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# MORE ON OSCILLATION OF $n$TH-ORDER EQUATIONS

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Abstract. In this paper we prove that a higher-order differential equation with one middle term has every bounded solution oscillatory. Moreover, the behavior of unbounded solutions is given. Two other results dealing with positive solutions are also given.

## 1. Introduction

Little is known about the behavior of solutions of differential equations which involve a nonlinear term $H(t,x)$, where $H:\mathbb{R}^+\times\mathbb{R}\to\mathbb{R}$ is continuous ($\mathbb{R}$ is the real line and $\mathbb{R}^+$ is the interval $(0,\infty)$), decreasing in its second variable and is such that $uH(t,u)<0$ for all $u\neq 0$. Some properties of solutions of such equations are given by the author in [1]–[4]. In [3] the author gave two oscillation results for odd-order equations of the form

$$x^{(n)}+p(t)x^{(n-1)}+q(t)x^{(n-2)}+H(t,x)=0. \qquad (1)$$

In [4] bounded and eventually positive solutions of (1) were studied for different choices of $p$ and $q$. The main difficulty in handling equations involving the indicated $H$ is the lack of tools for equations involving $H$ continuous, increasing in its second variable, and keeping the sign of the second variable. Some successful tools for equations with either $H$ proved to be nonlinear functions which were used by Erbe [5], Heidel [6], Kartsatos [7], Kartsatos and Kosmala [8], Kosmala [3], and others. In most cases the method used on an even-ordered equation does not apply to an odd-ordered equation, and vice versa.

## 2. Preliminaries

A function $x(t)$, $t\in[t_x,\infty)\subset\mathbb{R}^+$, is a solution of (1) if it is $n$ times continuously differentiable and satisfies (1) on $[t_x,\infty)$. The number $t_x>0$ depends on the particular solution $x(t)$ under consideration. We say that

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a function is "oscillatory" if it has an unbounded set of zeros. Moreover, a property $P$ holds "eventually" or "for all large $t$" if there exists $T > 0$ such that $P$ holds for all $t \geq T$. $C^n(I)$ denotes the space of all $n$ times continuously differentiable functions $f : I \to \mathbb{R}$. And we write $C(I)$ instead of $C^0(I)$. From [3] we quote the following lemma.

**Lemma 2.1.** *If $x$ is an eventually positive solution of* (1), $p \in C^1[t_0, \infty)$, $q \in C[t_0, \infty)$ *with* $t_0 > 0$ *where* $2q(t) \leq p'(t)$ *for all* $t \geq t_0$, *then either* $x^{(n-2)}(t)$
$\leq 0$ *or* $x^{(n-2)}(t) > 0$ *for all large* $t$.

The main result from [2] we state as our next lemma.

**Lemma 2.2.** *Suppose that* $x(t)$ *is a nonoscillatory solution of* (1) *with* $n$ *even. Suppose further that* $p \in C^1[t_0, \infty)$, $q \in C[t_0, \infty)$ *with* $t_0 > 0$ *where* $p(t) \leq 0$ *and* $2q(t) \leq p'(t) \leq 0$ *for all* $t \geq t_0$ *and* $\lim_{t\to\infty} p(t)$ *is finite. Moreover, suppose that*

$$-\int^{\infty} tH(t, \pm k)dt = \pm\infty$$

*for any constant* $k > 0$. *Suppose further that for* $v > 0$ *both*

$$I(v) \equiv \int_v^{\infty} tq(t)dt \quad \text{and} \quad \int^{\infty} I(t)dt$$

*are finite. Then* $x(t)x^{(n-2)} > 0$ *eventually.*

## 3. Main Results

**Theorem 3.1.** *Suppose that* $n$ *is odd,* $p \in C[t_0, \infty)$ *with* $t_0 > 0$ *where* $p(t)$ *for all* $t \geq t_0$, *and that*

$$\int_{t^*}^{\infty} t^i\Big(\exp\int_{t^*}^{t} p(s)ds\Big)H(t, k)dt = -\infty \tag{2}$$

*for any* $t^* \geq 0$, *every positive real constant* $k$, *and some integer* $i$ *where* $1 \leq i \leq n-1$.

