

PROPERTIES OF CERTAIN INTEGRAL OPERATORS

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ABSTRACT. Two integral operators P^α and Q_β^α for analytic functions in the open unit disk are introduced. The object of the present paper is to derive some properties of integral operators P^α and Q_β^α .

1. INTRODUCTION

Let A be the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disk $U = \{z : |z| < 1\}$. Recently, Jung, Kim, and Srivastava [1] have introduced the following one-parameter families of integral operators:

$$P^\alpha f = P^\alpha f(z) = \frac{2^\alpha}{z\Gamma(\alpha)} \int_0^z \left(\log \frac{z}{t}\right)^{\alpha-1} f(t) dt \quad (\alpha > 0), \tag{1.2}$$

$$Q_\beta^\alpha f = Q_\beta^\alpha f(z) = \left(\frac{\alpha + \beta}{\beta}\right) \frac{\alpha}{z^\beta} \int_0^z \left(1 - \frac{t}{z}\right)^{\alpha-1} t^{\beta-1} f(t) dt \tag{1.3}$$

($\alpha > 0, \beta > -1$)

and

$$J_\alpha f = J_\alpha f(z) = \frac{\alpha + 1}{z^\alpha} \int_0^z t^{\alpha-1} f(t) dt \quad (\alpha > -1), \tag{1.4}$$

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where $\Gamma(\alpha)$ is the familiar Gamma function, and (in general)

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha - \beta + 1)\Gamma(\beta + 1)} = \begin{bmatrix} \alpha \\ \alpha - \beta \end{bmatrix}. \quad (1.5)$$

For $\alpha \in \mathbb{N} = \{1, 2, 3, \dots\}$, the operators P^α , Q_1^α , and J_α were considered by Bernardi ([2], [3]). Further, for a real number $\alpha > -1$, the operator J_α was used by Owa and Srivastava [4], and by Srivastava and Owa ([8], [6]).

Remark 1. For $f(z) \in A$ given by (1.1), Jung, Kim, and Srivastava [1] have shown that

$$P^\alpha f(z) = z + \sum_{n=2}^{\infty} \left(\frac{2}{n+1}\right)^\alpha a_n z^n \quad (\alpha > 0), \quad (1.6)$$

$$Q_\beta^\alpha f(z) = z + \frac{\Gamma(\alpha + \beta + 1)}{\Gamma(\beta + 1)} \sum_{n=2}^{\infty} \frac{\Gamma(\beta + n)}{\Gamma(\alpha + \beta + n)} a_n z^n \quad (1.7)$$

$(\alpha > 0, \beta > -1)$

and

$$J_\alpha f(z) = z + \sum_{n=2}^{\infty} \left(\frac{\alpha + 1}{\alpha + n}\right) a_n z^n \quad (\alpha > -1). \quad (1.8)$$

By virtue of (1.6) and (1.8), we see that

$$J_\alpha f(z) = Q_\alpha^1 f(z) \quad (\alpha > -1). \quad (1.9)$$

2. AN APPLICATION OF THE MILLER–MOCANU LEMMA

To derive some properties of operators, we have to recall here the following lemma due to Miller and Mocanu [7].

Lemma 1. *Let $w(u, v)$ be a complex valued function,*

$$w : D \rightarrow \mathbb{C}, \quad D \subset \mathbb{C} \times \mathbb{C} \quad (\mathbb{C} \text{ is the complex plane}),$$

and let $u = u_1 + iu_2$ and $v = v_1 + iv_2$. Suppose that the function $w(u, v)$ satisfies the following conditions:

- (i) $w(u, v)$ is continuous in D ;
- (ii) $(1, 0) \in D$ and $\operatorname{Re}\{w(1, 0)\} > 0$;
- (iii) $\operatorname{Re}\{w(iu_2, v_1)\} \leq 0$ for all $(iu_2, v_1) \in D$ and such that $v_1 \leq -(1 + u_2^2)/2$.

