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ON THE NUMBER OF REPRESENTATIONS OF INTEGERS BY SOME QUADRATIC FORMS IN TEN VARIABLES

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ABSTRACT. A method of finding the so-called Liouville's type formulas for the number of representations of integers by

$a_1(x_1^2+x_2^2)+a_2(x_3^2+x_4^2)+a_3(x_5^2+x_6^2)+a_4(x_7^2+x_8^2)+a_5(x_9^2+x_{10}^2)$

quadratic forms is developed.

In the papers [4,5] four classes of entire modular forms of weight 5 for the congruence subgroup $\Gamma_0(4N)$ are constructed. The Fourier coefficients of these modular forms have a simple arithmetical sense. This allows one to get sometimes the so-called Liouville's type formulas for the number of representations of positive integers by positive quadratic forms in ten variables.

In the present paper we consider positive primitive quadratic forms

$$f=a_1(x_1^2+x_2^2)+a_2(x_3^2+x_4^2)+a_3(x_5^2+x_6^2)+\\+a_4(x_7^2+x_8^2)+a_5(x_9^2+x_{10}^2). \tag{1}$$

For the purpose of illustration we obtain a formula for the number of representations of positive integers by the form (1) for $a_1=\cdots=a_4=1$ $a_5=4$. In a similar way one can investigate as well other forms of the kind (1). As is well known, Liouville obtained in 1865 the corresponding formula for $a_1=\cdots=a_5=1$ only.

1. Some known results

1.1. In this paper N,a,d,k,n,q,r,s,λ denote positive integers; b,u,v are odd positive integers; p is a prime number; ν,l are non-negative integers; $H,c,g,h,j,m,x,y,\alpha,\beta,\gamma,\delta$ are integers; i is an imaginary unit; z,τ are

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complex variables ($\operatorname{Im}\tau > 0$); $e(z) = \exp 2\pi i z$; $Q = e(\tau)$; $(\frac{h}{u})$ is the generalized Jacobi symbol. Further, $\sum_{h \bmod q}$ and ${\sum'}_{h \bmod q}$ denote respectively sums in which h runs a complete and a reduced residue system modulo q.

Let

$$S(h,q) = \sum_{j \bmod q} e\Big(\frac{hj^2}{q}\Big) \quad \text{(Gaussian sum)}, \tag{1.1}$$

$$c(h,q) = {\sum_{j \bmod q}}' e\Big(\frac{hj^2}{q}\Big) \quad \text{(Ramanujan's sum)}, \tag{1.2}$$

$$\vartheta_{gh}(z|\tau; c, N) =$$
$$= \sum_{m \equiv c \pmod N} (-1)^{h(m-c)/N} e\Big(\frac{1}{2N}\big(m+\frac{g}{2}\big)^2\tau\Big)\, e\Big(\big(m+\frac{g}{2}\big)z\Big) \tag{1.3}$$

(theta-function with characteristics g, h),

hence

$$\frac{\partial^n}{\partial z^n}\vartheta_{gh}(z|\tau; c, N) = (\pi i)^n \sum_{m \equiv c \pmod N} (-1)^{h(m-c)/N}(2m+g)^n \times$$
$$\times e\Big(\frac{1}{2N}\big(m+\frac{g}{2}\big)^2\tau\Big)\, e\Big(\big(m+\frac{g}{2}\big)z\Big). \tag{1.4}$$

Put

$$\begin{gathered} \vartheta_{gh}(\tau; c, N) = \vartheta_{gh}(0|\tau; c, N), \\ \vartheta_{gh}^{(n)}(\tau; c, N) = \frac{\partial^n}{\partial z^n}\vartheta_{gh}(z|\tau; c, N)\big|_{z=0}. \end{gathered} \tag{1.5}$$

It is known (see, for example, [3], p. 112, formulas (2.3) and (2.5)) that

$$\begin{gathered} \vartheta_{g+2j,h}(\tau; c, N) = \vartheta_{gh}(\tau; c+j, N), \\ \vartheta_{g+2j,h}^{(n)}(\tau; c, N) = \vartheta_{gh}^{(n)}(\tau; c+j, N), \end{gathered} \tag{1.6}$$

$$\begin{gathered} \vartheta_{gh}(\tau; c+N_j, N) = (-1)^{hj}\vartheta_{gh}(\tau; c, N), \\ \vartheta_{gh}^{(n)}(\tau; c+N_j, N) = (-1)^{hj}\vartheta_{gh}^{(n)}(\tau; c, N). \end{gathered} \tag{1.7}$$

From (1.3), in particular, according to the notations (1.5), it follows that

$$\vartheta_{gh}(\tau; 0, N) = \sum_{m=-\infty}^{\infty} (-1)^{hm} Q^{(2Nm+g)^2/8N}, \tag{1.8}$$

$$\vartheta_{gh}^{(n)}(\tau; 0, N) = (\pi i)^n \sum_{m=-\infty}^{\infty} (-1)^{hm}(2Nm+g)^n Q^{(2Nm+g)^2/8N}. \tag{1.9}$$

From (1.8) and (1.9) it follows that

$$\vartheta_{-g,h}(\tau;0,N)=\vartheta_{gh}(\tau;0,N),\ \ \vartheta^{(n)}_{-g,h}(\tau;0,N)=(-1)^n\vartheta^{(n)}_{gh}(\tau;0,N). \quad (1.10)$$

Everywhere in this paper a denote a least common multiple of the coefficients a_k of the quadratic form (1) and $\Delta = \prod_{k=1}^5 a_k^2$ is its determinant.

Denoting by $r(n;f)$ the number of representations of n by the form (1), we get

$$\prod_{k=1}^{5} \vartheta^2_{00}(\tau;0,a_k) = 1 + \sum_{n=1}^{\infty} r(n;f)Q^n. \quad (1.11)$$

Further, put

$$\theta(\tau;f) = 1 + \sum_{n=1}^{\infty} \rho(n;f)Q^n, \quad (1.12)$$

where

$$\rho(n;f) = \frac{\pi^5}{4!\Delta^{1/2}}\, n^4 \sum_{q=1}^{\infty} A(q) \quad (1.13)$$

(singular series of the problem) and

$$A(q) = q^{-10} \sideset{}{'}\sum_{h \bmod q} e\Big(-\frac{hn}{q}\Big) \prod_{k=1}^{5} S^2(a_k h, q). \quad (1.14)$$

Finally let

$$\Gamma_0(4N) = \left\{\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\,\Big|\, \alpha\delta-\beta\gamma=1,\ \gamma\equiv 0 \pmod{4N}\right\}$$

(nonhomogeneous congruence subgroup).

1.2. For the convenience of references we quote some known results as the following lemmas.

Lemma 1. *If* $(h,q)=1$, *then*

$$S(kh,kq) = kS(h,q).$$

Lemma 2 (see, for example, [6], p. 13, Lemma 6). *If* $(h,q) = 1$, *then*

$$\begin{aligned} S^2(h,q) &= \Big(\frac{-1}{q}\Big)q \quad \textit{for} \quad q\equiv 1 \pmod 2,\\ &= 2i^h q \quad \textit{for} \quad q \equiv 0 \pmod 4,\\ &= 0 \quad \textit{for} \quad q\equiv 2 \pmod 4. \end{aligned}$$

Lemma 3 (see, for example, [6], p. 16, Lemma 8). *If $(h,q)=1$, then*

$$S(h,u)=\Big(\frac{h}{u}\Big)i^{(u-1)^2/4}u^{1/2}.$$

Lemma 4 (see, for example [6], p. 177, formula 20). *Let $q=p^\lambda$ and $p^\nu\|h$. Then*

$$\begin{aligned}c(h,q)&=0 \quad \textit{for} \quad \nu<\lambda-1,\\ &=-p^{\lambda-1} \quad \textit{for} \quad \nu=\lambda-1,\\ &=p^{\lambda-1}(p-1) \quad \textit{for} \quad \nu>\lambda-1.\end{aligned}$$

Lemma 5 (see, for example, [2], p. 14, Lemma 10). *Let*

$$\chi_p=1+A(p)+A(p)^2)+\cdots. \tag{1.15}$$

Then

$$\sum_{q=1}^{\infty}A(q)=\prod_p\chi_p.$$

Lemma 6 ([1], pp. 811 and 953). *The entire modular form $F(\tau)$ of weight r for the congruence subgroup $\Gamma_0(4N)$ is identically zero, if in its expansion in the series*

$$F(\tau)=\sum_{m=0}^{\infty}C_mQ^m,$$

$$C_m=0 \quad \textit{for all} \quad m\le\frac{r}{3}N\prod_{p|4N}\Big(1+\frac{1}{p}\Big).$$

Lemma 7 ([7], p. 18, Lemma 14). *The function $\theta(\tau;f)$ is an entire modular form of weight 5 and character $\chi(\delta)=\operatorname{sgn}\delta(\frac{-\Delta}{|\delta|})$ for $\Gamma_0(4a)$.*

Lemma 8 ([2], p. 21, the remark to Lemma 18). *The function $\prod_{k=1}^5\vartheta_{00}(\tau;2,a_k)$ is an entire modular form of weight 5 and character $\chi(\delta)=\operatorname{sgn}\delta(\frac{-\Delta}{|\delta|})$ for $\Gamma_0(4a)$.*

Lemma 9 ([4], p. 67, Theorem 1[1]and [5], p. 193, Theorem 1). *For a given N the functions*

(1) $$\begin{aligned}&\Psi_2(\tau;g_1,g_2;h_1,h_2;0,0;N_1,N_2)=\\ &=\tfrac{1}{N_1^2}\vartheta^{(4)}_{g_1h_1}(\tau;0,2N_1)\vartheta_{g_2h_2}(\tau;0,2N_2)+\\ &+\tfrac{1}{N_2^2}\vartheta_{g_1h_1}(\tau;0,2N_1)\vartheta^{(4)}_{g_2h_2}(\tau;0,2N_2)-\\ &-\tfrac{6}{N_1N_2}\vartheta''_{g_1h_1}(\tau;0,2N_1)\vartheta''_{g_2h_2}(\tau;0,2N_2),\end{aligned} \tag{1.16}$$

[1]There is a misprint in the formulation of Theorem 1 [4, p. 67] which can be easily corrected by substituting 5 for 10 and vice versa.

where

(a) $2|g_1,\ 2|g_2,\ N_1|N,\ N_2|N,\ 4|N\Big(\dfrac{h_1^2}{N_1}+\dfrac{h_2^2}{N_2}\Big),\ 4\Big|\dfrac{g_1^2}{4N_1}+\dfrac{g_2^2}{4N_2},$