(a) *Then every solution of*

$$x^{(n)} + p(t)x^{(n-1)} + H(t, x) = 0 \tag{3}$$

*with bounded* $(n-i-1)^{st}$ *derivative is oscillatory. In particular, every bounded solution of* (3) *is oscillatory.*

(b) *If* $i = 1$ *and* $x(t)$ *is an unbounded solution of* (3), *then* $x(t)x^{(j)}(t) > 0$ *for all* $j = 0, 1, 2, \cdots, n$ *eventually.*

*Proof of part* (a). First we assume that $x(t)$ is a positive solution of (3) such that $x^{(n-i-1)}(t)$ is bounded and $p(t) \leq 0$ for $t \geq t_1 \geq t_0$. Now we distinguish three cases.

*Case* 1. Suppose that $x^{(n-1)}(t_2)$ for some $t_2 \geq t_1$. Then, from (3) we have

$$x^{(n)}(t_2) = -H(t_2, x(t_2)) > 0.$$

Thus, $x^{(n-1)}(t)$ is increasing at any $t_2$, for which it is zero. Therefore, $x^{(n-1)}(t)$ cannot have any zeros larger than $t_2$.

*Case* 2. Suppose that $x^{(n-1)}(t) > 0$ for $t \geq t_3 \geq t_2$. Then, from (3) we have that

$$x^{(n)}(t) = -p(t)x^{(n-1)}(t) - H(t, x(t)) > 0.$$

But, $x^{(n)}(t) > 0$ and $x^{(n-1)}(t) > 0$ imply that $x^{(n-2)}(t)$ tends to $+\infty$ as $t$ tends to $+\infty$. But, $x^{(n-i-1)}(t)$ is bounded for some integer $i$, where $1 \leq i \leq n-1$, which contradicts $\lim_{t\to\infty} x^{(n-2)}(t) = +\infty$ no matter what $i$ is. This takes us to the final case.

*Case* 3. Suppose that $x^{(n-1)}(t) < 0$ for $t \geq t_3 \geq t_1$. Since $n$ is odd there exists $t_4 \geq t_3$ such that $x'(t) > 0$ for all $t \geq t_4$. Thus, we have $k \equiv x(t_4) \leq x(t)$ and hence

$$-t^i\Big(\exp\int_{t_4}^{t} p(s)ds\Big)H\big(t, x(t)\big) \geq -t^i\Big(\exp\int_{t_4}^{t} p(s)ds\Big)H(t, k) \tag{4}$$

for all $t \geq t_4$. Now, define

$$F(t) = t^i\Big(\exp\int_{t_4}^{t} p(s)ds\Big)x^{(n-1)}(t)$$

for $t \geq t_4$. Then, for $t \geq t_4$ we have

$$F'(t) = t^i\Big(\exp\int_{t_4}^{t} p(s)ds\Big)x^{(n)}(t) +$$

$$+t^i\Big(\exp\int_{t_4}^{t} p(s)ds\Big)p(t)x^{(n-1)}(t) + it^{i-1}\Big(\exp\int_{t_4}^{t} p(s)ds\Big)x^{(n-1)}(t) =$$

$$= t^i\Big(\exp\int_{t_4}^{t} p(s)ds\Big)\Big(-p(t)x^{(n-1)}(t) - H\big(t, x(t)\big)\Big) +$$

$$+t^i\Big(\exp\int_{t_4}^{t} p(s)ds\Big)p(t)x^{(n-1)}(t) + it^{i-1}\Big(\exp\int_{t_4}^{t} p(s)ds\Big)x^{(n-1)}(t) =$$

$$= it^{i-1}\Big(\exp\int_{t_4}^{t} p(s)ds\Big)x^{(n-1)}(t) - t^i\Big(\exp\int_{t_4}^{t} p(s)ds\Big)H\big(t, x(t)\big).$$

