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FIXED POINTS OF SEMIGROUPS OF LIPSCHITZIAN MAPPINGS DEFINED ON NONCONVEX DOMAINS

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Abstract. Certain fixed point theorems are established for nonlinear semigroups of Lipschitzian mappings defined on nonconvex domains in Hilbert and Banach spaces. Some known results are thus generalized.

1. Introduction

Let X be a Banach space and C be a nonempty subset of X. A mapping $T : C \to C$ is said to be a *Lipschitzian mapping* if, for each integer $n \geq 1$, there exists a constant $k_n \geq 0$ such that $\|T^n x - T^n y\| \leq k_n \|x - y\|$ $\forall x, y \in C$. A Lipschitzian mapping T is said to be *uniformly Lipschitzian* if $k_n = k$ for all $n \geq 1$, *nonexpansive* if $k_n = 1$ for all $n \geq 1$, and *asymptotically nonexpansive* if $\lim_{n\to\infty} k_n = 1$, respectively. Goebel and Kirk [1] initiated in 1973 the study of the fixed point theory for Lipschitzian mappings. They showed that if X is uniformly convex and C is a bounded closed convex subset of X, then every uniformly k-Lipschitzian mapping $T : C \to C$ with $k < \gamma$ has a fixed point, where $\gamma > 1$ is the unique solution of the equation

$$\gamma \left[1 - \delta_X \left(\frac{1}{\gamma} \right) \right] = 1,$$

with δ_X the modulus of convexity of X. Since then, much effort has been devoted to the existence theory for fixed points of Lipschitzian mappings in both Hilbert and Banach spaces; see [2], [4], [5], [7], [10], [11], and refereces cited there. Usually the domain C on which T is defined is assumed to be convex. Recently, Ishihara [3] and Takahashi [9] studied in Hilbert spaces the existence theory for fixed points of Lipschitzian mappings which are defined on nonconvex domains. However, their methods do not work outside Hilbert spaces.

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The purpose of the present paper is to investigate the existence theory for the fixed point theory of semigroups of Lipschitzian mappings defined on nonconvex domains in both Hilbert and Banach spaces.

2. Preliminaries

Let G be a semitopological semigroup, i.e., G is a semigroup with a Hausdorff topology such that for each $a \in G$, the mappings $t \to at$ and $t \to ta$ from G into itself are continuous. Let C be a nonempty subset of a Banach space X. Then a family $\mathcal{F} = \{T_t : t \in G\}$ of self-mappings of C is said to be a *Lipschitzian semigroup* on C if the following properties are satisfied:

(1) $T_{ts}x = T_tT_sx$ for all $t, s \in G$ and $x \in C$;
(2) for each $x \in C$, the mapping $t \to T_tx$ is continuous on G;
(3) for each $t \in G$, there is a constant $k_t > 0$ such that $\|T_tx - T_ty\| \leq k_t\|x - y\| \; \forall x, y \in C$.

A Lipschitzian semigroup $\mathcal{F}$ is called *uniformly k-Lipschitzian* if $k_t = k$ for all $t \in G$ and in particular, *nonexpansive* if $k_t = 1$ for all $t \in G$. We shall denote by $F(\mathcal{F})$ the set of common fixed points of $\mathcal{F} = \{T_t : t \in G\}$.

Recall that a semitopological semigroup G is said to be *left reversible* if any two closed right ideals of G have nonvoid intersection. In this case, $(G, \leq)$ is a directed system when the binary relation "$\leq$" on G is defined by $a \leq b$ if and only if $\{a\} \cup \overline{aG} \supseteq \{b\} \cup \overline{bG}$. Let $B(G)$ be the Banach space of all bounded real-valued functions on G with the supremum norm and let X be a subspace of $B(G)$ containing constants. Then an element μ of X^*, the dual space of X, is said to be a *mean* on X if $\|\mu\| = \mu(1) = 1$. It is known that $\mu \in X^*$ is a mean on X if and only if the inequalty

$$\inf\{f(t) : t \in G\} \leq \mu(f) \leq \sup\{f(t) : t \in G\}$$

holds for all $f \in X$. For a mean μ on X^* and an element $f \in X$, we use either $\mu_t(f(t))$ or $\mu(f)$ to denote the value of μ at f. For each $s \in G$, we define the left transformation ℓ_s from $B(G)$ into itself by $(\ell_sf)(t) = f(st)$, $t \in G$, for all $f \in B(G)$. The right transformation r_s is defined similarly. Let X be a subspace of $B(G)$ containing constants which is ℓ_G-invariant (r_G-invariant), i.e., $\ell_s(X) \subseteq X$ ($r_s(X) \subseteq X$) for all $s \in G$. Then a mean μ on X is said to be *left invariant* (*right invariant*) if $\mu(f) = \mu(\ell_sf)$ ($\mu(f) = \mu(r_sf)$) for all $f \in X$ and $s \in G$. An invariant mean is a mean that is both left and right invariant.