(b) *for all* α *and* δ *with* $\alpha\delta\equiv 1 \pmod{4N}$

$$\Big(\frac{N_1N_2}{|\delta|}\Big)\Psi_2(\tau;\alpha g_1,\alpha g_2;h_1,h_2;0,0;N_1,N_2)=$$
$$=\Big(\frac{\Delta}{|\delta|}\Big)\Psi_2(\tau;g_1,g_2;h_1,h_2;0,0;N_1,N_2),$$

$$\begin{aligned}(2)\ &\Psi_3(\tau;g_1,\dots,g_4;h_1,\dots,h_4;0,\dots,0;N_1,\dots,N_4)=\\ &=\Big\{\tfrac{1}{N_1}\vartheta''_{g_1h_1}(\tau;0,2N_1)\vartheta_{g_2h_2}(\tau;0,2N_2)-\\ &-\tfrac{1}{N_2}\vartheta_{g_1h_1}(\tau;0,2N_1)\vartheta''_{g_2h_2}(\tau;0,2N_2)\Big\}\times\\ &\times\vartheta'_{g_3h_3}(\tau;0,2N_3)\vartheta_{g_4h_4}(\tau;0,2N_4)\end{aligned}\tag{1.17}$$

and

$$\begin{aligned}(3)\ &\Psi_4(\tau;g_1,\dots,g_4;h_1,\dots,h_4;0,\dots,0;N_1,\dots,N_4)=\\ &=\textstyle\prod_{k=1}^{3}\vartheta'_{g_kh_k}(\tau;0,2N_k)\vartheta_{g_4h_4}(\tau;0,2N_4),\end{aligned}\tag{1.18}$$

where

(a) $2|g_k,\ N_k|N\ (k=1,2,3,4),\ 4|N\displaystyle\sum_{k=1}^{4}\frac{h_k^2}{N_k},\ 4|\sum_{k=1}^{4}\frac{g_k^2}{4N_k},$

(b) *for all* α *and* δ *with* $\alpha\delta\equiv 1 \pmod{4N}$

$$\left(\frac{\prod_{k=1}^4 N_k}{|\delta|}\right)\Psi_j(\tau;\alpha g_1,\dots,\alpha g_4;h_1,\dots,h_4;0,\dots,0;N_1,\dots,N_4)=$$
$$=\operatorname{sgn}\delta\Big(\frac{-\Delta}{|\delta|}\Big)\Psi_j(\tau;g_1,\dots,g_4;h_1,\dots,h_4;0,\dots,0;N_1,\dots,N_4)\quad(j=3,4),$$

are entire modular forms of weight 5 *and character* $\chi(\delta)=\operatorname{sgn}\delta(\frac{-\Delta}{|\delta|})$ *for* $\Gamma_0(4N)$.

2. Summation of the singular series $\rho(n;f)$

Everywhere in this section α,β,γ denote non-negative integers and m positive odd integers.

Lemma 10. *Let* $n=2^\alpha m$, $a_k=2^{\gamma_k}b_k$ $(k=1,2,\dots,5)$, $(b_1,\dots,b_5)=1$, $b=[b_1,\dots,b_5]$, $\gamma_5\ge\gamma_4\ge\gamma_3\ge\gamma_2\ge\gamma_1=0$, $\gamma=\sum_{k=1}^5\gamma_k$. *Then*

$$\begin{aligned}\chi_2&=1+(-1)^{(b_1-m)/2}\ \textit{for}\ 0\le\alpha\le\gamma_2-2,\\ &=1\ \textit{for}\ \alpha=\gamma_2-1,\ \alpha=\gamma_2<\gamma_3,\ \gamma_2=\gamma_3\le\alpha=\gamma_4-1,\\ &\qquad\gamma_2+1=\gamma_3\le\alpha=\gamma_4-1,\ \gamma_2=\gamma_3\le\alpha=\gamma_4<\gamma_5,\end{aligned}$$

$$
\begin{aligned}
&\qquad \gamma_2+1=\gamma_3\le\alpha=\gamma_4<\gamma_5,\\
&=1+(-1)^{(b_1+b_2)/2}(1-2^{\gamma_2-\alpha}\cdot 3)\ \textit{for}\ \gamma_2+1\le\alpha<\gamma_3,\\
&=1+(-1)^{\left(\sum_{k=1}^{3}b_k-m\right)/2}\cdot 2^{\gamma_2+\gamma_3-2\alpha-2}\ \textit{for}\ \gamma_2=\gamma_3\le\alpha<\gamma_4-2,\\
&\qquad \gamma_2+1=\gamma_3\le\alpha\le\gamma_4-2,\\
&=1+(-1)^{(b_1+b_2)/2}(1-2^{\gamma_2-\gamma_3+1})+(-1)^{\left(\sum_{k=1}^{3}b_k-m\right)/2}\cdot 2^{\gamma_2-\gamma_3-2\alpha-2}\\
&\qquad \textit{for}\ \gamma_2+2\le\gamma_3\le\alpha\le\gamma_4-2,\\
&=1+(-1)^{(b_1+b_2)/2}(1-2^{\gamma_2-\gamma_3+1})\ \textit{for}\ \gamma_2+2\le\gamma_3\le\alpha=\gamma_4-1,\\
&\qquad \gamma_2+2\le\gamma_3\le\alpha=\gamma_4<\gamma_5,\\
&=1+(-1)^{(b_1+b_2)/2}(1-2^{\gamma_2-\gamma_3+1})+(-1)^{\left(\sum_{k=1}^{4}b_k\right)/2}\times\\
&\times 2^{\gamma_2+\gamma_3-2\gamma_4}\Big\{\frac{1}{7}(1-2^{-3(\alpha-\gamma_4-1)})-2^{-3(\alpha-\gamma_4)}\Big\}\\
&\qquad \textit{for}\ \gamma_4+1\le\alpha<\gamma_5\ \textit{but}\ \gamma_4\ge\gamma_3\ge\gamma_2+2,\\
&=1+(-1)^{\left(\sum_{k=1}^{4}b_k\right)/2}\cdot 2^{\gamma_2+\gamma_3-2\gamma_4}\Big\{\frac{1}{7}(1-2^{-3(\alpha-\gamma_4-1)})-2^{-3(\alpha-\gamma_4)}\Big\}\\
&\qquad \textit{for}\ \gamma_4+1\le\alpha<\gamma_5\ \ \textit{but}\ \gamma_4\ge\gamma_3=\gamma_2\ \ \textit{or}\ \gamma_4\ge\gamma_3=\gamma_2+1,\\
&=1+(-1)^{\left(\sum_{k=1}^{5}b_k-m\right)/2}\cdot 2^{\gamma-4\alpha-4}\ \textit{for}\ \alpha\ge\gamma_5=\gamma_4\ge\gamma_3=\gamma_2,\\
&\qquad \alpha\ge\gamma_5=\gamma_4\ge\gamma_3=\gamma_2+1,\\
&=1+(-1)^{(b_1+b_2)/2}(1-2^{\gamma_2-\gamma_3+1})+(-1)^{\left(\sum_{k=1}^{5}b_k-m\right)/2}\cdot 2^{\gamma-4\alpha-4}\\
&\qquad \textit{for}\ \alpha\ge\gamma_5=\gamma_4\ge\gamma_3\ge\gamma_2+2,\\
&=1+(-1)^{(b_1+b_2)/2}(1-2^{\gamma_2-\gamma_3+1})+(-1)^{\left(\sum_{k=1}^{4}b_k\right)/2}\times\\
&\times 2^{\gamma_2+\gamma_3-2\gamma_4}(1-2^{-3(\gamma_5-\gamma_4-1)})/7+(-1)^{\left(\sum_{k=1}^{5}b_k-m\right)/2}\cdot 2^{\gamma-4\alpha-4}\\
&\qquad \textit{for}\ \alpha\ge\gamma_5>\gamma_4\ge\gamma_3\ge\gamma_2+2,\\
&=1+(-1)^{\left(\sum_{k=1}^{4}b_k\right)/2}\cdot 2^{\gamma_2+\gamma_3-2\gamma_4}(1-2^{-3(\gamma_5-\gamma_4-1)})/7+\\
&+(-1)^{\left(\sum_{k=1}^{5}b_k-m\right)/2}\cdot 2^{\gamma-4\alpha-4}\ \textit{for}\ \alpha\ge\gamma_5>\gamma_4\ge\gamma_3=\gamma_2,\\
&\qquad \alpha\ge\gamma_5>\gamma_4\ge\gamma_3=\gamma_2+1.
\end{aligned}
$$

Proof. I. If in (1.14) we put $q=2^\lambda$ and then instead of h introduce a new letter of summation y defined by the congruence $h\equiv by \pmod{2^\lambda}$, then we get

$$
A(2^\lambda)=2^{-10\lambda}\sum_{y \bmod 2^\lambda}{}' e(-2^{\alpha-\lambda}mby)\prod_{k=1}^{5}S^2(2^{\gamma_k}b_kby,2^\lambda). \tag{2.1}
$$

From (2.1), according to Lemmas 1, 2, and 4 it follows that
(1) for $\lambda = 1$, $\gamma_k + 1$ $(k = 2, 3, 4, 5)$

$$A(2^\lambda) = 0, \tag{2.2}$$

because $S^2(b_1 by, 2) = 0$, $S^2(2^{\gamma_k} b_k by, 2^{\gamma_k+1}) = 0$ $(k = 2, 3, 4, 5)$;
(2) for $2 \le \lambda \le \gamma_2$

$$\begin{aligned} A(2^\lambda) &= 2^{-10\lambda} \sum_{y \bmod 2^\lambda}{}' e(-2^{\alpha-\lambda} mby)(2i^{b_i by} \cdot 2^\lambda) 2^{8\lambda} = \\ &= 2^{1-\lambda} \sum_{y \bmod 2^\lambda}{}' e\Big(\frac{b_1 by}{4} - \frac{2^\alpha mby}{2^\lambda}\Big) = \\ &= 2^{1-\lambda} e\Big(\frac{(b_1 - 2^{\alpha-\lambda+2} m) b}{4}\Big) \sum_{y=0}^{2^{\lambda-1}-1} e\Big(\frac{(2^{\lambda-2} b_1 - 2^\alpha m) by}{2^{\lambda-1}}\Big) = \\ &= \begin{cases} e\big((b_1 - 2^{\alpha-\lambda+2} m) b/4\big) & \text{if } 2^{\lambda-1} | (2^{\lambda-2} b_1 - 2^\alpha m) b, \\ 0 & \text{if } 2^{\lambda-1} \dagger (2^{\lambda-2} b_1 - 2^\alpha m) b, \end{cases} \end{aligned}$$

i.e.,

$$A(2^\lambda) = (-1)^{b_1 - m)/2} \text{ if } \lambda = \alpha + 2, \tag{2.3}$$
$$= 0 \text{ if } \lambda \ne \alpha + 2; \tag{2.3$_1$}$$