Now we integrate above from $t_4$ to $t$, $t \geq t_4$,and then use (4) to obtain

$$F(t) - F(t_4) = \int_{t_4}^{t} is^{i-1}\Big(\exp\int_{t_4}^{s} p(u)du\Big)x^{(n-1)}(s)ds -$$

$$-\int_{t_4}^{t} s^i\Big(\exp\int_{t_4}^{s} p(u)du\Big)H\big(s, x(s)\big)ds \geq$$

$$\geq \int_{t_4}^{t} is^{i-1}\Big(\exp\int_{t_4}^{s} p(u)du\Big)x^{(n-1)}(s)ds -$$

$$-\int_{t_4}^{t} s^i\Big(\exp\int_{t_4}^{s} p(u)du\Big)H(s, k)ds.$$

Due to integral hypothesis (2) and the fact that $F(t) < 0$, we have that

$$\lim_{t\to\infty}\int_{t_4}^{t} s^{i-1}\Big(\exp\int_{t_4}^{s} p(u)du\Big)x^{(n-1)}(s)ds = -\infty.$$

But, since $p(t) \leq 0$ and $x^{(n-1)}(t) < 0$, which implies that $x^{(n-2)}(t) > 0$, we have that

$$t^{i-1}\Big(\exp\int_{t_4}^{t} p(s)ds\Big) \geq t^{i-1}x^{(n-1)}(t).$$

Thus,

$$\lim_{t\to\infty}\int_{t_4}^{t} s^{i-1}x^{(n-1)}(s)ds = -\infty. \tag{5}$$

Integration by parts gives

$$\lim_{t\to\infty}\Big[s^{i-1}x^{(n-2)}(s)\Big|_{t_4}^{t} - \int_{t_4}^{t}(i-1)s^{i-2}x^{(n-2)}(s)ds\Big] = -\infty.$$

Since $x^{(n-2)}(t)$ is positive, we have

$$\lim_{t\to\infty} \int_{t_4}^{t} s^{i-2} x^{(n-2)}(s)ds = +\infty.$$

Integration by parts gives

$$\lim_{t\to\infty} \int_{t_4}^{t} s^{i-3} x^{(n-3)}(s)ds = -\infty.$$

Continuing this process we get

$$(-1)^m \int_{t_4}^{\infty} s^{i-m} x^{(n-m)}(s)ds = +\infty,$$

where $m$ is an integer such that $1 \le m \le i$. (This can be proved by induction.) Setting $m = i$, we obtain

$$(-1)^i \int_{t_4}^{\infty} x^{(n-i)}(s)ds = +\infty,$$

which gives

$$\lim_{t\to\infty} (-1)^i \big[x^{(n-i-1)}(t) - x^{(n-i-1)}(t_4)\big] = +\infty.$$

But, $x^{(n-i-1)}(t)$ is bounded. This is a contradiction. Therefore assuming that $x(t)$ is eventually positive prevents $x^{(n-1)}(t)$ from existing. Similar steps to the above ones also bring a contradiction for an eventually negative solution of (3). Hence, the proof of part (a) is complete.

*Proof of part* (b). Suppose that $x(t)$ is an unbounded positive solution of (3) and $p(t) \le 0$ for $t \ge 0$. In view of case 1 above, $x^{(n-1)}(t)$ cannot be oscillatory. From (5) we have that

$$\int^{\infty} x^{(n-1)}(t)dt = -\infty,$$

which gives $\lim_{t\to\infty} x^{(n-2)}(t) = -\infty$. This is, however, not possible due to the positiveness of $x(t)$. Therefore, the only possibility is case 2 above, which yields the desired result. □

**Theorem 3.2.** *Suppose $n$ is odd, $p \in C^1[t_0,\infty)$, $q \in C[t_0,\infty)$ with $t_0>0$ where $p(t) \le 0$ and $q(t) \le p'(t) \le 0$ for all $t \ge t_0$, and $\lim_{t\to\infty} p(t)$ is finite. Suppose further that for $v>0$ both*

$$I(v) \equiv \int_v^\infty tq(t)dt \quad \text{and} \quad \int^\infty I(t)dt \tag{6}$$

*are finite. Also, $H^*(t,u) \equiv \frac{d}{du}H(t,u)$ is continuous negative, and decreasing for $u>0$, and*

$$\int^\infty H^*(t,0)dt = -\infty. \tag{7}$$

*If $u(t)$ and $v(t)$ are two bounded eventually positive solutions of* (1)*, then the difference $w(t)=u(t)-v(t)$ must be oscillatory.*

*Remark* 3.3. The conditions of Theorem 3.2 imply that $2q(t) \le p'(t)$ for $t \ge t_0$ and that

$$\int^\infty tH^*(t,0) = -\infty.$$

This is needed so that the proof of Lemma 2.2 can be used in Case 2 below. Furthermore, functions in the differential equation

$$x^{(5)} + \Big(\frac{1}{3t^3}-1\Big)x^{(4)} - \frac{2}{t^4}x''' + (1-e^{tx}) = 0$$

satisfy all the conditions of Theorem 3.2.