Let $C(G)$ be the Banach space of all bounded continuous real-valued functions on G, let $RUC(G)$ be the space of all bounded right uniformly continuous functions on G, i.e., all $f \in C(G)$ for which the mapping $s \to r_sf$ is continuous, and let $AP(G)$ be the space of all $f \in C(G)$ for which $\{\ell_sf : s \in G\}$ is relatively norm compact. Then $RUC(G)$ is a closed subalgebra of

$C(G)$ containing constants and both ℓ_G- and r_G-invariant; see [7] for details. If $\{x_s : s \in G\}$ is a bounded family of elements of a Banach space E, then for all $x \in E$ and $p \geq 1$, the functions $g(s) := \|x_s - x\|^p$ and $h(s) := \langle x_s, x\rangle$ (if E is a Hilbert space) are in $RUC(G)$.

3. The Hilbert Space Setting

In this section, we prove fixed point theorems for Lipschitzian semigroups defined on nonconvex domains in Hilbert spaces.

Theorem 3.1. *Let C be a nonempty subset of a real Hilbert space H, let G be a semitopological semigroup such that $RUC(G)$ has a left invariant mean μ, and let $\mathcal{F} = \{T_t : t \in G\}$ be a Lipschitzian semigroup on C such that $\mu_t(k_t^2) < 2$. Suppose that $\{T_t x : t \in G\}$ is bounded and $\cap_{s\in G}\, \overline{co}\{T_{st}x : t \in G\}$ is contained in C for all $x \in C$. Then there exists a point $z \in C$ for which $T_t z = z$ for all $t \in G$.*

Proof. Let $x_0 \in C$. It is easily seen that the functional $\mu_t\langle T_t x_0, x\rangle$, $x \in H$, is a continuous linear functional on H. By Riesz's representation theorem, there is a unique element $x_1 \in H$ satisfying $\mu_t\langle T_t x_0, x\rangle = \langle x_1, x\rangle$, $\forall x \in H$. By a routine argument (cf. [3] and [9]) via the separation theorem, we have $x_1 \in \bigcap_{s\in G} \overline{co}\{T_{st}x_0 : t \in G\}$ and hence by assumption, x_1 does remain in C. Therefore, we can continue the above procedure to obtain a sequence $\{x_n\}_{n=1}^{\infty}$ in C satisfying the following property:

$$\mu_t\langle T_t x_{n-1}, x\rangle = \langle x_n, x\rangle, \quad \forall x \in H, \quad \forall n \geq 1. \tag{3.1}$$

Noting the fact that for all $u, v \in H$, the function $h(t) := \|T_t u - v\|^2$ is in $RUC(G)$, it follows from (3.1) that

$$\begin{aligned}
&\mu_t\|T_t x_{n-1} - x\|^2 = \mu_t\|(T_t x_{n-1} - x_n) + (x_n - x)\|^2 = \mu_t(\|T_t x_{n-1} - x_n\|^2 + \\
&+ \|x_n - x\|^2 + 2\,\langle T_t x_{n-1} - x_n, x_n - x\rangle) = \mu_t\|T_t x_{n-1} - x_n\|^2 + \\
&+ \|x_n - x\|^2 + 2\,\mu_t\langle T_t x_{n-1} - x_n, x_n - x\rangle = \mu_t\|T_t x_{n-1} - x_n\|^2 + \\
&+ \|x_n - x\|^2 + 2\,\langle x_n - x_n, x_n - x\rangle = \mu_t\|T_t x_{n-1} - x_n\|^2 + \|x_n - x\|^2.
\end{aligned}$$

This shows that x_n is the unique minimizer of the convex function $\mu_t\|T_t x_{n-1} - x\|^2$ over H and in particular, taking $x = T_s x_n$ and noting that μ is left invariant, we get

$$\begin{aligned}
\mu_t\|T_t x_{n-1} - x_n\|^2 + \|x_n - T_s x_n\|^2 = \mu_t\|T_t x_{n-1} - T_s x_n\|^2 = \\
= \mu_t\|T_{st} x_{n-1} - T_s x_n\|^2 = \mu_t\|T_s T_t x_{n-1} - T_s x_n\|^2 \leq k_s^2 \mu_t\|T_t x_{n-1} - x_n\|^2.
\end{aligned}$$