(3) for $\gamma_2 + 2 \le \lambda \le \gamma_3$

$$\begin{aligned} A(2^\lambda) = 2^{-10\lambda} \sum_{y \bmod 2^\lambda}{}' e(-2^{\alpha-\lambda} mby)(2i^{b_i by} \cdot 2^\lambda) 2^{2\gamma_2} \cdot 2i^{b_2 by} \times \\ \times 2^{\lambda-\gamma_2} \cdot 2^{6\lambda} = 2^{\gamma_2 - 2\lambda + 2} (-1)^{(b_1+b_2)/2} c(2^\alpha mb, 2^\lambda), \end{aligned}$$

hence

$$A(2^\lambda) = (-1)^{(b_2+b_2)/2} \cdot 2^{\gamma_2+1-\lambda} \text{ if } \lambda < \alpha + 1, \tag{2.4}$$
$$= -(-1)^{(b_2+b_2)/2} \cdot 2^{\gamma_2-\alpha} \text{ if } \lambda = \alpha + 1, \tag{2.4$_1$}$$
$$= 0 \text{ if } \lambda > \alpha + 1; \tag{2.4$_2$}$$

(4) for $\gamma_3 + 2 \le \lambda \le \gamma_4$

$$\begin{aligned} A(2^\lambda) = 2^{-10\lambda} \sum_{y \bmod 2^\lambda}{}' e(-2^{\alpha-\lambda} mby)(2i^{b_1 by} \cdot 2^\lambda) 2^{2\gamma_2} \times \\ \times (2i^{b_2 by} \cdot 2^{\lambda-\gamma_2}) 2^{2\gamma_3} (2i^{b_3 by} \cdot 2^{\lambda-\gamma_3}) \cdot 2^{4\lambda} = \end{aligned}$$

$$= 2^{\gamma_2+\gamma_3-3\lambda+3} \sum_{y \bmod 2^\lambda}{}' e\Big(\frac{(b_1+b_2+b_3)by}{4} - \frac{2^\alpha mby}{2^\lambda}\Big) =$$
$$= 2^{\gamma_2+\gamma_3-3\lambda+3} e\Big(\frac{1}{4}\big((b_1+b_2+b_3)b - 2^{\alpha-\lambda+2}mb\big)\Big) \times$$
$$\times \sum_{y=0}^{2^{\lambda-1}-1} e\big(2^{1-\lambda}(2^{\lambda-2}(b_1+b_2+b_3) - 2^\alpha m)by\big),$$

hence, as in case (2)

$$A(2^\lambda) = (-1)^{\left(\sum_{k=1}^{3} b_k - m\right)/2} \cdot 2^{\gamma_2+\gamma_3-2\alpha-2} \text{ if } \lambda = \alpha+2, \tag{2.5}$$
$$= 0 \text{ if } \lambda \neq \alpha+2; \tag{2.5$_1$}$$

(5) for $\gamma_4 + 2 \le \lambda \le \gamma_5$

$$A(2^\lambda) = 2^{-10\lambda} \sum_{y \bmod 2^\lambda}{}' e(-2^{\alpha-\lambda}mby)(2i^{b_1by} \cdot 2^\lambda)(2^{2\gamma_2} \cdot 2i^{b_2by} \cdot 2^{\lambda-\gamma_2}) \times$$
$$\times (2^{2\gamma_3} \cdot 2i^{b_3by} \cdot 2^{\lambda-\gamma_3})(2^{2\gamma_4} \cdot 2i^{b_4by} \cdot 2^{\lambda-\gamma_4})2^{2\lambda} =$$
$$= 2^{\gamma_2+\gamma_3+\gamma_4-4\lambda+4}(-1)^{\left(\sum_{k=1}^{4} b_k\right)/2} c(2^\alpha bm, 2^\lambda),$$

hence

$$A(2^\lambda) = (-1)^{\left(\sum_{k=1}^{4} b_k\right)/2} \cdot 2^{\gamma-\gamma_5-3\lambda-3} \text{ if } \lambda < \alpha+1, \tag{2.6}$$
$$= -(-1)^{\left(\sum_{k=1}^{4} b_k\right)/2} \cdot 2^{\gamma-\gamma_5-3\alpha} \text{ if } \lambda = \alpha+1, \tag{2.6$_1$}$$
$$= 0 \text{ if } \lambda > \alpha+1; \tag{2.6$_2$}$$

(6) for $\lambda > \gamma_5 + 1$

$$A(2^\lambda) = 2^{-10\lambda} \sum_{y \bmod 2^\lambda}{}' e(-2^{\alpha-\lambda}mby)(2i^{b_1by} \cdot 2^\lambda)2^{2\gamma_2} \times$$
$$\times (2i^{b_2by} \cdot 2^{\lambda-\gamma_2})2^{2\gamma_3}(2i^{b_3by} \cdot 2^{\lambda-\gamma_3}) \times$$
$$\times (2^{2\gamma_4} \cdot 2i^{b_4by} \cdot 2^{\lambda-\gamma_4})(2^{2\gamma_5} \cdot 2i^{b_5by} \cdot 2^{\lambda-\gamma_5}) =$$
$$= 2^{\gamma-5\lambda+5} \sum_{y \bmod 2^\lambda}{}' e\left(\frac{\sum_{k=1}^{5} b_k by}{4} - \frac{2^\alpha mby}{2^\lambda}\right) =$$
$$= 2^{\gamma-5\lambda+5} e\Big(\frac{1}{4}\Big(\sum_{k=1}^{5} b_k b - 2^{\alpha-\lambda+2}mb\Big)\Big) \times$$

$$\times \sum_{y=0}^{2^{\lambda-1}-1} e\Big(2^{1-\lambda}\Big(2^{\lambda-2}\sum_{k=1}^{5} b_k - 2^{\alpha}m\Big)by\Big),$$

hence, as in case (2),

$$A(2^\lambda) = (-1)^{\left(\sum_{k=1}^{5} b_k - m\right)/2} \cdot 2^{\gamma-4\alpha-4} \text{ if } \lambda = \alpha + 2, \tag{2.7}$$
$$= 0 \text{ if } \lambda \neq \alpha + 2; \tag{2.7$_1$}$$

II. According to (1.15) and (2.2), we have

$$\chi_2 = 1 + \sum_{\lambda=2}^{\gamma_2} A(2^\lambda) + \sum_{\lambda=\gamma_2+2}^{\gamma_3} A(2^\lambda) + \sum_{\lambda=\gamma_3+2}^{\gamma_4} A(2^\lambda) + \\ + \sum_{\lambda=\gamma_4+2}^{\gamma_5} A(2^\lambda) + \sum_{\lambda=\gamma_5+2}^{\infty} A(2^\lambda). \tag{2.8}$$

Consider the following cases:

(1) Let $0 \leq \alpha \leq \gamma_2 - 2$. Then from (2.8), (2.3), (2.3$_1$), (2.4$_2$), (2.5$_1$), (2.6$_2$), and (2.7$_1$) we get

$$\chi_2 = 1 + \sum_{\lambda=2}^{\gamma_2} A(2^\lambda) = 1 + (-1)^{(b_1-m)/2}.$$

(2) Let $\alpha = \gamma_2 - 1$ or $\alpha = \gamma_2 < \gamma_3$. Then from (2.8), (2.3$_1$), (2.4$_2$), (2.5$_1$), (2.6$_2$), and (2.7$_1$) we get

$$\chi_2 = 1.$$

(3) Let $\gamma_2 + 1 \leq \alpha < \gamma_3$. Then from (2.8), (2.3$_1$), (2.4), (2.4$_1$), (2.4$_2$), (2.5$_1$), (2.6$_2$), and (2.7$_1$) we get

$$\chi_2 = 1 + \sum_{\lambda=\gamma_2+2}^{\gamma_3} A(2^\lambda) = 1 + \sum_{\lambda=\gamma_2+2}^{\alpha} (-1)^{(b_1+b_2)/2} \cdot 2^{\gamma_2+1-\lambda} - (-1)^{(b_1+b_2)/2} \cdot 2^{\gamma_2-\alpha}.$$

(4) Let $\gamma_2 = \gamma_3 \leq \alpha \leq \gamma_4 - 2$ or $\gamma_2 + 1 = \gamma_3 \leq \alpha \leq \gamma_4 - 2$. Then from (2.8), (2.3$_1$), (2.5), (2.5$_1$), (2.6$_2$), and (2.7$_1$) we get

$$\chi_2 = 1 + \sum_{\lambda=\gamma_3+2}^{\gamma_4} A(2^\lambda) = 1 + (-1)^{\left(\sum_{k=1}^{3} b_k - m\right)/2} \cdot 2^{\gamma_2+\gamma_3-2\alpha-2}.$$

(5) Let $\gamma_2+2 \le \gamma_3 \le \alpha \le \gamma_4-2$. Then from (2.8), ($2.3_1$), (2.4), (2.5), ($2.5_1$), ($2.6_2$), and ($2.7_1$) we get

$$\chi_2 = 1 + \sum_{\lambda=\gamma_2+2}^{\gamma_3} A(2^\lambda) + \sum_{\lambda=\gamma_3+2}^{\gamma_4} A(2^\lambda) =$$

$$= 1 + \sum_{\lambda=\gamma_2+2}^{\gamma_3} (-1)^{(b_1+b_2)/2} \cdot 2^{\gamma_2+1-\lambda} + (-1)^{\left(\sum_{k=1}^{3} b_k - m\right)/2} \cdot 2^{\gamma_2+\gamma_3-2\alpha-2}.$$

(6) Let $\gamma_2=\gamma_3 \le \alpha = \gamma_4 - 1$ or $\gamma_2+1=\gamma_3 \le \alpha=\gamma_4-1$ or $\gamma_2=\gamma_3 \le \alpha=\gamma_4<\gamma_5$ or $\gamma_2+1=\gamma_3 \le \alpha=\gamma_4<\gamma_5$. Then from (2.8), ($2.3_1$), ($2.5_1$), ($2.6_2$), and ($2.7_1$) we get

$$\chi_2 = 1.$$

(7) Let $\gamma_2+2 \le \gamma_3 \le \alpha=\gamma_4-1$ or $\gamma_2+2 \le \gamma_3 \le \alpha = \gamma_4 < \gamma_5$. Then from (2.8), ($2.3_1$), (2.4), ($2.4_2$), ($2.5_1$), ($2.6_1$), and ($2.7_1$) we get

$$\chi_2 = 1 + \sum_{\lambda=\gamma_2+2}^{\gamma_3} A(2^\lambda) = 1 + \sum_{\lambda=\gamma_2+2}^{\gamma_3} (-1)^{(b_1+b_2)/2} \cdot 2^{\gamma_2+1-\lambda}.$$