Let $p(z)$ be regular in U and $p(z) = 1 + p_1 z + p_2 z^2 + \dots$ such that $(p(z), zp'(z)) \in D$ for all $z \in U$. If $\operatorname{Re}\{w(p(z), zp'(z))\} > 0$ ($z \in U$), then $\operatorname{Re}\{p(z)\} > 0$ ($z \in U$).

Applying the above lemma, we derive

Theorem 1. *If $f(z) \in A$ satisfies*

$$\operatorname{Re} \left\{ \frac{P^{\alpha-2}f(z)}{P^{\alpha-1}f(z)} \right\} > \beta \quad (\alpha > 2; z \in U) \tag{2.1}$$

for some β ($\beta < 1$), then

$$\operatorname{Re} \left\{ \frac{P^{\alpha-1}f(z)}{P^\alpha f(z)} \right\} > \frac{4\beta - 1 + \sqrt{16\beta^2 - 8\beta + 17}}{8} \quad (z \in U). \tag{2.2}$$

Proof. Noting that

$$z(P^\alpha f(z))' = 2P^{\alpha-1}F(z) - P^\alpha f(z) \quad (\alpha > 1), \tag{2.3}$$

we have

$$\frac{z(P^\alpha f(z))'}{P^\alpha f(z)} = 2 \frac{P^{\alpha-1}f(z)}{P^\alpha f(z)} - 1 \quad (\alpha > 1). \tag{2.4}$$

Define the function $p(z)$ by

$$\frac{P^{\alpha-1}f(z)}{P^\alpha f(z)} = \gamma + (1 - \gamma)p(z) \tag{2.5}$$

with

$$\gamma = \frac{4\beta - 1 + \sqrt{16\beta^2 - 8\beta + 17}}{8}. \tag{2.6}$$

Then $p(z) = 1 + p_1z + p_2z^2 + \dots$ is analytic in the open unit disk U . Since

$$\frac{z(P^{\alpha-1}f(z))'}{P^{\alpha-1}f(z)} - \frac{z(P^\alpha f(z))'}{P^\alpha f(z)} = \frac{(1 - \gamma)zp'(z)}{\gamma + (1 - \gamma)p(z)}, \tag{2.7}$$

or

$$\frac{P^{\alpha-2}f(z)}{P^{\alpha-1}f(z)} = \frac{P^{\alpha-1}f(z)}{P^\alpha f(z)} + \frac{(1 - \gamma)zp'(z)}{2\{\gamma + (1 - \gamma)p(z)\}}, \tag{2.8}$$

we have

$$\begin{aligned} & \operatorname{Re} \left\{ \frac{P^{\alpha-2}f(z)}{P^{\alpha-1}f(z)} - \beta \right\} = \\ & = \operatorname{Re} \left\{ \gamma + (1 - \gamma)p(z) + \frac{(1 - \gamma)zp'(z)}{2\{\gamma + (1 - \gamma)p(z)\}} - \beta \right\} > 0. \end{aligned} \tag{2.9}$$

Therefore, if we define the function $w(u, v)$ by

$$w(u, v) = \gamma - \beta + (1 - \gamma)u + \frac{(1 - \gamma)v}{2\{\gamma + (1 - \gamma)u\}}, \tag{2.10}$$

then we see that

- (i) $w(u, v)$ is continuous in $D = \left(C - \left\{\frac{\gamma}{\gamma-1}\right\}\right) \times C$;
(ii) $(1, 0) \in D$ and $\operatorname{Re}\{w(1, 0)\} = 1 - \beta > 0$;
(iii) for all $(iu_2, v_1) \in D$ and such that $v_1 \leq -(1 + u_2^2)/2$,

$$\begin{aligned} \operatorname{Re}\{w(iu_2, v_1)\} &= \gamma - \beta + \frac{\gamma(1-\gamma)v_1}{2\{\gamma^2 + (1-\gamma)^2u_2^2\}} \leq \\ &\leq \gamma - \beta - \frac{\gamma(1-\gamma)(1+u_2^2)}{4\{\gamma^2 + (1-\gamma)^2u_2^2\}} = \\ &= -\frac{(1-\gamma)(\gamma + 4(1-\gamma)\beta)}{4\{\gamma^2 + (1-\gamma)^2u_2^2\}}u_2^2 \leq 0. \end{aligned}$$