It follows that

$$\|x_n - T_s x_n\|^2 \leq (k_s^2 - 1)\mu_t\|T_t x_{n-1} - x_n\|^2. \tag{3.2}$$

Set $A = \mu_s(k_s^2 - 1)$, $r_n = \mu_t \|T_t x_n - x_{n+1}\|^2$ and $R_n = \mu_t \|T_t x_n - x_n\|^2$. Then from (3.2) we have

$$R_n \leq A r_{n-1} \leq A R_{n-1} \leq \cdots \leq A^n R_0. \tag{3.3}$$

It then follows from (3.3) that

$$\begin{aligned} \|x_{n+1} - x_n\|^2 &= \mu_t \|(x_{n+1} - T_t x_n) + (T_t x_n - x_n)\|^2 \leq \\ &\leq 2\,\mu_t(\|T_t x_n - x_{n+1}\|^2 + \|T_t x_n - x_n\|^2) = \\ &= 2(r_n + R_n) \leq 4\,R_n \leq 4\,A^n R_0. \end{aligned}$$

Since $A < 1$, we see that $\{x_n\}$ is Cauchy and hence convergent in norm. Let z be the limit of $\{x_n\}$. We claim that z is a common fixed point of $\mathcal{F}$. In fact, for any $s \in G$, since μ is left invariant, we have

$$\begin{aligned} \|T_s z - z\|^2 &= \mu_t(\|(T_s z - T_{st} x_n) + (T_{st} x_n - z)\|^2 \leq \\ &\leq 2\,\mu_t(\|T_{st} x_n - T_s z\|^2 + \|T_{st} x_n - z\|^2) \leq \\ &\leq 2(k_s^2\,\mu_t \|T_t x_n - z\|^2 + \mu_t \|T_{st} x_n - z\|^2) = \\ &= 2(1 + k_s^2)\,\mu_t \|T_t x_n - z\|^2 = \\ &= 2(1 + k_s^2)\,\mu_t \|(T_t x_n - x_n) + (x_n - z)\|^2 \leq \\ &\leq 4(1 + k_s^2)(\mu_t \|T_t x_n - x_n\|^2 + \|x_n - z\|^2) = \\ &= 4(1 + k_s)(R_n + \|x_n - z\|^2) \to 0 \text{ as } n \to \infty. \end{aligned}$$

Therefore, $T_s z = z$. $\square$

Corollary 3.1 (Theorem 3.5 [4]). *If $\mathcal{F} = \{T_t : t \in G\}$ is a nonexpansive semigroup on a closed convex subset C of a Hilbert space H, $RUC(G)$ has a left invariant mean, and there exists an $x \in C$ such that $\{T_s(x) : s \in G\}$ is bounded, then $\mathcal{F}$ has a common fixed point in C.*

Let X be a subspace of $B(G)$ containing constants. Following Mizoguchi and Takahashi [7T], we say that a real valued function on X is a *submean* on X if the following conditions are fulfilled:

(1) $\mu(f + g) \leq \mu(f) + \mu(g), \quad \forall f, g \in X$;
(2) $\mu(\alpha f) \leq \alpha\,\mu(f), \quad \forall f \in X,\ \forall \alpha \geq 0$;
(3) $\forall\, f, g \in X, f \leq g \Longrightarrow \mu(f) \leq \mu(g)$;
(4) $\mu(c) = c$ for all constants c.

Theorem 3.2. *Let H be a real Hilbert space, C a nonempty subset of H, X an ℓ_G-invariant subspace of $B(G)$ containing constants that has a left invariant submean μ on X, and $\mathcal{F} = \{T_t : t \in G\}$ a Lipschitzian semigroup on C. Suppose that $\{T_t x : t \in G\}$ is bounded for some $x \in C$ and $\cap_{s \in G} \overline{co}\{T_{st} x : t \in G\} \subset C$ for all $x \in C$. Suppose also that for all $u, v \in C$, the function f on G defined by $f(t) = \|T_t u - v\|^2$ and the function* *h on G defined by* $h(t) = k_t^2$ *belong to* X *and* $\mu_t(k_t^2) < 2$. *Then there is a point* $z \in C$ *such that* $T_t z = z$ *for all* $t \in G$.

