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AN INTERPOLATION INEQUALITY INVOLVING HÖLDER NORMS

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ABSTRACT. An interpolation inequality of Nirenberg, involving Lebesgue-space norms of functions and their derivatives, is modified, replacing one of the norms by a Hölder norm.

0. INTRODUCTION

In his paper [1], L. Nirenberg derived the inequality

$$\|\nabla^j u\|_q \leqq C\|\nabla^m u\|_p^a \, \|u\|_r^{1-a} \tag{0.1}$$

which holds for all functions $u \in C_0^\infty(\mathbb{R}^N)$ with a constant $C > 0$ independent of u. Here $\|\cdot\|_s$ is the L^s-norm, $\nabla^k u$ is the vector of all derivatives $D^\alpha u$ of order $|\alpha| = k$, $k \in \mathbb{N}$, and the parameters p, q, r are connected, for $0 < a < 1$ and $0 < j < m$, by the "dilation formula"

$$-j + \frac{N}{q} = a\Big(-m + \frac{N}{p}\Big) + (1-a)\frac{N}{r}. \tag{0.2}$$

Moreover, it is shown that the parameter a has to satisfy the condition

$$a \geqq \frac{j}{m}.$$

Inequality (0.1) was, among others, a very important tool in the description of properties of Sobolev spaces $W^{m,p}(\mathbb{R}^n)$. For example, for the limiting cases $j = 0$ and $a = 1$, we obtain from (0.1) the famous Sobolev Imbedding theorem

$$\|u\|_q \leqq C\|\nabla^m u\|_p \quad \text{with} \quad \frac{1}{q} = \frac{1}{p} - \frac{m}{N}.$$

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The aim of this note is to modify inequality (0.1) replacing the L^r-norm of u, $\|u\|_r$ on the right-hand side by the Hölder quotient

$$[u]_{H(\lambda)} = \sup_{x \neq y} \frac{|u(x) - u(y)|}{|x - y|^{\lambda}}, \quad 0 < \lambda < 1, \tag{0.3}$$

i.e., to derive inequalities of the form

$$\|\nabla^j u\|_q \leqq C \|\nabla^m u\|_p^a \, [u]_{H(\lambda)}^{1-a} \tag{0.4}$$

for appropriate values of the parameters j, m, p, q, λ, a.

First, let us note that the formula

$$-j + \frac{N}{q} = a\Big(- m + \frac{N}{p}\Big) + (1-a)(-\lambda) \tag{0.5}$$

is an analogue of formula (0.2) for the case of inequality (0.4). Indeed, if (0.4) holds for every function $u = u(x) \in C_0^{\infty}(\mathbb{R}^n)$ with a constant $C > 0$ independent of u, then it holds necessarily for the function $U(x) = u(Rx)$ with $R > 0$, which again belongs to $C_0^{\infty}(\mathbb{R}^n)$. From (0.4) we obtain that

$$\|\nabla^j U\|_q R^{-j+\frac{N}{p}} \leqq C \|\nabla^m U\|_p^a R^{a(-m+\frac{N}{p})} [u]_{H(\lambda)}^{1-a} R^{-\lambda(1-a)}$$

and (0.5) follows since $R > 0$ is arbitrary.

The paper is organized as follows: in Section 1, we will derive an important auxiliary estimate (Lemma 1). In Section 2, we will first deal with inequality (0.4) for the one-dimensional case (Theorem 1) and then, in Section 3, the result will be extended to functions defined on $\mathbb{R}^N$, $N > 1$, but under certain more restrictive conditions on the parameters (Theorem 2).