This implies that the function $w(u, v)$ satisfies the conditions in Lemma 1. Thus, applying Lemma 1, we conclude that

$$\begin{aligned} \operatorname{Re}\left\{\frac{P^{\alpha-1}f(z)}{P^\alpha f(z)}\right\} &> \gamma = \\ &= \frac{4\beta - 1 + \sqrt{16\beta^2 - 8\beta + 17}}{8} \quad (z \in U). \quad \square \quad (2.11) \end{aligned}$$

Taking the special values for β in Theorem 1, we have

Corollary 1. *Let $f(z)$ be in the class A . Then*

- (i) $\operatorname{Re}\left\{\frac{P^{\alpha-2}f(z)}{P^{\alpha-1}f(z)}\right\} > -\frac{1}{2} \quad (z \in U)$
 $\implies \operatorname{Re}\left\{\frac{P^{\alpha-1}f(z)}{P^\alpha f(z)}\right\} > \frac{1}{4} \quad (z \in U),$
(ii) $\operatorname{Re}\left\{\frac{P^{\alpha-2}f(z)}{P^{\alpha-1}f(z)}\right\} > -\frac{1}{4} \quad (z \in U)$
 $\implies \operatorname{Re}\left\{\frac{P^{\alpha-1}f(z)}{P^\alpha f(z)}\right\} > \frac{\sqrt{5}-1}{4} \quad (z \in U),$
(iii) $\operatorname{Re}\left\{\frac{P^{\alpha-2}f(z)}{P^{\alpha-1}f(z)}\right\} > 0 \quad (z \in U)$
 $\implies \operatorname{Re}\left\{\frac{P^{\alpha-1}f(z)}{P^\alpha f(z)}\right\} > \frac{\sqrt{17}-1}{8} \quad (z \in U),$
(iv) $\operatorname{Re}\left\{\frac{P^{\alpha-2}f(z)}{P^{\alpha-1}f(z)}\right\} > \frac{1}{4} \quad (z \in U)$
 $\implies \operatorname{Re}\left\{\frac{P^{\alpha-1}f(z)}{P^\alpha f(z)}\right\} > \frac{1}{2} \quad (z \in U),$

and

- (v) $\operatorname{Re}\left\{\frac{P^{\alpha-2}f(z)}{P^{\alpha-1}f(z)}\right\} > \frac{1}{2} \quad (z \in U)$
 $\implies \operatorname{Re}\left\{\frac{P^{\alpha-1}f(z)}{P^\alpha f(z)}\right\} > \frac{\sqrt{17}+1}{8} \quad (z \in U).$

Next, we have

Theorem 2. *If $f(z) \in A$ satisfies*

$$\operatorname{Re} \left\{ \frac{Q_\beta^{\alpha-2} f(z)}{Q_\beta^{\alpha-1} f(z)} \right\} > \gamma \quad (\alpha > 2, \beta > -1; z \in U) \tag{2.12}$$

for some γ $((\alpha + \beta - 3)/2(\alpha + \beta - 1) \leq \gamma < 1)$, then

$$\operatorname{Re} \left\{ \frac{Q_\beta^{\alpha-1} f(z)}{Q_\beta^\alpha f(z)} \right\} > \delta \quad (z \in U), \tag{2.13}$$

where

$$\delta = \frac{1 + 2\gamma(\alpha + \beta - 1) + \sqrt{(1 + 2\gamma(\alpha + \beta - 1))^2 + 8(\alpha + \beta)}}{4(\alpha + \beta)}. \tag{2.14}$$

Proof. By the definition of $Q_\beta^\alpha f(z)$, we know that

$$z(Q_\beta^\alpha f(z))' = (\alpha + \beta)Q_\beta^{\alpha-1} f(z) - (\alpha + \beta - 1)Q_\beta^\alpha f(z) \tag{2.15}$$