Proof. Let $x_0 \in C$ and define the function r_0 on H by $r_0(x) = \mu_t\|T_t x_0 - x\|^2$, $x \in H$. Note that r_0 is well defined since by assumption, the function $t \mapsto \|T_t x_0 - x\|^2$ is in X for every $x \in H$. As r_0 is strictly convex and continuous and $r_0(x) \to \infty$ as $\|x\| \to \infty$, there is a unique element $x_1 \in H$ such that $r_0 := r_0(x_1) = \inf\{r_0(x) : x \in H\}$. We claim that this x_1 belongs to $\cap_{s\in G} \overline{co}\{T_{st}x_0 : t \in G\}$ and thus to C by our hypothesis. Indeed, if we denote by P_s the nearest point projection of H onto the set $\overline{co}\{T_{st}x_0; t \in G\}$, then, as P_s is nonexpansive and μ is left invariant, we get

$$\begin{aligned} r_0(P_s x_1) &= \mu_t\|T_t x_0 - P_s x_1\|^2 = \mu_t\|T_{st}x_0 - P_s x_1\|^2 = \\ &= \mu_t\|P_s T_{st}x_0 - P_s x_1\|^2 \le \mu_t\|T_{st}x_0 - x_1\|^2 = \\ &= \mu_t\|T_t x_0 - x_1\|^2 = r_0, \end{aligned}$$

which shows that $P_s x_1$ is also a minimizer of r_0 and hence by uniqueness, $P_s x_1 = x_1$, i.e., $x_1 \in \cap_{s\in G} \overline{co}\{T_{st}x_0 : t \in G\}$. This proves the claim. Repeating the above process, we obtain a sequence $\{x_n\}$ in C with the following property:

$$x_n \in \bigcap_{s\in G} \overline{co}\{T_{st}x_{n-1} : t \in G\} \quad \forall n \ge 1$$

and x_n is the unique minimizer over H of the functional $r_n(\cdot)$ defined by $r_n(x) = \mu_t\|T_t x_{n-1} - x\|^2$, $x \in H$. Now by the same argument as in the proof of Theorem 3.1, we conclude that $\{x_n\}$ converges strongly to a common fixed point $z \in C$. $\square$

Corollary 3.2 (Theorem 1 [8]). *Let* C *be a closed convex subset of a Hilbert space* H *and* X *be an* ℓ_G*-invariant subspace of* $B(G)$ *containing constants which has a left invariant submean* μ. *Let* $\mathcal{F} = \{T_t : t \in G\}$ *be a Lipschitzian semigroup on* C *such that* $\{T_s x : s \in G\}$ *is bounded for some* $x \in C$. *If for each* $u, v \in C$, *the function* $f(t) := \|T_t u - v\|^2$ *and the function* $g(t) := k_t^2$ $(t \in G)$ *belong to* X *and* $\mu_s(k_s^2) < 2$, *then there is* $z \in C$ *such that* $T_s z = z$ *for all* $s \in G$.

We now extend Theorem 3.3 of Lau [4] to a wider class of Lipschitzian semigroups which are defined on nonconvex domains.

Theorem 3.3. *Suppose* H *is a real Hilbert space,* C *is a nonempty subset of* H, *and* $\mathcal{F} = \{T_t : t \in G\}$ *is a Lipschitzian semigroup on* C. *Suppose also* $AP(G)$ *has a left invariant mean* μ. *If* $\mu_t(k_t^2) \le 1$ *and if there exists an* $x \in C$ *such that* $\{T_t x : t \in G\}$ *is relatively compact in norm and* $\cap_{s\in G} \overline{co}\{T_{st}x : t \in G\}$ *is contained in* C, *then* $\mathcal{F}$ *has a common fixed point.*

Proof. Since $\{T_t x : t \in G\}$ is relatively compact, by Lemma 3.1 of Lau [4], for all $y \in H$, the functions h and g defined on G by $h(t) = \langle y, T_t x\rangle$ and $g(t) = \|y - T_t x\|^2$ are both in $AP(G)$. So we have a unique $z \in H$ such that $\mu_t\langle T_t x, y\rangle = \langle z, y\rangle$, $\forall y \in H$. As seen before, we have (i) $z \in \cap_{s\in G} \overline{co}\{T_{st}x : t \in G\}$ and hence $z \in C$ and (ii) $\mu_t\|T_t x - y\|^2 = \mu_t\|T_t x - z\|^2 + \|y - z\|^2$ $\forall y \in H$. In particular, we have for all $s \in G$,

$$\mu_t\|T_t x - T_s z\|^2 = \mu_t\|T_t x - z\|^2 + \|T_s z - z\|^2.$$

Noting that

$$\mu_t\|T_t x - T_s z\|^2 = \mu_t\|T_{st} x - T_s z\|^2 = \mu_t\|T_s T_t x - T_s z\|^2 \le k_s^2\mu_t\|T_t x - z\|^2,$$

we get for all $s \in G$, $\|T_s z - z\|^2 \le (k_s^2 - 1)\mu_t\|T_t x - z\|^2$. Hence