1. An auxiliary result

Lemma 1. *Let $u = u(t)$ be a smooth function on the finite closed interval $I \subset \mathbb{R}$. Suppose $m, j \in \mathbb{N}$, $0 < j < m$, $0 < \lambda \leqq 1$ and denote*

$$[u]_{\lambda, I} = \sup \Big\{ \frac{|u(t) - u(s)|}{|t - s|^{\lambda}}; t, s \in I, t \neq s \Big\}.$$

Then the estimate

$$|u^{(j)}(t)| \leqq K \Big\{ |I|^{m-j-1} \int_I |u^{(m)}(s)| ds + |I|^{\lambda - j} [u]_{\lambda, I} \Big\} \tag{1.1}$$

holds for every $t \in I$ with $K > 0$ independent of u, t and the length $|I|$ of the interval I: $K = K(j, m, \lambda)$.

Proof. Without loss of generality, we can assume that $I = [0, b]$, $0 < b < \infty$.

(i) Take $\xi \in [0, \frac{1}{3}b]$, $\eta \in [\frac{2}{3}b, b]$. Then there is an $x \in [\xi, \eta]$ such that

$$u(\xi) - u(\eta) = u'(x)(\xi - \eta),$$

i.e.,

$$|u'(x)| = \frac{|u(\xi) - u(\eta)|}{|\xi - \eta|} = \frac{|u(\xi) - u(\eta)|}{|\xi - \eta|^{\lambda}} |\xi - \eta|^{\lambda - 1},$$

and since $|\xi - \eta| \geqq \frac{1}{3}b$ and $\lambda - 1 \leqq 0$, we have

$$|u'(x)| \leqq [u]_{\lambda, I} \left(\frac{b}{3}\right)^{\lambda - 1}. \tag{1.2}$$

Let us fix this x and take any $t \in [0, b]$. Then

$$u'(t) = \int_x^t u''(s)ds + u'(x)$$

and consequently

$$|u'(t)| \leqq \int_0^b |u''(s)|ds + |u'(x)| \leqq \int_0^b |u''(s)|ds + 3^{1-\lambda} b^{\lambda - 1} [u]_{\lambda, I} \tag{1.3}$$

due to (1.2). But (1.3) is (1.1) for $j = 1$, $m = 2$.

(ii) Take $\xi_0 \in [0, \frac{1}{9}b]$, $\xi_1 \in [\frac{2}{9}b, \frac{1}{3}b]$. Then there is a $\xi \in [\xi_0, \xi_1]$ – i.e., $\xi \in [0, \frac{1}{3}b]$ – such that

$$u(\xi_0) - u(\xi_1) = u'(\xi)(\xi_0 - \xi_1).$$

Further, take $\eta_0 \in [\frac{2}{3}b, \frac{7}{9}b]$, $\eta_1 \in [\frac{8}{9}b, b]$. Then there is an $\eta \in [\eta_0, \eta_1]$ – i.e., $\eta \in [\frac{2}{3}b, b]$ – such that

$$u(\eta_0) - u(\eta_1) = u'(\eta)(\eta_0 - \eta_1).$$

Moreover, there is an $x \in [\xi, \eta]$ such that

$$u'(\xi) - u'(\eta) = u''(x)(\xi - \eta).$$

Consequently,

$$u''(x) = \frac{u'(\xi) - u'(\eta)}{\xi - \eta} = \frac{1}{\xi - \eta} \Big[\frac{u(\xi_0) - u(\xi_1)}{\xi_0 - \xi_1} - \frac{u(\eta_0) - u(\eta_1)}{\eta_0 - \eta_1} \Big],$$

and since $|\xi - \eta| \geqq \frac{1}{3}b$, $|\xi_0 - \xi_1| \geqq \frac{1}{9}b$, $|\eta_0 - \eta_1| \geqq \frac{1}{9}b$, we have

$$|u''(x)| \leqq$$

$$\frac{1}{|\xi - \eta|} \Big[\frac{|u(\xi_0) - u(\xi_1)|}{|\xi_0 - \xi_1|^{\lambda}} |\xi_0 - \xi_1|^{\lambda - 1} + \frac{|u(\eta_0) - u(\eta_1)|}{|\eta_0 - \eta_1|^{\lambda}} |\eta_0 - \eta_1|^{\lambda - 1} \Big] \leqq$$

$$\leqq \frac{3}{b} 2[u]_{\lambda,I} \left(\frac{b}{9}\right)^{\lambda-1} = 6 \cdot 9^{1-\lambda} b^{\lambda-2} [u]_{\lambda,I}. \tag{1.4}$$