8) Let $\gamma_4+1 \le \alpha < \gamma_5$, but $\gamma_4 \ge \gamma_3 \ge \gamma_2+2$. Then from (2.8), ($2.3_1$), (2.4), ($2.5_1$), (2.6), ($2.6_1$), and ($2.7_1$) we get

$$\chi_2 = 1 + \sum_{\lambda=\gamma_2+2}^{\gamma_3} A(2^\lambda) + \sum_{\lambda=\gamma_4+2}^{\gamma_5} A(2^\lambda) =$$

$$= 1 + \sum_{\lambda=\gamma_2+2}^{\gamma_3} (-1)^{(b_1+b_2)/2} \cdot 2^{\gamma_2+1-\lambda} +$$

$$+ \sum_{\lambda=\gamma_4+2}^{\alpha} (-1)^{\left(\sum_{k=1}^{4} b_k\right)/2} \cdot 2^{\gamma-\gamma_5-3\lambda+3} - (-1)^{\left(\sum_{k=1}^{4} b_k\right)/2} \cdot 2^{\gamma-\gamma_5-3\alpha}.$$

(9) Let $\gamma_4+1 \le \alpha < \gamma_5$, but $\gamma_4 \ge \gamma_3 = \gamma_2$ or $\gamma_4 \ge \gamma_3 = \gamma_2+1$. Then from (2.8), ($2.3_1$), ($2.5_1$), (2.6), ($2.6_1$), and ($2.7_1$) we get

$$\chi_2 = 1 + \sum_{\lambda=\gamma_4+2}^{\gamma_5} A(2^\lambda) + \sum_{\lambda=\gamma_5+2}^{\infty} A(2^\lambda) =$$

$$= 1 + \sum_{\lambda=\gamma_4+2}^{\alpha} (-1)^{\left(\sum_{k=1}^{4} b_k\right)/2} \cdot 2^{\gamma-\gamma_5-3\lambda+3} - (-1)^{\left(\sum_{k=1}^{4} b_k\right)/2} \cdot 2^{\gamma-\gamma_5-3\alpha}.$$

(10) Let $\alpha \geq \gamma_5 = \gamma_4 \geq \gamma_3 = \gamma_2$ or $\alpha \geq \gamma_5 = \gamma_4 \geq \gamma_3 = \gamma_2 + 1$. Then from (2.8), (2.3_1), (2.5_1), and (2.7_1) we get

$$\chi_2 = 1 + \sum_{\lambda=\gamma_5+2}^{\infty} A(2^\lambda) = 1 + (-1)^{\left(\sum_{k=1}^{5} b_k - m\right)/2} \cdot 2^{\gamma-4\alpha-4}.$$

(11) Let $\alpha \geq \gamma_5 = \gamma_4 \geq \gamma_3 \geq \gamma_2 + 2$. Then from (2.8), (2.3_1), (2.4), (2.5_1), (2.7), and (2.7_1) we get

$$\chi_2 = 1 + \sum_{\lambda=\gamma_2+2}^{\gamma_3} A(2^\lambda) + \sum_{\lambda=\gamma_5+2}^{\infty} A(2^\lambda) =$$

$$= 1 + \sum_{\lambda=\gamma_2+2}^{\gamma_3} (-1)^{(b_1+b_2)/2} \cdot 2^{\gamma_2+1-\lambda} + (-1)^{\left(\sum_{k=1}^{5} b_k - m\right)/2} \cdot 2^{\gamma-4\alpha-4}.$$

(12) Let $\alpha \geq \gamma_5 > \gamma_4 \geq \gamma_3 \geq \gamma_2 + 2$. Then from (2.8), (2.3_1), (2.4), (2.5_1), (2.6), (2.7), and (2.7_1) we get

$$\chi_2 = 1 + \sum_{\lambda=\gamma_2+2}^{\gamma_3} A(2^\lambda) + \sum_{\lambda=\gamma_4+2}^{\gamma_5} A(2^\lambda) +$$

$$+ \sum_{\lambda=\gamma_5+2}^{\infty} A(2^\lambda) = 1 + \sum_{\lambda=\gamma_2+2}^{\gamma_3} (-1)^{(b_1+b_2)/2} \cdot 2^{\gamma_2+1-\lambda} +$$

$$+ \sum_{\lambda=\gamma_4+2}^{\gamma_5} (-1)^{\left(\sum_{k=1}^{4} b_k\right)/2} \cdot 2^{\gamma-\gamma_5-3\lambda+3} + (-1)^{\left(\sum_{k=1}^{5} b_k - m\right)/2} \cdot 2^{\gamma-4\alpha-4}.$$

(13) Let $\alpha \geq \gamma_5 > \gamma_4 \geq \gamma_3 = \gamma_2$ or $\alpha \geq \gamma_5 > \gamma_4 \geq \gamma_3 = \gamma_2 + 1$. Then from (2.8), (2.3_1), (2.5_1), (2.6), (2.7), and (2.7_1) we get

$$\chi_2 = 1 + \sum_{\lambda=\gamma_4+2}^{\gamma_5} A(2^\lambda) + \sum_{\lambda=\gamma_5+2}^{\infty} A(2^\lambda) =$$

$$= 1 + \sum_{\lambda=\gamma_4+2}^{\gamma_5} (-1)^{\left(\sum_{k=1}^{4} b_k\right)/2} \cdot 2^{\sum_{k=2}^{4} \gamma_k - 3\lambda + 3} + (-1)^{\left(\sum_{k=1}^{5} b_k - m\right)/2} \cdot 2^{\gamma-4\alpha-4}.$$

Calculating the sums in the right-hand sides of the above given equalities, we complete the proof of the lemma. □

Lemma 11. *Let* $p > 2$, $p^\beta \| n$, $p^{\ell_k} \| a_k$ $(k=1,\ldots,5)$, $\max \ell_k = \overline{\ell}$, $\min \ell_k = \underline{\ell} = 0$, $\overline{\ell} \geq \ell''' \geq \ell'' \geq \ell' \geq \underline{\ell} = 0$, $\ell = \sum_{k=1}^{5} \ell_k = \overline{\ell} + \ell''' + \ell'' + \ell' + \underline{\ell}$, $\eta(\ell') = 1$ *if* $2|\ell'$ *and* $\eta(\ell') = 0$ *if* $2\nmid\ell'$. *Then*

$\chi_p = (1 - p^{-1})(\beta + 1)$ *for* $\ell' \geq \beta + 1$, $p \equiv 1 \pmod 4$,

$$= (1+p^{-1}) \ \textit{for} \ \ell' \geq \beta+1, \ p \equiv 3 \pmod 4, \ 2|\beta,$$
$$= 0 \ \textit{for} \ \ell' \geq \beta+1, \ p \equiv 3 \pmod 4, \ 2\dagger\beta;$$
$$= (1-p^{-1})\ell' + (1+p^{-1})(1-p^{-(\beta-\ell'+1)})$$
$$\textit{for} \ \ell' \leq \beta < \ell'', \ p \equiv 1 \pmod 4,$$
$$= (1+p^{-1})\eta(\ell') - (-1)^{\ell'}(1+p)p^{-(\beta-\ell'+2)})$$
$$\textit{for} \ \ell' \leq \beta < \ell'', \ p \equiv 3 \pmod 4,$$
$$= (1-p^{-1})\ell' + (1+p^{-1}) - p^{-(\ell''-\ell')}\big\{(1+p^{-2(\beta-\ell'')-1)})(p+1)^{-1} +$$
$$+ p^{-2(\beta-\ell'')-3}\big\} \ \textit{for} \ \ell'' \leq \beta < \ell''', \ p \equiv 1 \pmod 4,$$
$$= (1+p^{-1})\eta(\ell') - (-1)^{\ell'}p^{-(\ell''-\ell'+1)}\big\{(1+(p-1)(p^2+1)^{-1} \times$$
$$\times (1-(-1)^{\beta+\ell''}p^{-2(\beta-\ell'')}) - (-1)^{\beta+\ell''}p^{-2(\beta-\ell''+1)}\big\}$$
$$\textit{for} \ \ell'' \leq \beta < \ell''', \ p \equiv 3 \pmod 4,$$
$$= (1-p^{-1})\ell' + (1+p^{-1}) - p^{-(\ell''-\ell')}\big\{(1+p^{-2(\ell'''-\ell'')-1)})(p+1)^{-1} -$$
$$- p^{-2(\ell'''-\ell'')-1}(1-p^{-3(\beta-\ell''')}(p^2+p+1)^{-1}\big\} - p^{-3(\beta+1)+\ell-\overline{\ell}-1}$$
$$\textit{for} \ \ell''' \leq \beta < \overline{\ell}, \ p \equiv 1 \pmod 4,$$
$$= (1+p^{-1})\eta(\ell') - (-1)^{\ell'}p^{-(\ell''-\ell'+1)}\big\{(1+(p-1)(p^2+1)^{-1} \times$$
$$\times (1-(-1)^{\ell'''+\ell''}p^{-2(\beta-\ell'')})\big\} + (-1)^{\ell-\overline{\ell}}p^{\ell-\overline{\ell}-3\ell'''-1} \times$$
$$\times \big\{(1-p^{-3(\beta-\ell''')})(p^2+p+1)^{-1} - p^{-3(\beta-\ell'''+1)}\big\}$$
$$\textit{for} \ \ell''' \leq \beta < \overline{\ell}, \ p \equiv 3 \pmod 4,$$
$$= (1-p^{-1})\ell' + (1+p^{-1}) - p^{-(\ell''-\ell')}\big\{(1+p^{-2(\ell'''-\ell'')-1)})(p+1)^{-1} -$$
$$- p^{-2(\ell'''-\ell'')-1}(1-p^{-3(\overline{\ell}-\ell''')}(p^2+p+1)^{-1}\big\} +$$
$$+ p^{\ell-4\overline{\ell}-1}\big\{(1-p^{-4(\beta-\overline{\ell})})(p^3+p^2+p+1)^{-1} - p^{-4(\beta-\overline{\ell}+1)}\big\}$$
$$\textit{for} \ \beta \geq \overline{\ell}, \ p \equiv 1 \pmod 4,$$
$$= (1+p^{-1})\eta(\ell') - (-1)^{\ell'}p^{-(\ell''-\ell'+1)}\big\{(1+(p-1)(p^2+1)^{-1} \times$$
$$\times (1-(-1)^{\ell'''+\ell''}p^{-2(\ell'''-\ell'')})\big\} + (-1)^{\ell-\overline{\ell}}p^{\ell-\overline{\ell}-3\ell'''-1} \times$$
$$\times \big\{(1-p^{-3(\overline{\ell}-\ell''')})(p^2+p+1)^{-1} - p^{-3(\overline{\ell}-\ell''')}(1-(-1)^{\overline{\ell}+\beta}p^{-4(\beta-\overline{\ell})} \times$$
$$\times (p-1)(p^4+1)^{-1}\big\} + (-1)^{\beta+\ell}p^{\ell-4\beta-5} \ \textit{for} \ \beta \geq \overline{\ell}, \ p \equiv 3 \pmod 4.$$