$$\mu_s\|T_s z - z\|^2 \le (\mu_s(k_s^2 - 1))\mu_t\|T_t x - z\|^2 \le 0$$

for $\mu_s(k_s^2) \le 1$; namely, $\mu_s\|T_s z - z\|^2 = 0$. Now for $a \in G$, we have

$$\begin{aligned}\|T_a z - z\|^2 &= \mu_s\|(T_a z - T_s z) + (T_s z - z)\|^2 \le 2\,\mu_s(\|T_a z - T_s z\|^2 + \\ &+ \|T_s z - z\|^2) = 2(\mu_s\|T_s z - T_a z\|^2 + \mu_s\|T_s z - z\|^2) = \\ &= 2\,\mu_s\|T_{as} z - T_a z\|^2 = 2\,\mu\|T_a T_s z - T_a z\|^2 \le \\ &\le 2\,k_a^2\mu_s\|T_s z - z\|^2 = 0.\end{aligned}$$

Therefore $T_a z = z$. □

Corollary 3.3 (Theorem 3.2 [4]). *If C is a closed convex subset of a Hilbert space H, $\mathcal{F} = \{T_t : t \in G\}$ is a nonexpansive semigroup on C, $AP(G)$ has a left invariant mean, and $x \in C$ such that $\{T_s x : s \in G\}$ is relatively compact, then C contains a common fixed point for $\mathcal{F}$.*

By the same proof as in Theorem 3.1, we get immediately the following result.

Theorem 3.4. *Suppose H and C are as in Theorem 3.3, suppose $AP(G)$ has a left invariant mean μ, and suppose $\mathcal{F} = \{T_t : t \in G\}$ is a Lipschitzian semigroup on C such that $\mu_t(k_t^2) < 2$. If, for every $x \in C$, $\{T_t x : t \in G\}$ is relatively compact in norm and $\cap_{s\in G} \overline{co}\{T_{st}x : t \in G\}$ is contained in C, then $\mathcal{F}$ has a common fixed point.*

4. The Banach Space Setting

In this section we study the existence of fixed points for Lipschitzian mappings defined on nonconvex domains in Banach spaces. So suppose C is a nonempty subset of a Banach space X and $\mathcal{F} = \{T_t : t \in G\}$ is a Lipschitzian semigroup defined on C. (Here G is as in Section 3 a semitopological semigroup.) We will employ the following condition introduced by Goebel, Kirk, and Thele in [2]: A nonempty subset E of C is said to satisfy the property

(P): For every $x \in E$ and $\varepsilon > 0$, there exists $s \in G$ such that $\text{dist}(T_t x, E) < \varepsilon$ for all $t \geq s$.

Recall that the modulus of convexity of a Banach space X is defined as the function

$$\delta_X(\varepsilon) = \inf\left\{1 - \frac{1}{2}\|x+y\| : x, y \in B_X \text{ with } \|x-y\| \geq \varepsilon\right\}, \quad 0 \leq \varepsilon \leq 2,$$

where B_X is the closed unit ball of X. A Banach space X is said to be *uniformly convex* if $\delta_X(\varepsilon) > 0$ for all $0 < \varepsilon \leq 2$. It is said to be p-uniformly convex for some $p \geq 2$ if there exists a constant $d > 0$ such that $\delta_X(\varepsilon) \geq d\,\varepsilon^p$ for $0 \leq \varepsilon \leq 2$. It is known that a Hilbert space is 2-uniformly convex and an L^p $(1 < p < \infty)$ space is max(2,p)-uniformly convex. We shall need the following characterization of a p-uniformly convex Banach space.

Proposition ([cf. [12]). *Given a number $p \geq 2$. A Banach space X is p-uniformly convex if and only if there exists a constant $d = d_p > 0$ such that*

$$\|\lambda x + (1-\lambda)y\|^p \leq \lambda\|x\|^p + (1-\lambda)\|y\|^p - dW_p(\lambda)\|x-y\|^p \tag{4.1}$$

for all $x, y \in X$ and $0 \leq \lambda \leq 1$, where $W_p(\lambda) = \lambda^p(1-\lambda) + \lambda(1-\lambda)^p$.