$(\alpha > 1, \beta > -1),$

so that

$$\frac{z(Q_\beta^\alpha f(z))'}{Q_\beta^\alpha f(z)} = (\alpha + \beta) \frac{Q_\beta^{\alpha-1} f(z)}{Q_\beta^\alpha f(z)} - (\alpha + \beta - 1). \tag{2.16}$$

We define the function $p(z)$ by

$$\frac{Q_\beta^{\alpha-1} f(z)}{Q_\beta^\alpha f(z)} = \delta + (1 - \delta)p(z). \tag{2.17}$$

Then $p(z) = 1 + p_1z + p_2z^2 + \dots$ is analytic in U . Making use of the logarithmic differentiations in both sides of (2.17), we have

$$\frac{z(Q_\beta^{\alpha-1} f(z))'}{Q_\beta^{\alpha-1} f(z)} = \frac{z(Q_\beta^\alpha f(z))'}{Q_\beta^\alpha f(z)} + \frac{(1 - \delta)zp'(z)}{\delta + (1 - \delta)p(z)}. \tag{2.18}$$

Applying (2.15) to (2.18), we obtain that

$$\begin{aligned} \frac{Q_\beta^{\alpha-2} f(z)}{Q_\beta^{\alpha-1} f(z)} &= \frac{1}{\alpha + \beta - 1} \left\{ (\alpha + \beta) \frac{Q_\beta^{\alpha-1} f(z)}{Q_\beta^\alpha f(z)} - 1 + \right. \\ &+ \left. \frac{(1 - \delta)zp'(z)}{\delta + (1 - \delta)p(z)} \right\} = \frac{1}{\alpha + \beta - 1} \left\{ \delta(\alpha + \beta) - 1 + \right. \\ &+ \left. (\alpha + \beta)(1 - \delta)p(z) + \frac{(1 - \delta)zp'(z)}{\delta + (1 - \delta)p(z)} \right\}, \end{aligned} \tag{2.19}$$

that is, that

$$\begin{aligned} \operatorname{Re} \left\{ \frac{Q_{\beta}^{\alpha-2} f(z)}{Q_{\beta}^{\alpha-1} f(z)} - \gamma \right\} &= \frac{1}{\alpha + \beta - 1} \operatorname{Re} \left\{ \delta(\alpha + \beta) - 1 + \right. \\ &\left. + (\alpha + \beta)(1 - \delta)p(z) + \frac{(1 - \delta)zp'(z)}{\delta + (1 - \delta)p(z)} \right\} - \gamma > 0. \end{aligned} \quad (2.20)$$

Now, we let

$$\begin{aligned} w(u, v) &= \frac{1}{\alpha + \beta - 1} \left\{ \delta(\alpha + \beta) - 1 + \right. \\ &\left. + (\alpha + \beta)(1 - \delta)u + \frac{(1 - \delta)v}{\delta + (1 - \delta)u} \right\} - \gamma \end{aligned} \quad (2.21)$$

with $u = u_1 + iu_2$ and $v = v_1 + iv_2$. Then $w(u, v)$ satisfies that

- (i) $w(u, v)$ is continuous in $D = \left(\mathbb{C} - \left\{ \frac{\delta}{\delta-1} \right\} \right) \times \mathbb{C}$;
- (ii) $(1, 0) \in D$ and $\operatorname{Re}\{w(1, 0)\} = 1 - \gamma > 0$;
- (iii) for all $(iu_2, v_1) \in D$ and such that $v_1 \leq -(1 + u_2^2)/2$,

$$\begin{aligned} \operatorname{Re}\{w(iu_2, v_1)\} &= \frac{1}{\alpha + \beta - 1} \left\{ \delta(\alpha + \beta) - 1 + \right. \\ &\left. + \frac{\delta(1 - \delta)v_1}{\delta^2 + (1 - \delta)^2 u_2^2} \right\} - \gamma \leq \frac{1}{\alpha + \beta - 1} \left\{ \delta(\alpha + \beta) - 1 - \right. \\ &\left. - \gamma(\alpha + \beta - 1) - \frac{\delta(1 - \delta)(1 + u_2^2)}{2\{\delta^2 + (1 - \delta)^2 u_2^2\}} \right\} \leq 0. \end{aligned}$$