Proof. I. Let $2\dagger q$ and $q=(q,a_k)q_k$ $(k=1,\dots,5)$. Then from (1.14) and Lemmas 1 and 2 it follows that

$$A(q)=q^{-10}\sideset{}{'}\sum_{h \bmod q} e\Big(-\frac{hn}{q}\Big)\prod_{k=1}^{5}(q,a_k)^2 S^2\Big(\frac{a_k}{(q,a_k)}\,h,q_k\Big)=$$
$$=q^{-10}\sideset{}{'}\sum_{h \bmod q} e\Big(-\frac{hn}{q}\Big)\prod_{k=1}^{5}(q,a_k)^2\Big(\frac{-1}{q_k}\Big)q_k=$$
$$=q^{-5}\sideset{}{'}\sum_{h \bmod q} e\Big(-\frac{hn}{q}\Big)\prod_{k=1}^{5}(q,a_k)\Big(\frac{-1}{q_k}\Big),$$

where, putting $q=p^\lambda$ and taking into account that $(a_1,\dots,a_5)=1$, we get

$$A(p^\lambda)=p^{-5\lambda}\sideset{}{'}\sum_{h \bmod p^\lambda} e\Big(-\frac{hn}{p^\lambda}\Big)\prod_{k=1}^{5}(p^\lambda,a_k)\Big(\frac{-1}{p^\lambda/(p^\lambda,a_k)}\Big)=$$
$$=\Big(\frac{-1}{p}\Big)^{\lambda+\min(\lambda,\overline{\ell})+\min(\lambda,\ell''')+\min(\lambda,\ell'')+\min(\lambda,\ell')}\times$$
$$\times\, p^{\min(\lambda,\overline{\ell})+\min(\lambda,\ell''')+\min(\lambda,\ell'')+\min(\lambda,\ell')}p^{-5\lambda}c(n,p^\lambda). \tag{2.9}$$

It follows from (2.9) and Lemma 4 that
(1) for $\lambda\le\ell'$

$$A(p^\lambda)=\Big(\frac{-1}{p}\Big)^{\lambda}(1-p^{-1})\ \text{ if }\ \lambda<\beta+1, \tag{2.10}$$
$$=-\Big(\frac{-1}{p}\Big)^{\beta+1}p^{-1}\ \text{ if }\ \lambda=\beta+1; \tag{2.10$_1$}$$

(2) for $\ell'<\lambda\le\ell''$

$$A(p^\lambda)=\Big(\frac{-1}{p}\Big)^{\ell'}p^{-(\lambda-\ell')}(1-p^{-1})\ \text{ if }\ \lambda<\beta+1, \tag{2.11}$$
$$=-\Big(\frac{-1}{p}\Big)^{\ell'}p^{\ell'-\beta-2}\ \text{ if }\ \lambda=\beta+1; \tag{2.11$_1$}$$

(3) for $\ell''<\lambda\le\ell'''$

$$A(p^\lambda)=\Big(\frac{-1}{p}\Big)^{\lambda+\ell''+\ell'}p^{-2\lambda+\ell''\ell'}(1-p^{-1})\text{ if }\lambda<\beta+1, \tag{2.12}$$
$$=-\Big(\frac{-1}{p}\Big)^{\beta+1+\ell''+\ell'}p^{-2\beta+\ell''+\ell'-3}\text{ if }\lambda=\beta+1; \tag{2.12$_1$}$$

(4) for $\ell''' < \lambda \leq \overline{\ell}$

$$A(p^\lambda)=\Big(\frac{-1}{p}\Big)^{\ell-\overline{\ell}}p^{\ell-\overline{\ell}-3\lambda}(1-p^{-1}) \text{ if } \lambda<\beta+1, \tag{2.13}$$

$$=-\Big(\frac{-1}{p}\Big)^{\ell-\overline{\ell}}p^{\ell-\overline{\ell}} \text{ if } \lambda=\beta+1; \tag{2.13$_1$}$$

(5) for $\lambda > \overline{\ell}$

$$A(p^\lambda) = \Big(\frac{-1}{p}\Big)^{\lambda+\ell}p^{\ell-4\lambda}(1-p^{-1}) \ \text{ if } \ \lambda < \beta+1, \tag{2.14}$$

$$= -\Big(\frac{-1}{p}\Big)^{\beta+\ell+1}p^{\ell-4\beta-5} \ \text{ if } \ \lambda = \beta+1. \tag{2.14$_1$}$$

In all the above-mentioned cases

$$A(p^\lambda) = 0 \quad \text{if} \quad \lambda > \beta+1. \tag{2.15}$$

II. According to (1.15) we have

$$\chi_p = 1+\sum_{\lambda=1}^{\ell'}A(p^\lambda)+\sum_{\lambda=\ell'+1}^{\ell''}A(p^\lambda)+\sum_{\lambda=\ell''+1}^{\ell'''}A(p^\lambda)+$$
$$+\sum_{\lambda=\ell'''+1}^{\overline{\ell}}A(p^\lambda)+\sum_{\lambda=\overline{\ell}+1}^{\infty}A(p^\lambda). \tag{2.16}$$

Consider the following cases:

(1) Let $= \ell' \geq \beta+1$. Then from (2.16), (2.10), (2.10$_1$), and (2.15) we get

$$\chi_p = 1+\sum_{\lambda=1}^{\beta}\Big(\frac{-1}{p}\Big)^{\lambda}(1-p^{-1})-\Big(\frac{-1}{p}\Big)^{\beta+1}p^{-1}.$$

(2) Let $\ell' \leq \beta < \ell''$. Then from (2.16), (2.10) (2.11), (2.11$_1$), and (2.15) we get

$$\chi_p = 1+\sum_{\lambda=1}^{\ell'}\Big(\frac{-1}{p}\Big)^{\lambda}(1-p^{-1})+\sum_{\lambda=\ell'+1}^{\beta}\Big(\frac{-1}{p}\Big)^{\ell'}p^{-(\lambda-\ell')}(1-p^{-1})-$$
$$-\Big(\frac{-1}{p}\Big)^{\ell'}p^{-(\beta-\ell'+1)}.$$

(3) Let $\ell'' \le \beta < \ell'''$. Then from (2.16), (2.10), (2.11), (2.12), (2.12_1), and (2.15) we get

$$\chi_p = 1 + \sum_{\lambda=1}^{\ell'} \left(\frac{-1}{p}\right)^{\lambda}(1-p^{-1}) + \sum_{\lambda=\ell'+1}^{\ell''} \left(\frac{-1}{p}\right)^{\ell'} p^{-(\lambda-\ell')}(1-p^{-1}) +$$
$$+ \sum_{\lambda=\ell''+1}^{\beta} \left(\frac{-1}{p}\right)^{\lambda+\ell''+\ell'} p^{-2\lambda+\ell''+\ell'}(1-p^{-1}) - \left(\frac{-1}{p}\right)^{\beta+1+\ell''+\ell'} p^{-2\beta+\ell''+\ell'+3}.$$

(4) Let $\ell''' \le \beta < \overline{\ell}$. Then from (2.16), (2.10), (2.11), (2.12), (2.13), (2.13_1), and (2.15) we get

$$\chi_p = 1 + \sum_{\lambda=1}^{\ell'} \left(\frac{-1}{p}\right)^{\lambda}(1-p^{-1}) + \sum_{\lambda=\ell'+1}^{\ell''} \left(\frac{-1}{p}\right)^{\ell'} p^{-(\lambda-\ell')}(1-p^{-1}) +$$
$$+ \sum_{\lambda=\ell''+1}^{\ell'''} \left(\frac{-1}{p}\right)^{\lambda+\ell''+\ell'} p^{-2\lambda+\ell''+\ell'}(1-p^{-1}) +$$
$$+ \sum_{\lambda=\ell'''+1}^{\beta} \left(\frac{-1}{p}\right)^{\ell-\overline{\ell}} p^{\ell-\overline{\ell}-3\lambda}(1-p^{-1}) - \left(\frac{-1}{p}\right)^{\ell-\overline{\ell}} p^{\ell-\overline{\ell}-3\beta-4}.$$

(5) Let $\beta \ge \overline{\ell}$. Then from (2.16), (2.10), (2.11), (2.12), (2.13), (2.14), (2.14_1), and (2.15) we get

$$\chi_p = 1 + \sum_{\lambda=1}^{\ell'} \left(\frac{-1}{p}\right)^{\lambda}(1-p^{-1}) + \sum_{\lambda=\ell'+1}^{\ell''} \left(\frac{-1}{p}\right)^{\ell'} (1-p^{-1})p^{-(\lambda-\ell')} +$$
$$+ \sum_{\lambda=\ell''+1}^{\ell'''} \left(\frac{-1}{p}\right)^{\lambda+\ell''+\ell'} p^{\ell''+\ell'-2\lambda}(1-p^{-1}) + \sum_{\lambda=\ell'''+1}^{\overline{\ell}} \left(\frac{-1}{p}\right)^{\ell-\overline{\ell}} p^{\ell-\overline{\ell}-3\lambda}(1-p^{-1}) +$$
$$+ \sum_{\lambda=\overline{\ell}+1}^{\beta} \left(\frac{-1}{p}\right)^{\lambda+\ell} p^{\ell-4\lambda}(1-p^{-1}) - \left(\frac{-1}{p}\right)^{\beta+\ell+1} p^{\ell-4\beta-5}. \qquad (2.17)$$

Calculating the sums in the right-hand sides of the above-mentioned equalities, we complete the proof of the lemma. □

Theorem 1. *Let* $n = 2^{\alpha}m = 2^{\alpha}uv$, $u = \prod_{\substack{p\mid n\\ p\nmid 2\Delta}} p^{\beta}$, $v = \prod_{\substack{p\mid n\\ p\mid\Delta, p>2}} p^{\beta}$. *Then*

$$\rho(n;f) = \frac{2^{4\alpha+6}v^4}{5\Delta^{1/2}}\,\chi_2 \prod_{\substack{p\mid\Delta\\ p>2}} \chi_p \prod_{p\mid\Delta, p>2} \left(1 - \left(\frac{-1}{p}\right)p^{-5}\right)^{-1} \sum_{d_1d_2=u} \left(\frac{-1}{d_1}\right) d_2^4,$$

where the values of χ_2 *and* χ_p *are given in Lemmas* 10 *and* 11.