Theorem 4.1. *Suppose X is a p-uniformly convex Banach space for some $p \geq 2$, C is a nonempty subset of X, G is a semitopological semigroup that is left reversible, and $\mathcal{F} = \{T_t : t \in G\}$ is a uniformly k-Lipschitzian semigroup on C with $k < (1+d)^{\frac{1}{p}}$, d being the constant appearing in* (4.1). *Suppose also there exist an $\bar{x} \in C$ such that $\{T_t\bar{x} : t \in G\}$ is bounded and a nonempty bounded closed convex subset E of C with Property (P). Then there exists a point $z \in E$ such that $T_t z = z$ for all $t \in G$.*

Proof. Since $\{T_t\bar{x} : t \in G\}$ is bounded, it is easily seen that for all $x \in C$, $\{T_t x : t \in G\}$ is bounded. Now choose any $x_0 \in E$ and define a functional f on E by $f(x) = \inf_s \sup_{t \geq s} \|T_t x_0 - x\|^p$, $x \in E$. By Lemma 3 of [11], we have a unique $x_1 \in E$ such that $f(x_1) \leq f(x) - d\|x - x_1\|^p$, $\forall x \in E$.

Continuing this procedure, we construct a sequence $\{x_n\}_{n=1}^{\infty}$ in E such that for every integer $n \geq 1$,

$$\inf_s \sup_{t \geq s} \|T_t x_{n-1} - x_n\|^p \leq \inf_s \sup_{t \geq s} \|T_t x_{n-1} - x\|^p - \\ - d\|x - x_n\|^p, \quad \forall x \in E. \tag{4.2}$$

Now by Property (P), we can find for each $n \geq 1$ an $a_n \in G$ such that

$$\operatorname{dist}(T_a x_n, E) < A^n \quad \text{for all } a \geq a_n, \tag{4.3}$$

where $A = \frac{k^p - 1}{d} < 1$. For each fixed $a \geq a_n$, from (4.3) we can thus find a $y_n \in E$ such that

$$\|T_a x_n - y_n\| < A^n. \tag{4.4}$$

Using the mean value theorem, it is easy to see that for all $x, y \in X$,

$$|\|y\|^p - \|x\|^p| \leq p\,[\max\{\|x\|, \|y\|\}]^{p-1} |\|y\| - \|x\|| \leq p(\|x\| + \|y\|)^{p-1} \|y - x\|.$$

Since all the involved sequences are bounded, we can find a constant M big enough so that for all $t, a \in G$ and $n \geq 1$,

$$p(\|T_t x_{n-1} - y_n\| + \|T_t x_{n-1} - T_a x_n\|)^{p-1} \leq M/2$$

and

$$p(\|T_a x_n - x_n\| + \|y_n - x_n\|)^{p-1} \leq M/2.$$

It thus follows from (4.2) and (4.4) that

$$\begin{aligned}
&\inf_s \sup_{t \geq s} \|T_t x_{n-1} - x_n\|^p \leq \inf_s \sup_{t \geq s} \|T_t x_{n-1} - y_n\|^p - d\|y_n - x_n\|^p \leq \\
&\leq \inf_s \sup_{t \geq s} \|T_t x_{n-1} - T_a x_n\|^p - d\|T_a x_n - x_n\|^p + \sup_t (\|T_t x_{n-1} - \\
&- y_n\|^p - \|T_t x_{n-1} - T_a x_n\|^p) + d(\|T_a x_n - x_n\|^p - \|y_n - x_n\|^p) \leq \\
&\leq \inf_s \sup_{t \geq s} \|T_t x_{n-1} - T_a x_n\|^p - d\|T_a x_n - x_n\|^p + M\|T_a x_n - y_n\| \leq \\
&\leq \inf_s \sup_{t \geq s} \|T_t x_{n-1} - T_a x_n\|^p - d\|T_a x_n - x_n\|^p + MA^n \leq \\
&\leq \inf_s \sup_{t \geq s} \|T_{at} x_{n-1} - T_a x_n\|^p - d\|T_a x_n - x_n\|^p + MA^n \leq \\
&\leq k_a^p \inf_s \sup_{t \geq s} \|T_t x_{n-1} - x_n\|^p - d\|T_a x_n - x_n\|^p + MA^n.
\end{aligned}$$

Hence for all $a \geq a_n$,

$$d\|T_a x_n - x_n\|^p \leq (k^p - 1) \inf_s \sup_{t \geq s} \|T_t x_{n-1} - x_n\|^p + MA^n,$$

which results in the conclusion

$$\inf_s \sup_{t\geq s} \|T_t x_n - x_n\|^p \leq A \inf_s \sup_{t\geq s} \|T_t x_{n-1} - x_n\|^p + M' A^n,$$

where $M' = M/d$. Write $R_n = \inf_s \sup_{t\geq s} \|T_t x_n - x_n\|^p$ and $r_n = \inf_s \sup_{t\geq s} \|T_t x_n - x_{n+1}\|^p$. Then $r_n \leq R_n$ by (4.2) and

$$R_n \leq A r_{n-1} + M' A^n \leq A R_{n-1} + M' A^n \leq A(A R_{n-2} + M' A^{n-1}) + \\ + M' A^n = A^2 R_{n-2} + 2M' A^n \leq \cdots \leq (R_0 + nM') A^n.$$

