It is clear that  $f$  and  $g$  are the equimeasurable functions. Let  $x \in \mathbb{R}^2$ ,  $\gamma \neq \gamma^0$ , and  $R \in B_2^\gamma(x)$ . By using inequality (3) it is not difficult to show that there exists a cubic interval  $Q_x \in B_1(x)$  such that for any  $\lambda$  we have the relation

$$\frac{1}{|R|} |R \cap I_\lambda| \leq 4c(\gamma) \frac{1}{|Q_{x/3}|} |Q_{x/3} \cap I_\lambda| + c(\gamma)|I_\lambda|,$$

where  $Q_{x/3}$  denotes the image of the interval  $Q_x$  under the homothety with center at the origin and coefficient  $1/3$ . Since  $R$  is arbitrary, we obtain (see [9, p. 649])

$$M_{B_2^\gamma} f(x) \leq 4c(\gamma)M_{B_1} f(x/3) + c(\gamma) \int_S f.$$

Finally, for directions  $\gamma$  for which  $\varepsilon < \alpha(\gamma) < \frac{\pi}{2} - \varepsilon$  we have (see (30))

$$M_{B_2^\gamma} f(x) \leq 4c(\gamma)M_{B_1} f(x/3) + 2^{-1}.$$



By virtue of Stein's theorem [10] and the fact that  $f \in L \log^+ L(U)$  (see (29)) we obtain

$$\int_{\{M_{B_2^\gamma} f > 1\}} M_{B_2^\gamma} f(x) dx < < 4c \int_{\{x: M_{B_1} f(x/3) > 1/8c\}} M_{B_1} f(x/3) dx + 2^{-1} |\{x : M_{B_1} f(x/3) > 1/8c\}| < \infty.$$

Now we shall prove assertion (a). Define the function  $\Phi(x_1)$  as

$$\Phi(x_1) = f(x_1, 0).$$

Note that if  $x_1 \in [-1, 1]$  and  $x_2 \geq 1$  then

$$M_{B_2} f(x_1, x_2) \geq \frac{2}{x_2 + 1} M\Phi(x_1),$$

where  $M\Phi(x_1)$  is the maximal Hardy–Littlewood function on the straight line. By performing transformations and using the fact that  $f$  does not belong to the class  $L(\log^+ L)^2(U)$  we arrive at

$$\begin{aligned} & \int_{\{M_{B_2} f > 1\}} M_{B_2} f(x_1, x_2) dx_1 dx_2 \geq \\ & \geq 2 \int_1^\infty dx_2 \left( \int_{\{x_1 \in (-1, 1): M\Phi(x_1) \geq \frac{x_2 + 1}{2}\}} \frac{1}{x_2 + 1} M\Phi(x_1) dx_1 \right) = \\ & = 2 \int_{\{x_1 \in (-1, 1): M\Phi(x_1) > 1\}} M\Phi(x_1) \log^+ (M\Phi(x_1)) dx_1 = +\infty. \end{aligned}$$

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(Received 17.01.1994)

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