*Proof of Theorem* 3.2. Suppose that $u(t)$ and $v(t)$ are bounded, positive solutions of (1) and that $w(t)=u(t)-v(t)$ for all $t \ge t_1 \ge t_0$. The case where $w(t)$ is negative follows similar steps and is therefore omitted. Now, from (1) we have

$$\begin{aligned} w^{(n)}(t) + p(t)w^{(n-1)}(t) + q(t)w^{(n-2)}(t) &+ \\ +H(t,u(t)) - H(t,v(t)) &= 0. \end{aligned} \tag{8}$$

Note that since $w(t)>0$, we have $H(t,u(t)) < H(t,v(t))$. Hence, we can follow the proof of Lemma 2.1 to reach the conclusion that $w^{(n-2)}(t) \le 0$ or $w^{(n-2)}(t) > 0$ for all $t \ge t_2 \ge t_1$. In order to prove this theorem, we need to consider both cases and find a contradiction in each.

*Case* 1. Suppose that $w^{(n-2)}(t) > 0$ for $t \ge t_2$. Since $n$ is odd we have $w'(t) > 0$ for all $t \ge t_3 \ge t_2$.Now, if $w^{(n-2)}(t_4) > 0$ for some $t_4 \ge t_3$, then from (8) we have

$$w^{(n)}(t_4) = -q(t_4)w^{(n-2)}(t_4) - \big(H\big(t_4\big)\big) - H\big(t_4, v(t_4)\big)\big) > 0.$$

Therefore, $w^{(n-1)}(t)$ is increasing at each of its zeros. Thus, $w^{(n-1)}(t) > 0$ or $w^{(n-1)}(t) < 0$ for all $t \geq t_5 > t_4$.If $w^{(n-1)}(t) > 0$, then together with $w^{(n-2)}(t) > 0$ we contradict the fact that $w(t)$ is bounded. Thus, we must have $w^{(n-1)}(t) < 0$ for all $t \geq t_5$. Since $w'(t) > 0$, we have $0 < k \equiv w(t_5) \leq w(t)$ for all $t \geq t_5$. So, by the mean value theorem there exists a function $\lambda$ between $u$ and $v$, hence positive, such that

$$\frac{H(t,u(t)) - H(t,v(t))}{u(t)-v(t)} = H^*(t,\lambda(t)).$$

Therefore,

$$\begin{aligned} H(t,u(t)) - H(t,v(t)) = (u(t)-v(t))H^*(t,\lambda(t)) = \\ = w(t)H^*(t,\lambda(t)) \leq kH^*(t,0) \end{aligned} \tag{9}$$

for all $t \geq t_5$. Now, integrate (8) from $t_5$ to $t \geq t_5$ to obtain

$$\begin{aligned} w^{(n-1)}(t) + p(t)w^{(n-2)} = w^{(n-1)}(t_5) + p(t_5)w^{(n-2)}(t_5) + \\ + \int_{t_5}^{t} \big(p'(s) - q(s)\big)w^{(n-2)}(s)ds - \int_{t_5}^{t} \big(H\big(s,u(s)\big) - H\big(s,v(s)\big)\big)ds = \\ = M + f(t) - \int_{t_5}^{t} \big(H\big(s,u\big)\big) - H\big(s,v(s)\big)\big)ds, \end{aligned}$$

where $M$ is a constant and $f(t)$ the first integral in the above expression. If $z(t) = w^{(n-2)}(t)$, then $z$ satisfies a first-order linear differential equation and thus can be written as

$$\begin{aligned} z(t) = \exp\Big[-\int_{t_5}^{t} p(s)ds\Big]\Big\{z(t_5) + \int_{t_5}^{t}\Big[\exp\int_{t_5}^{s} p(r)dr\Big]\Big[M + f(s) - \\ - \int_{t_5}^{s}\big(H\big(r,u(r)\big) - H\big(r,v(r)\big)\big)dr\Big]ds\Big\}. \end{aligned}$$