Therefore,

$$\|x_n - x_{n-1}\|^p = \inf_s \sup_{t\geq s} \|(x_n - T_t x_{n-1}) + (T_t x_{n-1} - x_{n-1})\|^p \leq \\ \leq 2^{p-1} \inf_s \sup_{t\geq s} (\|x_n - T_t x_{n-1}\|^p + \|T_t x_{n-1} - x_{n-1}\|^p) \leq \\ \leq 2^{p-1}(r_{n-1} + R_{n-1}) \leq 2^p R_{n-1} \leq 2^p (R_0 + (n-1)M') A^{n-1},$$

which shows that $\{x_n\}$ is Cauchy and hence convergent to some z strongly. We now show that this z is a common fixed point of $\mathcal{F}$. In fact, noting that the inequality (cf. [11])

$$\inf_s \sup_{t\geq s} \|T_t x - y\|^p \leq \inf_s \sup_{t\geq s} \|T_{at} x - y\|^p$$

is valid for all $x, y \in C$ and $a \in G$, we have for all $a \in G$,

$$\|z - T_a z\|^p \leq \inf_s \sup_{t\geq s} (\|z - x_n\| + \|x_n - T_t x_n\| + \\ + \|T_t x_n - T_a x_n\| + \|T_a x_n - T_a z\|)^p \leq 4^p \inf_s \sup_{t\geq s} ((1 + k^p)\|z - x_n\|^p + \\ + \|x_n - T_t x_n\|^p + \|T_t x_n - T_a x_n\|^p) \leq 4^p ((1 + k^p)\|z - x_n\|^p + R_n + \\ + \inf_s \sup_{t\geq s} \|T_{at} x_n - T_a x_n\|^p) \leq 4^p (1 + k^p)(\|z - x_n\|^p + R_n) \to 0 \ (n \to \infty).$$

Hence $T_a z = z$. □

Theorem 4.2. *Let X be a p-uniformly convex Banach space for some $p \geq 2$, C a nonempty subset of X, G a semitopological semigroup for which $RUC(G)$ has a left invariant mean μ, and $\mathcal{F} = \{T_t : t \in G\}$ a uniformly k-Lipschitzian semigroup on C such that $k < (1 + d)^{\frac{1}{p}}$ with d being the constant appearing in (4.1). Suppose there is a point $\bar{x} \in C$ for which the orbit $\{T_t \bar{x} : t \in G\}$ is bounded. Suppose also there exists a nonempty bounded closed convex subset E of C with the following property:*

(P^*) *For every $x \in E$ and $\varepsilon > 0$, there exists $s \in G$ such that dist$(T_{st} x, E) < \varepsilon$, $\forall t \in G$.*

Then there exists some $z \in E$ such that $T_t z = z$ for all $t \in G$.

Proof. Choose an arbitrary $x_0 \in E$. As the function $t \mapsto \|T_t u - v\|^p$ is in $RUC(G)$ for any $u \in C$ and $v \in X$, we can inductively construct a sequence $\{x_n\}_{n=1}^{\infty}$ in E in the following way: For each integer $n \geq 1$, $x_n \in E$ is the unique minimizer of the functional $\mu_t \|T_t x_{n-1} - x\|^2$ over E. Then by Lemma 2 of [11], we have

$$\mu_t \|T_t x_{n-1} - x_n\|^p \leq \mu_t \|T_t x_{n-1} - x\|^p - d\|x - x_n\|^p, \quad \forall x \in E. \tag{4.5}$$

Write $r_n = \inf\{\mu_t \|T_t x_n - x\|^p : x \in E\} = \mu_t \|T_t x_n - x_{n+1}\|^p$ and $R_n = \mu_t \|T_t x_n - x_n\|^p$. Then for a fixed integer $n \geq 1$, condition (P^*) yields an $s_n \in G$ such that

$$\operatorname{dist}(T_{s_n r} x_n, E) < A^n, \quad \forall r \in G, \tag{4.6}$$

where $A = (k^p - 1)/d < 1$. From (4.6) we can find a $y_n \in E$ (depending on r) such that $\|T_{s_n r} x_n - y_n\| < A^n$. It then follows from (4.5) that