Thus, the function $w(u, v)$ satisfies the conditions in Lemma 1. This shows that $\operatorname{Re}\{p(z)\} > 0$, or

$$\operatorname{Re} \left\{ \frac{Q_{\beta}^{\alpha-1} f(z)}{Q_{\beta}^{\alpha} f(z)} \right\} > \delta \quad (z \in U). \quad \square \quad (2.22)$$

If we take $\gamma = (\alpha + \beta - 3)/2(\alpha + \beta - 1)$ in Theorem 2, then we have

Corollary 2. *If $f(z) \in A$ satisfies*

$$\begin{aligned} \operatorname{Re} \left\{ \frac{Q_{\beta}^{\alpha-2} f(z)}{Q_{\beta}^{\alpha-1} f(z)} \right\} &> \frac{\alpha + \beta - 3}{2(\alpha + \beta - 1)} \\ &(\alpha > 2, \beta > -1; z \in U), \end{aligned}$$

then

$$\operatorname{Re} \left\{ \frac{Q_{\beta}^{\alpha-1} f(z)}{Q_{\beta}^{\alpha} f(z)} \right\} > \frac{1}{2} \quad (z \in U).$$

Further, letting $\alpha = 2 - \beta$ and $\gamma = 1/2$ in Theorem 2, we have

Corollary 3. *If $f(z) \in A$ satisfies*

$$\operatorname{Re} \left\{ \frac{Q_{\beta}^{-\beta} f(z)}{Q_{\beta}^{1-\beta} f(z)} \right\} > \frac{1}{2} \quad (\beta > -1; z \in U),$$

then

$$\operatorname{Re} \left\{ \frac{Q_{\beta}^{1-\beta} f(z)}{Q_{\beta}^{2-\beta} f(z)} \right\} > \frac{1 + \sqrt{5}}{4} \quad (z \in U).$$

3. AN APPLICATION OF JACK'S LEMMA

We need the following lemma due to Jack [8], (also, due to Miller and Mocanu [7]).

Lemma 2. *Let $w(z)$ be analytic in U with $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r < 1$ at a point $z_0 \in U$, then we can write*

$$z_0 w'(z_0) = k w(z_0), \tag{3.1}$$

where k is a real number and $k \geq 1$.

Applying Lemma 2 for the operator P^{α} , we have

Theorem 3. *If $f(z) \in A$ satisfies*

$$\operatorname{Re} \left\{ \frac{P^{\alpha-2} f(z)}{P^{\alpha-1} f(z)} \right\} > \beta \quad (\alpha > 2, \beta \leq 1/4; z \in U), \tag{3.2}$$

then

$$\operatorname{Re} \left\{ \frac{P^{\alpha-1} f(z)}{P^{\alpha} f(z)} \right\} > \gamma \quad (z \in U), \tag{3.3}$$

where

$$\gamma = \frac{3 + 4\beta + \sqrt{16\beta^2 - 40\beta + 9}}{8}. \tag{3.4}$$

Proof. Defining the function $w(u, v)$ by

$$\frac{P^{\alpha-1} f(z)}{P^{\alpha} f(z)} = \frac{1 - (1 - 2\gamma)w(z)}{1 + w(z)}, \tag{3.5}$$

we see that $w(z)$ is analytic in U and $w(0) = 0$. It follows from (3.5) that

$$\begin{aligned} \frac{z(P^{\alpha-1} f(z))'}{P^{\alpha-1} f(z)} &= \frac{z(P^{\alpha} f(z))'}{P^{\alpha} f(z)} - \\ &- \frac{(1 - 2\gamma)zw'(z)}{1 - (1 - 2\gamma)w(z)} - \frac{zw'(z)}{1 + w(z)}. \end{aligned} \tag{3.6}$$