Proof. Let $p>2$ $p^\beta\|n$, $p\dagger\Delta$ (i.e., $\ell=0$). In (2.17) putting $\ell=0$, we get

$$\chi_p = 1+\sum_{\lambda=1}^{\beta}\Big(\frac{-1}{p}\Big)^{\lambda}p^{-4\lambda}(1-p^{-1})-\Big(\frac{-1}{p}\Big)^{\beta+1}p^{-4\beta-5} =$$

$$= 1+\sum_{\lambda=1}^{\beta}\Big(\frac{-1}{p}\Big)^{\lambda}p^{-4\lambda}-\Big(\frac{-1}{p}\Big)p^{-5}\sum_{\lambda=1}^{\beta}\Big(\frac{-1}{p}\Big)^{\lambda-1}p^{-4(\lambda-1)}-$$

$$-\Big(\frac{-1}{p}\Big)^{\beta+1}p^{-4\beta-5} = \sum_{\lambda=0}^{\beta}\Big(\frac{-1}{p}\Big)^{\lambda}p^{-4\lambda}-\Big(\frac{-1}{p}\Big)p^{-5}\sum_{\lambda=0}^{\beta}\Big(\frac{-1}{p}\Big)^{\lambda}p^{-4\lambda} =$$

$$= \Big(1-\Big(\frac{-1}{p}\Big)p^{-5}\Big)\sum_{\lambda=0}^{\beta}\Big(\frac{-1}{p}\Big)^{\lambda}p^{-4\lambda} =$$

$$= \Big(1-\Big(\frac{-1}{p}\Big)p^{-5}\Big)\sum_{d\mid p^\beta}\Big(\frac{-1}{d}\Big)p^{-4}. \tag{2.18}$$

For $p\dagger\Delta n$, i.e., for $\beta=0$, from (2.18) we get

$$\chi_p = 1-\Big(\frac{-1}{p}\Big)p^{-5}. \tag{2.19}$$

Thus it follows from Lemma 5, (2.18), and (2.19) that

$$\sum_{q=1}^{\infty}A(q) = \chi_2\prod_{\substack{p\mid\Delta\\ p>2}}\chi_p\prod_{\substack{p\mid n\\ p\dagger 2\Delta}}\chi_p\prod_{\substack{p\dagger\Delta n\\ p>2}}\chi_p =$$

$$= \chi_2\prod_{\substack{p\mid\Delta\\ p>2}}\chi_p\prod_{\substack{p\mid n\\ p\dagger 2\Delta}}\Big\{\Big(1-\Big(\frac{-1}{p}\Big)p^{-5}\Big)\sum_{d\mid p^\beta}\Big(\frac{-1}{d}\Big)d^{-4}\Big\}\prod_{\substack{p\dagger\Delta n\\ p>2}}\Big(1-\Big(\frac{-1}{p}\Big)p^{-5}\Big) =$$

$$= \chi_2\prod_{\substack{p\mid\Delta\\ p>2}}\chi_p\prod_{p>2}\Big(1-\Big(\frac{-1}{p}\Big)p^{-5}\Big)\prod_{\substack{p\mid\Delta\\ p>2}}\Big(1-\Big(\frac{-1}{p}\Big)p^{-5}\Big)^{-1}\sum_{d\mid u}\Big(\frac{-1}{d}\Big)d^{4} =$$

$$= \chi_2\prod_{\substack{p\mid\Delta\\ p>2}}\chi_p\mathcal{L}^{-1}(5,-1)\prod_{\substack{p\mid\Delta\\ p>2}}\Big(1-\Big(\frac{-1}{p}\Big)p^{-5}\Big)^{-1}\frac{1}{u^4}\sum_{d_1d_2=u}\Big(\frac{-1}{d_1}\Big)d_2^4, \tag{2.20}$$

where $\mathcal{L}(5,-1)$ is the Dirichlet $\mathcal{L}$-function and it is well known that

$$\mathcal{L}(5,-1) = \frac{5\pi^4}{2^6\cdot 4!}. \tag{2.21}$$

Thus the lemma follows from (1.13), (2.20), and (2.21). □

3. Formulas for $r(n;f)$

As an example let us consider the quadratic form

$$f = x_1^2 + \cdots + x_8^2 + 4(x_9^2 + x_{10}^2)$$

in which

$$a_1 = \cdots a_4 = 1,\ \ a_5 = 4,\ \ b_1 = \cdots = b_5 = 1,$$
$$\gamma_5 = 2,\ \ \gamma_4 = \gamma_3 = \gamma_2 = \gamma_1 = 0,\ \ \gamma = 2,\ \ a = 4,\ \ \Delta = 2^4.$$

Lemma 12. *The function*

$$\psi(\tau;f) = \vartheta_{00}^8(\tau;0,2)\vartheta_{00}^2(\tau;0,8) - \theta(\tau;f) + \frac{42}{5\cdot 512\pi^4}\Psi_2(\tau;8,8;0,0;0,0;4,4) -$$
$$-\frac{544}{5\cdot 8192\pi^4}\Psi_2(\tau;0,0;0,0;0,0;4,4) + \frac{3}{5\cdot 128\pi^4}\Psi_2(\tau;4,4;0,0;0,0;2,2) +$$
$$+\frac{2}{32\pi^3 i}\Psi_3(\tau;4,0,4,0;0,1,1,1;0,0,0,0;2,4,2,4) +$$
$$+\frac{24}{512\pi^3 i}\Psi_4(\tau;8,8,8,0;1,1,1,1;0,0,0,0;4,4,4,4) \qquad (3.1)$$

is an entire modular form of weight 5 *and character* $\chi(\delta) = \operatorname{sgn}\delta(\frac{-1}{|\delta|})$ *for* $\Gamma_0(16)$.

Proof. According to Lemmas 7 and 8, the first two summands in (3.1) are entire modular forms of weight 5 and character $\chi(\delta) = \operatorname{sgn}\delta(\frac{-1}{|\delta|})$ for $\Gamma_0(16)$.

Put $N = 4$. Then it is obvious that the following five summands in (3.1) satisfy the condition (a) from Lemma 9.

If $\alpha\delta \equiv 1 \pmod{16}$, then $\alpha\delta \equiv 1 \pmod 4$, i.e.,

$$\alpha \equiv \pm 1 \pmod 4 \quad \text{and respectively} \quad \delta \equiv \pm 1 \pmod 4. \qquad (3.2)$$

In the third, fourth, and fifth summands from (3.1) we have

$$\left(\frac{N_1 N_2}{|\delta|}\right) = 1 \quad \text{and} \quad \left(\frac{\Delta}{|\delta|}\right) = 1. \qquad (3.3)$$

By (1.16), (1.6), (1.7), and (1.10) we have

$$\Psi_2(\tau;8\alpha,8\alpha;0,0;0,0;4,4) = \frac{1}{8}\vartheta_{8\alpha,0}^{(4)}(\tau;0,8)\vartheta_{8\alpha,0}(\tau;0,8) - \frac{3}{8}\vartheta_{8\alpha,0}^{''2}(\tau;0,8) =$$
$$= \frac{1}{8}\vartheta_{\pm 8+8(\alpha\mp 1),0}^{(4)}(\tau;0,8)\vartheta_{\pm 8+8(\alpha\mp 1),0}(\tau;0,8) - \frac{3}{8}\vartheta_{\pm 8+8(\alpha\mp 1),0}^{''2}(\tau;0,8) =$$
$$= \frac{1}{8}\vartheta_{\pm 8,0}^{(4)}(\tau;4(\alpha\mp 1),8)\vartheta_{\pm 8,0}(\tau;4(\alpha\mp 1),8) - \frac{3}{8}\vartheta_{\pm 8,0}^{''2}(\tau;4(\alpha\mp 1),8) =$$
$$= \frac{1}{8}\vartheta_{\pm 8,0}^{(4)}(\tau;0,8)\vartheta_{\pm 8,0}(\tau;0,8) - \frac{3}{8}\vartheta_{\pm 8,0}^{''2}(\tau;0,8) = \frac{1}{8}\vartheta_{80}^{(4)}(\tau;0,8)\times$$

$$\times\vartheta_{80}(\tau;0,8)-\frac{3}{8}\vartheta''^{2}_{80}(\tau;0,8)=\Psi_2(\tau;8,8;0,0;0,0;4,4) \tag{3.4}$$

and similarly

$$\Psi_2(\tau;4\alpha,4\alpha;0,0;0,0;2,2)=\frac{1}{2}\vartheta^{(4)}_{4\alpha,0}(\tau;0,4)\vartheta_{4\alpha,0}(\tau;0,4)-\frac{3}{2}\vartheta''^{2}_{4\alpha,0}(\tau;0,4)=$$
$$=\frac{1}{2}\vartheta^{(4)}_{40}(\tau;0,4)\vartheta_{40}(\tau;0,4)-\frac{3}{2}\vartheta''^{2}_{40}(\tau;0,4)=$$
$$=\Psi_2(\tau;4,4;0,0;0,0;2,2). \tag{3.5}$$

Hence, according to (3.2)–(3.5) the functions $\Psi_2(\tau;8,8;0,0;0,0;4,4)$ and $\Psi_2(\tau;4,4;0,0;0,0;2,2)$ satisfy also the condition (b) from Lemma 9. Thus they are entire modular forms of weight 5 and character $\chi(\delta)=\operatorname{sgn}\delta(\frac{-1}{|\delta|})$ for $\Gamma_0(16)$.