$$\begin{aligned} r_{n-1} &\leq \mu_t \|T_t x_{n-1} - y_n\|^p - d\|y_n - x_n\|^p = \mu_t \|(T_t x_{n-1} - T_{s_n r} x_n) + \\ &+ (T_{s_n r} x_n - y_n)\|^p - d\|(y_n - T_{s_n r} x_n) + (T_{s_n r} x_n - x_n)\|^p \leq \\ &\leq \mu_t \|T_t x_{n-1} - T_{s_n r} x_n\|^p - d\|T_{s_n r} x_n - x_n\| + MA^n = \\ &= \mu_t \|T_{s_n r t} x_{n-1} - T_{s_n r} x_n\|^p - d\|T_{s_n r} x_n - x_n\|^p + MA^n \leq \\ &\leq k^p \, \mu_t \|T_t x_{n-1} - x_n\|^p - d\|T_{s_n r} x_n - x_n\|^p + MA^n, \end{aligned}$$

where $M > 0$ is some appropriate constant independent of $r \in G$, which can be found similarly to the proof of Theorem 4.2. Hence

$$\|T_{s_n r} x_n - x_n\|^p \leq \frac{k^p - 1}{d} r_{n-1} + \frac{M}{d} A^n = A r_{n-1} + M' A^n,$$

where $M' = M/d$, and

$$\begin{aligned} R_n = \mu_t \|T_t x_n - x_n\|^p &= \mu_r \|T_{s_n r} x_n - x_n\|^p \leq \\ &\leq A r_{n-1} + M' A^n \leq \cdots \leq (R_0 + nM') A^n. \end{aligned}$$

Therefore, we have

$$\begin{aligned} \|x_n - x_{n-1}\|^p &\leq \mu_t \|(x_n - T_t x_{n-1}) + (T_t x_{n-1} - x_{n-1})\|^p \leq \\ &\leq 2^p \, \mu(\|x_n - T_t x_{n-1}\|^p + \|T_t x_{n-1} - x_{n-1}\|^p) = \\ &= 2^p (r_{n-1} + R_{n-1}) \leq 2^{p+1} R_{n-1} \leq \\ &\leq 2^{p+1} (R_0 + (n-1)M') A^{n-1}, \end{aligned}$$

showing that $\{x_n\}$ is Cauchy and hence strongly convergent. Let z be the limit. Now for all $a \in G$, we have

$$\|z - T_a z\|^p \le \mu_t(\|z - x_n\| + \|x_n - T_t x_n\| + \|T_t x_n - T_a x_n\| + \\ + \|T_a x_n - T_a z\|)^p \le 4^p\, \mu_t(\|z - x_n\|^p + \|x_n - T_t x_n\|^p + \|T_t x_n - T_a x_n\|^p + \\ + \|T_a x_n - T_a z\|^p) \le 4^p((1 + k^p)\|x_n - z\|^p + R_n + \mu_t\|T_{at} x_n - T_a z_n\|^p) \le \\ \le 4^p(1 + k^p)(\|x_n - z\|^p + R_n) \ \to 0 \ (n \to \infty).$$

Hence $T_a z = z$. $\square$

In L^p $(1 < p < \infty)$, we have the following inequalities (cf. [5], [6], [12]):

$$\|\lambda x + (1-\lambda)y\|^q \le \lambda\|x\|^q + (1-\lambda)\|y\|^q - d_p W_q(\lambda)\|x - y\|^q$$

for all $x, y \in L^p$ and $\lambda \in [0, 1]$, where $q = \max\{2, p\}$, $W_q(\lambda) = \lambda^q(1-\lambda) + \lambda(1-\lambda)^q$, and

$$d_p = \begin{cases} \frac{(1+t_p^{p-1})}{(1+t_p)^{p-1}}, & \text{if } 2 < p < \infty; \\ p - 1, & \text{if } 1 < p \le 2. \end{cases}$$

t_p is the unique solution of the equation $(p-2)t^{p-1} + (p-1)t^{p-2} - 1 = 0$, $t \in (0, 1)$. Thus we have the following consequence of Theorems 4.1 and 4.2.

Corollary 4.1. *Let C be a nonempty subset of L^p $(1 < p < \infty)$, G a semitopological semigroup which is left reversible or for which the space $RUC(G)$ has an invariant mean, and $\mathcal{F} = \{T_t : t \in G\}$ a uniformly k-Lipschitzian semigroup on C with $k < \sqrt{p}$ if $1 < p \le 2$ or $k < 1 + (1 + t_p^{p-1})$ $(1+t_p)^{1-p}]^{\frac{1}{p}}$ if $2 < p < \infty$. Suppose there exists an $x \in C$ such that the orbit $\{T_t x : t \in G\}$ is bounded. Suppose also there exists a nonempty bounded closed convex subset of C which possesses Property (P) in the case where G is left reversible or Property (P^*) in the case $RUC(G)$ has an invariant mean. Then there exists a $z \in E$ such that $T_s z = z$ for all $s \in G$.*

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