Let us fix this x and take any $t \in [0, b]$. Then

$$u''(t) = \int_x^t u'''(s)ds + u''(x)$$

and consequently, due to (1.4)

$$|u''(t)| \leqq \int_0^b |u'''(s)|ds + 6 \cdot 9^{1-\lambda} b^{\lambda-2} [u]_{\lambda,I}. \tag{1.5}$$

But this is (1.1) for $j = 2$, $m = 3$.

(iii) Integrating (1.5) with respect to t over the interval $[0, b]$, we obtain that

$$\int_0^b |u''(t)|dt \leqq b\Big[\int_0^b |u'''(s)|ds + 6 \cdot 9^{1-\lambda} b^{\lambda-2} [u]_{\lambda,I}\Big] =$$

$$= b\int_0^b |u'''(s)|ds + 6 \cdot 9^{1-\lambda} b^{\lambda-1} [u]_{\lambda,I}.$$

Using this estimate in (1.3), we see that

$$|u'(t)| \leqq b\int_0^b |u'''(s)|ds + 6 \cdot 9^{1-\lambda} b^{\lambda-1} [u]_{\lambda} + 3^{1-\lambda} b^{\lambda-1} [u]_{\lambda,I} =$$

$$= b\int_0^b |u'''(s)|ds + K b^{\lambda-1} [u]_{\lambda,I}$$

with $K = 6 \cdot 9^{1-\lambda} + 3^{1-\lambda}$. But this is (1.1) for $j = 1$, $m = 3$.

(iv) The proof for general $j, m \in \mathbb{N}$ ($j < m$) proceeds by induction. First, we show that there is an $x \in [0, b]$ such that

$$|u^{(j)}(x)| \leqq K(j)[u]_{\lambda,I} b^{\lambda-j}$$

with $K(j) = 2^{j-1} 3^{\frac{j}{2}(j-2\lambda+1)}$ [compare with (1.2) and (1.4) for $j = 1$ and $j = 2$, respectively].

Putting this x fixed and taking any $x \in [0,b]$, we obtain from

$$u^{(j)}(t) = \int_x^t u^{(j+1)}(s)ds + u^{(j)}(x)$$

that

$$|u^{(j)}(t)| \leqq \int_0^b |u^{(j+1)}(s)|ds + K(j)b^{\lambda-j}[u]_{\lambda,I} \tag{1.6}$$

and integration with respect to t over $[0,b]$ yields

$$\int_0^b |u^{(j)}(t)| \leqq b\int_0^b |u^{(j+1)}(s)|ds + K(j)b^{\lambda-j+1}[u]_{\lambda,I}. \tag{1.7}$$

For $j = m-1$, (1.6) is the estimate (1.1).

For $j = m-2$, estimate (1.6) yields

$$|u^{(m-2)}(t)| \leqq \int_0^b |u^{(m-1)}(s)|ds + K(m-2)b^{\lambda-m+2}[u]_{\lambda,I} \tag{1.8}$$

while (1.7) yields, for $j = m-1$, that

$$\int_0^b |u^{(m-1)}(s)|ds \leqq b\int_0^b |u^{(m)}(s)|ds + K(m-1)b^{\lambda-m+2}[u]_{\lambda,I}.$$

Using this estimate in (1.8), we immediately obtain (1.1) for $j = m-2$ with $K = K(m-1) + K(m-2)$.