Using (2.3), we obtain that

$$\begin{aligned} \frac{P^{\alpha-2}f(z)}{P^{\alpha-1}f(z)} &= \frac{1 - (1 - 2\gamma)w(z)}{1 + w(z)} - \\ &- \frac{1}{2} \frac{zw'(z)}{w(z)} \left(\frac{(1 - 2\gamma)w(z)}{1 - (1 - 2\gamma)w(z)} + \frac{w(z)}{1 + w(z)} \right). \end{aligned} \quad (3.7)$$

If we suppose that there exists a point $z_0 \in U$ such that

$$\max_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = 1 \quad (w(z_0) \neq -1),$$

then Lemma 2 gives us that

$$z_0 w'(z_0) = k w(z_0) \quad (k \geq 1).$$

Therefore, letting $w(z_0) = e^{i\theta}$ ($\theta \neq \pi$), we have

$$\begin{aligned} \operatorname{Re} \left\{ \frac{P^{\alpha-2}f(z_0)}{P^{\alpha-1}f(z_0)} \right\} &= \operatorname{Re} \left\{ \frac{1 - (1 - 2\gamma)w(z_0)}{1 + w(z_0)} - \right. \\ &\quad \left. - \frac{k}{2} \left(\frac{(1 - 2\gamma)w(z_0)}{1 - (1 - 2\gamma)w(z_0)} + \frac{w(z_0)}{1 + w(z_0)} \right) \right\} = \\ &= \operatorname{Re} \left\{ \frac{1 - (1 - 2\gamma)e^{i\theta}}{1 + e^{i\theta}} - \frac{k}{2} \left(\frac{(1 - 2\gamma)e^{i\theta}}{1 - (1 - 2\gamma)e^{i\theta}} + \frac{e^{i\theta}}{1 + e^{i\theta}} \right) \right\} = \\ &= \gamma - \frac{k}{2} \left(\frac{(1 - 2\gamma)(\cos \theta - (1 - 2\gamma))}{1 + (1 - 2\gamma)^2 - 2(1 - 2\gamma)\cos \theta} + \frac{1}{2} \right) \leq \\ &\leq \gamma - \frac{k}{2} \left(\frac{1}{2} - \frac{1 - 2\gamma}{2(1 - \gamma)} \right) \leq \frac{\gamma(3 - 4\gamma)}{4(1 - \gamma)} = \beta. \end{aligned} \quad (3.8)$$

This contradicts our condition (3.2). Therefore, we have

$$|w(z)| = \left| \frac{\frac{P^{\alpha-1}f(z)}{P^\alpha f(z)} - 1}{\frac{P^{\alpha-1}f(z)}{P^\alpha f(z)} + (1 - 2\gamma)} \right| < 1 \quad (z \in U), \quad (3.9)$$

which implies (3.3). \square

Taking $\beta = 0$ and $\beta = 1/4$ in Theorem 3, we have

Corollary 4. *Let $f(z)$ be in the class A . Then*

$$\begin{aligned} \operatorname{Re} \left\{ \frac{P^{\alpha-2}f(z)}{P^{\alpha-1}f(z)} \right\} &> 0 \quad (\alpha > 2; z \in U) \\ \implies \operatorname{Re} \left\{ \frac{P^{\alpha-1}f(z)}{P^\alpha f(z)} \right\} &> \frac{3}{4} \quad (z \in U) \end{aligned}$$

and

$$\begin{aligned} \operatorname{Re} \left\{ \frac{P^{\alpha-2} f(z)}{P^{\alpha-1} f(z)} \right\} &> -\frac{1}{4} \quad (\alpha > 2; z \in U) \\ \implies \operatorname{Re} \left\{ \frac{P^{\alpha-1} f(z)}{P^\alpha f(z)} \right\} &> \frac{1}{2} \quad (z \in U). \end{aligned}$$