Analogously, by (1.17), (1.18), (1.6), (1.7), and (1.10) we obtain

$$\Psi_3(\tau;4\alpha,0,4\alpha,0;0,1,1,1;0,0,0,0;2,4,2,4)=\Big\{\frac{1}{2}\vartheta''_{4\alpha,0}(\tau;0,4)\vartheta_{01}(\tau;0,8)-$$
$$-\frac{1}{4}\vartheta_{4\alpha,0}(\tau;0,4)\vartheta''_{01}(\tau;0,8)\Big\}\vartheta''_{4\alpha,1}(\tau;0,4)\vartheta_{01}(\tau;0,8)=$$
$$=\pm\Big\{\frac{1}{2}\vartheta''_{40}(\tau;0,4)\vartheta_{01}(\tau;0,8)-\frac{1}{4}\vartheta_{40}(\tau;0,4)\vartheta''_{01}(\tau;0,8)\Big\}\vartheta'_{41}(\tau;0,4)\times$$
$$\times\vartheta_{01}(\tau;0,8)=\pm\Psi_3(\tau;4,0,4,0;0,1,1,1;0,0,0,0;2,4,2,4), \tag{3.6}$$
$$\Psi_4(\tau;8\alpha,8\alpha,8\alpha,0;1,1,1,1;0,0,0,0;4,4,4,4)=$$
$$=\vartheta'^{3}_{8\alpha,1}(\tau;0,8)\vartheta_{01}(\tau;0,8)=\pm\vartheta'^{3}_{81}(\tau;0,8)\vartheta_{01}(\tau;0,8)=$$
$$=\pm\Psi_4(\tau;8,8,8,0;1,1,1,1;0,0,0,0;4,4,4,4). \tag{3.7}$$

In ((3.6) and (3.7) we have "+" if $\alpha\equiv 1 \pmod 4$ and "−" if $\alpha\equiv -1$ (mod 4). Hence the functions $\Psi_3(\tau;4,0,4,0;0,1,1,1;0,0,0,0;2,4,2,4)$ and $\Psi_4(\tau;8,8,8,0;1,1,1,1;0,0,0,0;4,4,4,4)$ also satisfy the condition (b) from Lemma 9, since

$$\left(\frac{\prod_{k=1}^{4}N_k}{|\delta|}\right)=1 \quad\text{and}\quad \operatorname{sgn}\delta\Big(\frac{-1}{|\delta|}\Big)=\begin{cases}1 & \text{if } \delta\equiv 1 \pmod 4,\\ -1 & \text{if } \delta\equiv -1 \pmod 4).\end{cases}$$

Thus, they are entire modular forms of weight 5 and character $\chi(\delta)=\operatorname{sgn}\delta(\frac{-1}{|\delta|})$ for $\Gamma_0(16)$. □

Theorem 2.

$$\vartheta^{8}_{00}(\tau;0,2)\vartheta^{2}_{00}(\tau;0,8)=\theta(\tau;f)-\frac{42}{5\cdot 512\pi^4}\Psi_2(\tau;8,8;0,0;0,0;4,4)+$$
$$+\frac{544}{5\cdot 8192\pi^4}\Psi_2(\tau;0,0;0,0;0,0;4,4)-\frac{3}{5\cdot 128\pi^4}\Psi_2(\tau;4,4;0,0;0,0;2,2)-$$

$$-\frac{2}{32\pi^3 i}\Psi_3(\tau;4,0,4,0;0,1,1,1;0,0,0,0;2,4,2,4)-$$
$$-\frac{24}{512\pi^3 i}\Psi_4(\tau;8,8,8,0;1,1,1,1;0,0,0,0;4,4,4,4). \tag{3.8}$$

Proof. According to Lemma 6, the function $\psi(r;f)$ will be identically zero if all coefficients by Q^n $(n \le 10)$ in its expansion by powers of Q are zero.

I. In Theorem 1 and Lemma 10 put $n = 2^\alpha u$, $m = u$, $v = 1$, $\Delta = 2^4$. Then

$$\begin{aligned}\rho(n;f) &= \frac{16}{5}\sum_{d_1 d_2 = u}\Big(\frac{-1}{d_1}\Big)d_2^4 \ \text{ for } \ \alpha = 0,\\ &= \frac{224}{5}\sum_{d_1 d_2 = u}\Big(\frac{-1}{d_1}\Big)d_2^4 \ \text{ for } \ \alpha = 1,\\ &= \frac{4}{5}\Big(2^{4\alpha-1}\cdot 9 + \Big(\frac{-1}{u}\Big)\Big)\sum_{d_1 d_2 = u}\Big(\frac{-1}{d_1}\Big)d_2^4 \ \text{ for } \ \alpha \ge 2.\end{aligned}$$

Calculating the values of $\rho(n;f)$ by the above formulas for all $n \le 10$, we get

$$\theta(\tau;f) = 1 + \frac{16}{5}Q + \frac{224}{5}Q^2 + 256Q^3 + \frac{4612}{5}Q^4 + \frac{10016}{5}Q^5 + 3584Q^6 +$$
$$+ 7680Q^7 + \frac{73732}{5}Q^8 + \frac{103696}{5}Q^9 + \frac{140224}{5}Q^{10} + \cdots. \tag{3.9}$$

It follows from (1.8) that

$$\vartheta_{00}^8(\tau;0,2)\vartheta_{00}^2(\tau;0,8) = 1 + 16Q + 112Q^2 + 448Q^3 + 1140Q^4 + 2080Q^5 +$$
$$+3584Q^6 + 7296Q^7 + 13876Q^8 + 20240Q^9 + 27104Q^{10} + \cdots. \tag{3.10}$$

II. From (1.8) and (1.9) we have

(1) $$\frac{1}{8}\vartheta_{80}^{(4)}(\tau;0,8)\vartheta_{80}(\tau;0,8) = \frac{1}{8}4096\pi^4\sum_{m_1=-\infty}^{\infty}(2m_1+1)^4 Q^{(2m_1+1)^2}\times$$
$$\times\sum_{m_2=-\infty}^{\infty}Q^{(2m_1+1)^2} = 512\pi^4\cdot 4Q^2(1 + 81Q^8 + 625Q^{24} + \cdots)\times$$
$$\times(1 + Q^8 + Q^{24} + \cdots) = 512\pi^4(4Q^2 + 328Q^{10} + \cdots), \tag{3.11}$$
$$\frac{3}{8}\vartheta_{80}''^2(\tau;0,8) = \frac{3}{8}4096\pi^4\sum_{m_1=-\infty}^{\infty}(2m_1+1)^4 Q^{(2m_1+1)^2}\times$$

$$\times \sum_{m_2=-\infty}^{\infty} (2m_2+1)^4 Q^{(2m_1+1)^2} = 3 \cdot 512\pi^4 \cdot 4Q^2 \times$$

$$\times (1+9Q^8+25Q^{24}+\cdots)^2 = 512\pi^4(12Q^2+216Q^{10}+\cdots); \quad (3.12)$$

$$(2) \ \ \frac{1}{8}\vartheta_{00}^{(4)}(\tau;0,8)\vartheta_{00}(\tau;0,8) = \frac{1}{8}\cdot 2^{16}\pi^4 \sum_{m_1=-\infty}^{\infty} m_1^4 Q^{4m_1^2} \sum_{m_2=-\infty}^{\infty} Q^{4m_2^2} =$$

$$= 8192\pi^4(2Q^4+4Q^8+32Q^{16}+\cdots), \quad (3.13)$$

$$\frac{3}{8}\vartheta_{\infty}''^{2}(\tau;0,8) = \frac{3}{8}\cdot 2^{16}\pi^4 \sum_{m_1=-\infty}^{\infty} m_1^2 Q^{4m_1^2} \sum_{m_2=-\infty}^{\infty} m_2^2 Q^{4m_2^2} =$$

$$= 8192\pi^4(12Q^8+96Q^{20}+\cdots); \quad (3.14)$$

$$(3) \ \ \frac{1}{2}\vartheta_{40}^{(4)}(\tau;0,4)\vartheta_{40}(\tau;0,4) = \frac{1}{2}\cdot 256\pi^4 \sum_{m_1=-\infty}^{\infty} (2m_1+1)^4 Q^{(2m_1+1)^2/2} \times$$

$$\times \sum_{m_2=-\infty}^{\infty} Q^{(2m_2+1)^2/2} =$$

$$= 128\pi^4(4Q+328Q^5+324Q^9+2504Q^{13}+\cdots), \quad (3.15)$$

$$\frac{3}{2}\vartheta_{40}''^{2}(\tau;0,4) = \frac{3}{2}\cdot 256\pi^4 \sum_{m_1=-\infty}^{\infty} (2m_1+1)^2 Q^{(2m_1+1)^2/2} \times$$

$$\times \sum_{m_2=-\infty}^{\infty} (2m_2+1)^2 Q^{(2m_2+1)^2/2} =$$

$$= 128\pi^4(12Q+216Q^5+972Q^9+600Q^{13}+\cdots); \quad (3.16)$$

$$(4) \ \ \frac{1}{2}\vartheta_{40}''(\tau;0,4)\vartheta_{01}(\tau;0,8) = \frac{1}{2}\cdot 16\pi^2 \sum_{m_1=-\infty}^{\infty} (2m_1+1)^2 Q^{(2m_1+1)^2/2} \times$$

$$\times \sum_{m_3=-\infty}^{\infty} (-1)^{m_3} Q^{4m_3^2} = -8\pi^2 \cdot 2Q^{1/2}(1+9Q^4+25Q^{12}+\cdots)(1-2Q^4+$$

$$+2Q^{16}-\cdots) = -8\pi^2\cdot 2Q^{1/2}(1+7Q^4-18Q^8+25Q^{12}+\cdots), \ (3.17)$$

$$\frac{1}{4}\vartheta_{40}(\tau;0,4)\vartheta_{01}''(\tau;0,8) = -\frac{1}{4}\sum_{m_1=-\infty}^{\infty} Q^{(2m_1+1)^2/2}\cdot 256\pi^2 \times$$

$$\times \sum_{m_3=-\infty}^{\infty} (-1)^{m_3} m_3^2 Q^{4m_3^2} = -64\pi^2\cdot 2Q^{1/2}(1+Q^4+Q^{12}+\cdots)(-2Q^4+$$

$$+8Q^{16}-\cdots) = -64\pi^2\cdot 2Q^{1/2}(-2Q^4-2Q^8+6Q^{16}+\cdots), \quad (3.18)$$

$$\vartheta'_{41}(\tau;0,4)\vartheta_{01}(\tau;0,8) = -4\pi i \sum_{m_2=-\infty}^{\infty} (-1)^{m_2} Q^{(2m_2+1)^2/2} \times$$
$$\times \sum_{m_4=-\infty}^{\infty} (-1)^{m_4} Q^{4m_4^2} = 4\pi i \cdot 2Q^{1/2}(1 - 3Q^4 + 5Q^{12} - \cdots)(1 - 2Q^4 +$$
$$+2Q^{16} - \cdots) = 4\pi i \cdot 2Q^{1/2}(1 - 5Q^4 + 6Q^8 + 5Q^{12} - \cdots), \quad (3.19)$$