Analogously we proceed for $j = m-3, m-4, \ldots$. □

Remark. Inequality (1.1) is a counterpart of the inequality

$$|u^{(j)}(t)| \leqq K\Big\{|I|^{m-j-1}\int_I |u^{(m)}(s)ds + |I|^{-j-1}\int_I |u(s)|ds\Big\}$$

which is a useful tool when deriving interpolation inequalities in (weighted) L^s-norms (see, e.g., R.C. Brown and D.B. Hinton [2]).

Suppose $1 < p, q < \infty$. Then we can immediately derive from Lemma 1 the following

Corollary. *Under the assumptions of Lemma* 1, *the estimate*

$$\int_I |u^{(j)}(t)|^q dt \leqq$$
$$\leqq \widetilde{K}\Big\{|I|^{(m-j)q+1-\frac{q}{p}}\Big(\int_I |u^{(m)}(s)|^p ds\Big)^{q/p} + |I|^{1+(\lambda-j)q}[u]^q_{\lambda,I}\Big\} \quad (1.9)$$

holds.

Proof. The Hölder inequality yields for $1 < p < \infty$ that

$$\int_I |u^{(m)}(s)|ds \leqq \Big(\int_I |u^{(m)}(s)|^p ds\Big)^{1/p}|I|^{1-\frac{1}{p}}. \quad (1.10)$$

For $1 < q < \infty$, if follows from (1.1) that

$$|u^{(j)}(t)|^q \leqq$$
$$\leqq 2^{q-1}K\Big\{|I|^{(m-j-1)q}\Big(\int_I |u^{(m)}(s)|ds\Big)^q + |I|^{(\lambda-j)q}[u]^q_{\lambda,I}\Big\}$$

holds for every $t \in I$. Integrating this inequality with respect to t over I and using (1.10), we obtain the estimate (1.9). □

2. The one-dimensional case

Let us assume that $u = u(t)$ is defined on $\mathbb{R}_+$, that $0 < j < \infty$, and that $u^{(m)} \in L^p(\mathbb{R}_+)$, $u^{(j)} \in L^q(\mathbb{R}_+)$, and $[u]_{\lambda,\mathbb{R}_+}$ is finite.

Consider first the interval $[0, L]$, $0 < L < \infty$. Following the idea of L. Nirenberg [2], we will cover this interval by a finite number of successive intervals $I_1, I_2, \dots$ where the initial point of I_{i+1} coincides with the endpoint of I_i.

Take a fixed $k \in \mathbb{N}$ and consider the estimate (1.9) for the special interval $I = [0, L/k]$. If the first term on the right-hand side of (1.9) is greater than the second, then we set $I_1 = I$ and hence we have the estimate

$$\int_{I_1} |u^{(j)}(s)|^q ds \leqq 2\widetilde{K}\Big(\frac{L}{k}\Big)^{(m-j-\frac{1}{p})q+1}\Big(\int_{I_1} |u^{(m)}(s)|^p ds\Big)^{q/p}. \quad (2.1)$$

On the other hand, if the second term is greater, we proceed in the following way: We suppose that

$$1 + (\lambda - j)q < 0 \quad (2.2)$$

[in fact, this means that we have to suppose $\lambda < 1 - 1/q$ if $j = 1$, since for $j = 2, 3, \dots$ the condition (2.2) is satisfied due to the assumption $0 < \lambda \leqq 1$], while

$$\Big(m - j - \frac{1}{p}\Big)q + 1 > 0, \tag{2.3}$$

and we introduce a parameter a, $0 < a < 1$.