But, $f(t) \geq 0$ and $\exp\left[-\int_{t_5}^{t} p(s)ds\right] \geq 1$; thus in view of (9) we have

$$z(t) \geq \int_{t_5}^{t}\Big[\exp\Big(-\int_{s}^{t} p(r)dr\Big)\Big]\Big[M - k\int_{t_5}^{s} H^*(r,0)dr\Big]ds. \tag{10}$$

Due to condition (7), there exists $s_0 \in \mathbb{R}^+$ such that

$$M - k \int_{t_5}^{s_0} H^*(t,0)dt > 0.$$

Therefore, the right hand side of (10) tends to $+\infty$. This implies that the left hand side of (10) also tends to $+\infty$. But this is a contradiction to the positiveness of $w(t)$. Hence, $w^{(n-2)}(t)$ cannot be eventually positive.

*Case* 2. Suppose that $w^{(n-2)}(t) \le 0$ for $t \ge t_2$. With a few minor changes, the proof of Lemma 2.2 goes through for equation (8), contradicting this situation as well. In this proof we need assumption (6). We will not give the detailed proof since it is very long and tedious. Also, see Remark 3.3. □

**Theorem 3.4.** *Suppose that $n$ is odd, $p \in C^1[t_0,\infty)$ for $t_0 > 0$, $p(t) > 0$ for all $t \ge t_0$, with $\lim_{t\to\infty} p(t)$ finite and*

$$\int^{\infty} \frac{1}{p(s)} ds = +\infty.$$

*Also,*

$$\int^{\infty} H(t,k)dt = -\infty$$

*for any positive real constant $k$. Then, the differential inequality*

$$\big(p(t)x^{(n-1)}\big)' + H(t,x) \ge 0 \tag{11}$$

*has no bounded eventually positive solutions.*

*Proof.* Suppose that $x(t) > 0$ is a bounded solution of (11) and $p(t) > 0$ for $t \ge t_0 > 0$. Then

$$\big(p(t)x^{(n-1)}\big)' \ge -H(t,x) > 0. \tag{12}$$

Therefore, if $v(t) \equiv p(t)x^{(n-1)}(t)$, then $v(t)$ is increasing for all $t \ge t_0$. We distinguish two cases.

*Case* 1. Suppose $v(t) > 0$ for all $t \ge t_1 \ge t_0$. Then, from (12) we have $v'(t) > 0$, and so $v(t) \ge v(t_1) > 0$ for all $t \ge t_1$. Therefore,

$$x^{(n-1)}(t) = \frac{v(t)}{p(t)} \ge \frac{v(t_1)}{p(t)}$$

for all $t \ge t_1$. Integrating the above from $t$ to $t_1$, $t \ge t_1$ we obtain

$$x^{(n-2)}(t) - x^{(n-2)}(t_1) \ge v(t_1) \int_{t_1}^{\infty} \frac{1}{p(s)} ds \to +\infty \quad \text{as} \quad t \to +\infty.$$

Therefore, $\lim_{t\to\infty} x^{(n-2)}(t) = +\infty$, which is a contradiction to the boundedness of $x(t)$.

*Case* 2. Suppose $v(t) < 0$ for all $t \geq t_2 \geq t_0$. Then, from (12) since $p(t) > 0$, we have $x^{(n-1)}(t) < 0$. Now, since $n$ is odd we have $x'(t) > 0$ which gives $H(t, x(t)) \leq H(t, k)$ for all $t \geq t_2 \equiv k$. Therefore, integrating (11) from $t_2$ to $t$, $t \geq t_2$, we get

$$p(t)x^{(n-1)}(t) - p(t_2)x^{(n-1)}(t_2) \geq$$

$$\geq -\int_{t_2}^{t} H(s, x(s))ds \geq -\int_{t_2}^{t} H(s, k)ds \to +\infty$$

as $t \to +\infty$. Thus, $\lim_{t\to\infty} p(t)x^{(n-1)}(t) = +\infty$. Since $\lim_{t\to\infty} p(t)$ is finite, we have $\lim_{t\to\infty} x^{(n-1)}(t) = +\infty$, which is a contradiction to the positiveness of $x(t)$. Hence, the proof is complete. $\square$

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