Finally, we prove

Theorem 4. *If $f(z) \in A$ satisfies*

$$\operatorname{Re} \left\{ \frac{Q^{\alpha-2} f(z)}{Q^{\alpha-1} f(z)} \right\} > \gamma \quad (\alpha > 2; \beta > -1; z \in U) \tag{3.10}$$

for some γ ($\gamma < 1$), then

$$\operatorname{Re} \left\{ \frac{Q_\beta^{\alpha-1} f(z)}{Q_\beta^\alpha f(z)} \right\} > \delta \quad (z \in U), \tag{3.11}$$

where δ ($0 \leq \delta < 1$) is the smallest positive root of the equation

$$\begin{aligned} 2(\alpha + \beta)\delta^2 - \{2(\alpha + \beta)(\gamma + 1) - \\ -(2\gamma - 1)\}\delta + 2\{(\alpha + \beta - 1)\gamma + 1\} = 0. \end{aligned} \tag{3.12}$$

Proof. Defining the function $w(z)$ by

$$\frac{Q_\beta^{\alpha-1} f(z)}{Q_\beta^\alpha f(z)} = \frac{1 - (1 - 2\delta)w(z)}{1 + w(z)}, \tag{3.13}$$

we obtain that

$$\begin{aligned} \frac{Q_\beta^{\alpha-2} f(z)}{Q_\beta^{\alpha-1} f(z)} = \frac{1}{\alpha + \beta - 1} \left\{ (\alpha + \beta) \frac{1 - (1 - 2\delta)w(z)}{1 + w(z)} - 1 - \right. \\ \left. - \frac{zw'(z)}{w(z)} \left(\frac{(1 - 2\delta)w(z)}{1 - (1 - 2\delta)w(z)} + \frac{w(z)}{1 + w(z)} \right) \right\}. \end{aligned} \tag{3.14}$$

Therefore, supposing that there exists a point $z_0 \in U$ such that

$$\max_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = 1 \quad (w(z_0) \neq -1),$$

we see that

$$\begin{aligned} \operatorname{Re} \left\{ \frac{Q_\beta^{\alpha-2} f(z_0)}{Q_\beta^{\alpha-1} f(z_0)} \right\} &\leq \\ &\leq \frac{1}{\alpha + \beta - 1} \left\{ (\alpha + \beta)\delta - 1 - \frac{\delta}{2(1 - \delta)} \right\} = \gamma. \quad \square \end{aligned} \tag{3.15}$$

Making $\gamma = 0$ in Theorem 4, we have

Corollary 5. *If $f(z) \in A$ satisfies*

$$\operatorname{Re} \left\{ \frac{Q_{\beta}^{\alpha-2} f(z)}{Q_{\beta}^{\alpha-1} f(z)} \right\} > 0 \quad (\alpha > 2; \beta > -1; z \in U), \quad (3.16)$$

then

$$\operatorname{Re} \left\{ \frac{Q_{\beta}^{\alpha-1} f(z)}{Q_{\beta}^{\alpha} f(z)} \right\} > \frac{-1}{\alpha + \beta - 1} \quad (z \in U). \quad (3.17)$$

Letting $\gamma = 1/2$ in Theorem 4, we have

Corollary 6. *If $f(z) \in A$ satisfies*

$$\operatorname{Re} \left\{ \frac{Q_{\beta}^{\alpha-2} f(z)}{Q_{\beta}^{\alpha-1} f(z)} \right\} > \frac{1}{2} \quad (\alpha > 2; \beta > -1; z \in U), \quad (3.18)$$

then

$$\operatorname{Re} \left\{ \frac{Q_{\beta}^{\alpha-1} f(z)}{Q_{\beta}^{\alpha} f(z)} \right\} > \frac{\alpha + \beta - 3}{\alpha + \beta - 1} \quad (z \in U). \quad (3.19)$$

Further, if $\alpha = 3 - \beta$, we have

$$\begin{aligned} \operatorname{Re} \left\{ \frac{Q_{\beta}^{1-\beta} f(z)}{Q_{\beta}^{2-\beta} f(z)} \right\} &> \frac{1}{2} \quad (\beta > -1; z \in U) \\ \implies \operatorname{Re} \left\{ \frac{Q_{\beta}^{2-\beta} f(z)}{Q_{\beta}^{3-\beta} f(z)} \right\} &> 0 \quad (z \in U). \end{aligned}$$

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