Now we extend the interval I (keeping the left endpoint fixed) until the a-multiple of the second term becomes equal to the $(1-a)$-multiple of the first term. This must occur for a finite value of $|I|$, since the exponent on $|I|$ in the first term is positive due to (2.3), but the exponent on $|I|$ is negative due to (2.2). Denoting I_1 the resulting interval and using the identity

$$A + B = \Big(\frac{1}{a}\Big)^a \Big(\frac{1}{1-a}\Big)^{1-a} A^a B^{1-a} \quad \text{if} \quad aB = (1-a)A,$$

we then have

$$\int\limits_{I_1} |u^{(j)}(s)|^q ds \leqq \widetilde{K}\Big(\frac{1}{a}\Big)^a \Big(\frac{1}{1-a}\Big)^{1-a} |I_1|^{(m-j-\frac{1}{p})qa+a} \times$$
$$\times \Big(\int\limits_{I_1} |u^{(m)}(t)|^p dt\Big)^{aq/p} \cdot |I_1|^{(1-a)(1+\lambda q-jq)} [u]^{q(1-a)}_{\lambda, I_1}.$$

If we choose

$$a = \frac{j - \frac{1}{q} - \lambda}{m - \frac{1}{p} - \lambda} \tag{2.4}$$

then the foregoing estimate becomes simple:

$$\int\limits_{I_1} |u^{(j)}(s)|^q ds \leqq \widetilde{K}_a \Big(\int\limits_{I_1} |u^{(m)}(s)|^p ds\Big)^{aq/p} \cdot [u]^{q(1-a)}_{\lambda, I_1}. \tag{2.5}$$

Keeping k fixed, we now start at the endpoint of I_1 and repeat this process [beginning with an interval of length L/k, comparing the two terms on the right-hand side of the corresponding inequality (1.9), etc.] choosing $I_2, I_3, \dots$ until the interval $[0, l]$ is covered. There are at most k such intervals, and if we now sum up our estimates of

$$\int\limits_{I_i} |u^{(j)}(s)|^q ds$$

which are of the form (2.1) or (2.5), we finally find that

$$\int_0^L |u^{(j)}(s)|^q ds \leqq \sum_i \int_{I_i} |u^{(j)}(s)|^q ds \leqq$$

$$\leqq k \cdot 2\widetilde{K}\Big(\frac{L}{k}\Big)^{(m-j-\frac{1}{p})q+1}\Big(\int_0^\infty |u^{(m)}(s)|^p ds\Big)^{q/p} +$$

$$+\widetilde{K}_a \sum_i \Big(\int_{I_i} |u^{(m)}(t)|^p dt\Big)^{aq/p} \cdot [u]^{q(1-a)}_{\lambda, I_i}. \tag{2.6}$$

If we suppose

$$\frac{aq}{p} \geqq 1, \tag{2.7}$$

which in fact means that

$$\lambda \leqq \frac{jq - mp}{q - p} \tag{2.8}$$

and which contains the assumption $jq - mp > 0$, i.e.,

$$q > \frac{m}{j} p, \tag{2.9}$$

then

$$\sum_i \Big(\int_{I_i} |u^{(m)}(t)|^p dt\Big)^{aq/p} \cdot [u]^{q(1-a)}_{\lambda, I_i} \leqq$$

$$\leqq \Big\{\sum_i \Big(\int_{I_i} |u^{(m)}(t)|^p dt\Big)^{aq/p}\Big\} \cdot [u]^{q(1-a)}_{\lambda, \mathbb{R}_+} \leqq$$

$$\leqq \Big\{\sum_i \Big(\int_{I_i} |u^{(m)}(t)|^p dt\Big)\Big\}^{aq/p} \cdot [u]^{q(1-a)}_{\lambda, \mathbb{R}_+} \leqq$$

$$\leqq \Big(\int_0^\infty |u^{(m)}(t)|^p dt\Big)^{aq/p} \cdot [u]^{q(1-a)}_{\lambda, \mathbb{R}_+}.$$

This is a (global) bound for the second term on the right-hand side of (2.6). If we now let $k \to \infty$, then the first term tends to zero, since $(m - j - \frac{1}{p})q + 1 > 1$, and we obtain the interpolation inequality

$$\Big(\int_0^\infty |u^{(j)}(t)|^q dt\Big)^{1/q} \leqq C\Big(\int_0^\infty |u^{(m)}(t)|^p dt\Big)^{a/p} \cdot [u]^{1-a}_{\lambda, \mathbb{R}_+} \tag{2.10}$$

since the number L on the left-hand side of (2.6) was arbitrary.

Let us summarize the result.

Theorem 1. *Suppose* $m, j \in \mathbb{N}$, $0 < j < m$, $1 < p < q < \infty$, $0 < \lambda \leqq 1$, $0 < \lambda < 1 - \frac{1}{q}$, *if* $j = 1$. *Further suppose that*

$$q > \frac{m}{j} p$$

and

$$\lambda \leqq \frac{jq - mp}{q - p}.$$

Then the interpolation inequality

$$\|u^{(j)}\|_q \leqq C \|u^{(m)}\|_p^a \cdot [u]_{H(\lambda)}^{1-a} \tag{2.11}$$

holds for every $u \in C_0^\infty(\mathbb{R}_+)$ *with*

$$a = \frac{j - \frac{1}{p} - \lambda}{m - \frac{1}{p} - \lambda}.$$

3. The N-dimensional case

Theorem 2. *Suppose* $N, m, j \in \mathbb{N}$, $N \geqq 2$, $0 < j < m$, $1 < p < q < \infty$. *Further, let*

$$\frac{m}{j} p < q \leqq \frac{m-1}{j-1} p \tag{3.1}$$

and

$$\lambda = \frac{jq - mp}{q - p}. \tag{3.2}$$

Then the interpolation inequality (0.4),

$$\|\nabla^j u\|_q \leqq C \|\nabla^m u\|_p^a \cdot [u]_{H(\lambda)}^{1-a}, \tag{3.3}$$

holds for every $u \in C_0^\infty(\mathbb{R}^N)$ *with*

$$a = \frac{p}{q}. \tag{3.4}$$

Proof. For $x \in \mathbb{R}^N$ denote $x = (t, x')$ with $t \in \mathbb{R}$ and $x' \in \mathbb{R}^{N-1}$. For any fixed x' we can rewrite the inequality (2.11) [i.e., (2.10), but now on $\mathbb{R}$ instead of $\mathbb{R}_+$] in the form

$$\int_{-\infty}^{+\infty} \Big| \frac{\partial^j u}{\partial t^j}(x', t) \Big|^q dt \leqq C^q \Big(\int_{-\infty}^{+\infty} \Big| \frac{\partial^m u}{\partial t^m}(x', t) \Big|^p dt \Big)^{aq/p} \cdot [u(x', \cdot)]_{\lambda, \mathbb{R}_+}^{(1-a)q}.$$

Estimating $[u](x',\cdot)]_{\lambda,\mathbb{R}}$ by $[u]_{H(\lambda)}$ and integrating the resulting inequality with respect to $x' \in \mathbb{R}^{N-1}$, we obtain that

$$\int\limits_{\mathbb{R}^N} \Big|\frac{\partial^j u}{\partial t^j}(x)\Big|^q dx \leqq C\Big(\int\limits_{\mathbb{R}^{N-1}} \Big[\int\limits_{\mathbb{R}} \Big|\frac{\partial^m u}{\partial t^m}(x',t)\Big|^p dt\Big]^{aq/p} dx'\Big) \cdot [u]_{H(\lambda)}^{(1-a)q} =$$

$$= C^p \Big(\int\limits_{\mathbb{R}^N} \Big|\frac{\partial^m u}{\partial t^m}(x)\Big|^p dx\Big)^{aq/p} \cdot [u]_{H(\lambda)}^{(1-a)q}$$

since due to (3.4), $aq/p = 1$. Now (3.3) follows immediately, taking the $1/q$th power of both sides.

Due to (3.4), the "dilation formula" (0.5) has now the form

$$-j + \frac{N}{q} = \frac{p}{q}\Big(-m + \frac{N}{p}\Big) + \frac{p-q}{q}\lambda$$

which leads to formula (3.2), and since $0 < \lambda \leqq 1$, we obtain the conditions (3.1). $\square$

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