(5) $$\vartheta'^3_{81}(\tau;0,8)\vartheta_{01}(\tau;0,8) = -512\pi^3 i \sum_{m_1,m_2,m_3=-\infty}^{\infty} (-1)^{m_1+m_2+m_3}(2m_1+1) \times$$
$$\times(2m_2+1)(2m_3+1)Q^{(2m_1+1)^2+(2m_2+1)^2+(2m_3+1)^2} \sum_{m_4=-\infty}^{\infty} (-1)^{m_4} Q^{4m_4^2} =$$
$$= -512\pi^3 i(8Q^3 - 16Q^7 - 72Q^{11} + \cdots). \quad (3.20)$$

III. From (3.4), (3.11), and (3.12) it follows that

$$\frac{1}{512\pi^4}\Psi_2(\tau;8,8;0,0;0,0;4,4) = -8Q^2 + 112Q^{10} + \cdots. \quad (3.21)$$

From (1.16), (3.13), and (3.14) it follows that

$$\frac{1}{8192\pi^4}\Psi_2(\tau;0,0;0,0;0,0;4,4) = 2Q^4 - 8Q^8 + 31Q^{16} + \cdots. \quad (3.22)$$

From (3.5), (3.15), and (3.16) it follows that

$$\frac{1}{128\pi^4}\Psi_2(\tau;4,4;0,0;0,0;2,2) = -8Q + 112Q^5 + 1904Q^{13} - \cdots. \quad (3.23)$$

From (3.6), (3.17)–(3.19) it follows that

$$-\frac{1}{32\pi^3 i}\Psi_3(\tau;4,0,4,0;0,1,1,1;0,0,0,0;2,4,2,4) =$$
$$= 4Q + 72Q^5 - 444Q^9 + 712Q^{13} + \cdots. \quad (3.24)$$

From (3.7) and (3.20) follows

$$-\frac{1}{512\pi^3 i}\Psi_4(\tau;8,8,8,0;1,1,1,1;0,0,0,0;4,4,4,4) =$$
$$= 8Q^3 - 16Q^7 - 72Q^{11} + \cdots. \quad (3.25)$$

According to (3.1), (3.9), (3.10), and (3.21)–(3.25) it is easy to verify that all coefficients by Q^n ($n \le 10$) in the expansion of the function $\psi(\tau; f)$ by powers of Q are zero. □

Theorem 2a. *Let $n = 2^\alpha u$. Then*

$$r(n;f) = \frac{16}{5}\sum_{d_1d_2=u}\Big(\frac{-1}{d}\Big)d_2^4 - \frac{12}{5}\sum_{\substack{x^2+y^2=2u\\ 2\nmid x, 2\nmid y, x>0 y>0}}(x^4-3x^2y^2)+$$

$$+8\sum_{\substack{x^2+y^2+8(z^2+t^2)=2u\\ 2\nmid x, 2\nmid y, x>0, y>0}}(-1)^{(y-1)/2+z+t}(x^2-8z^2)y)$$

$$\textit{for } \ \alpha=0,\ u\equiv 1 \pmod 4,$$

$$= \frac{16}{5}\sum_{d_1d_2=u}\Big(\frac{-1}{d}\Big)d_2^4 + 192\sum_{\substack{x^2+y^2+z^2+4t^2=u\\ 2\nmid x, 2\nmid y, 2\nmid z,\\ x>0, y>0, z>0}}(-1)^{(x+y+z-3)/1+t}xyz$$

$$\textit{for } \ \alpha=0,\ u\equiv 3 \pmod 4,$$

$$= \frac{224}{5}\sum_{d_1d_2=u}\Big(\frac{-1}{d}\Big)d_2^4 - \frac{168}{5}\sum_{\substack{x^2+y^2=2u\\ 2\nmid x, 2\nmid y, x>0, y>0}}(x^4-3x^2y^2)$$

$$\textit{for } \ \alpha=0=0,\ u\equiv 1 \pmod 4,$$

$$= \frac{224}{5}\sum_{d_1d_2=u}\Big(\frac{-1}{d}\Big)d_2^4 \ \ \textit{for } \ \alpha=1,\ u\equiv 3 \pmod 4,$$

$$= \frac{4612}{5}\sum_{d_1d_2=u}\Big(\frac{-1}{d}\Big)d_2^4 + \frac{1088}{5}\sum_{\substack{x^2+y^2=u\\ 2\nmid x, x>0, y=0}}(x^4-3x^2y^2)+$$

$$+\frac{2176}{5}\sum_{\substack{x^2+y^2=u\\ x\not\equiv y \pmod 2\\ x>0, y>0}}(x^4-3x^2y^2) \ \ \textit{for } \ \alpha=2,\ u\equiv 1 \pmod 8,$$

$$= \frac{4612}{5}\sum_{d_1d_2=u}\Big(\frac{-1}{d}\Big)d_2^4 + \frac{2176}{5}\sum_{\substack{x^2+y^2=u\\ x\not\equiv y \pmod 2\\ x>0, y>0}}(x^4-3x^2y^2)$$

$$\textit{for } \ \alpha=2,\ u\equiv 5 \pmod 8,$$

$$= \frac{4}{5}(2^{4\alpha-1}\cdot 9-1)\sum_{d_1d_2=u}\Big(\frac{-1}{d}\Big)d_2^4 \ \ \textit{for } \ \alpha\geq 2,\ u\equiv 3 \pmod 4,$$

$$= \frac{73732}{5}\sum_{d_1d_2=u}\Big(\frac{-1}{d}\Big)d_2^4 + \frac{2176}{5}\sum_{\substack{x^2+y^2=2u\\ 2\nmid x, 2\nmid y, x>0, y>0}}(x^4-3x^2y^2)$$

$$\textit{for } \ \alpha=3,\ u\equiv 1 \pmod 4,$$

$$= \frac{4}{5}(2^{4\alpha-1}\cdot 9+1)\sum_{d_1d_2=u}\Big(\frac{-1}{d}\Big)d_2^4+\frac{2176}{5}\sum_{\substack{x^2+y^2=2^{\alpha-2}u\\ 2|x,2|y,x>0,y>0}}(x^4-3x^2y^2)$$

$$\textit{for } 2\dagger\alpha,\ \alpha>3,\ u\equiv 1 \pmod 4,$$

$$= \frac{4}{5}(2^{4\alpha-1}\cdot 9+1)\sum_{d_1d_2=u}\Big(\frac{-1}{d}\Big)d_2^4+\frac{1088}{5}\sum_{\substack{x^2+y^2=2^{\alpha-2}u\\ 2|x,x>0,y=0}}(x^4-3x^2y^2)+$$

$$+\frac{2176}{5}\sum_{\substack{x^2+y^2=2^{\alpha-2}u\\ 2|x,2|y,x>0,y>0}}(x^4-3x^2y^2)\ \textit{ for } 2|\alpha,\ \alpha>3,\ u\equiv 1 \pmod 8,$$

$$= \frac{4}{5}(2^{4\alpha-1}\cdot 9+1)\sum_{d_1d_2=u}\Big(\frac{-1}{d}\Big)d_2^4+\frac{2176}{5}\sum_{\substack{x^2+y^2=2^{\alpha-2}u\\ 2|x,2|y,x>0,y>0}}(x^4-3x^2y^2)$$

$$\textit{for } 2|\alpha,\ \alpha>3,\ u\equiv 5 \pmod 8.$$

Proof. Equating the coefficients by Q^n in both sides of (3.8), we obtain

$$r(n;f)=\rho(n;f)-\frac{42}{5}\nu_1(n)+\frac{544}{5}\nu_2(n)-\\ -\frac{3}{5}\nu_3(n)+2\nu_4(n)+24\nu_5(n), \tag{3.26}$$

where $\nu_k(n)$ $(k=1,2,3,4,5)$ denote respectively the coefficients by Q^n in the expansions of the functions $\frac{1}{512\pi^4}\Psi_2(\tau;8,8;0,0;0,0;4,4)$, $\frac{1}{8192\pi^4}\Psi_2(\tau;0,0;0,0;0,0;4,4)$, $\frac{1}{128\pi^4}\Psi_2(\tau;4,4;0,0;0,0;2,2)$, $-\frac{1}{32\pi^3 i}\Psi_3(\tau;4,0,4,0;0,1,1,1;0,0,0,0;2,4,2,4)$, $-\frac{1}{512\pi^3 i}\Psi_4(\tau;8,8,8,0;1,1,1,1;0,0,0,0;4,4,4,4)$ by powers of Q.

From (3.4), (3.11), and (3.12) it follows that

$$\frac{1}{512\pi^4}\Psi_2(\tau;8,8;0,0;0,0;4,4)=\sum_{m_1,m_2=-\infty}^{\infty}(2m_1+1)^4Q^{(2m_1+1)^2+(2m_2+1)^2}-\\ -3\sum_{m_1,m_2=-\infty}^{\infty}(2m_1+1)^2(2m_2+1)^2Q^{(2m_1+1)^2+(2m_2+1)^2},$$

i.e.,

$$\nu_1(n)=4\sum_{\substack{x^2+y^2=n\\ 2\dagger x,2\dagger y,x>0,y>0}}(x^4-3x^2y^2); \tag{3.27}$$

it is not difficult to verify that

$$\nu_1(n) \neq 0 \quad \text{for} \quad \alpha = 1, \quad u \equiv 1 \pmod 4 \ \text{only}.$$

From (1.16), (3.13), and (3.14) follows

$$\frac{1}{8192\pi^4}\Psi_2(\tau;0,0;0,0;0,0;4,4) = \sum_{m_1,m_2=-\infty}^{\infty} m_1^4 Q^{4m_1+4m_2^2} -$$

$$-3\sum_{m_1,m_2=-\infty}^{\infty} m_1^2 m_2^2 Q^{4m_1^2+4m_2^2},$$

i.e.,

$$\nu_2(n) = 4\sum_{4(x^2+y^2)=n}(x^4-3x^2y^2) =$$

$$= \begin{cases} \sum_{x^2+y^2=2^{\alpha-2}u}(x^4-3x^2y^2) & \text{for } \alpha \geq 2, \\ 0 & \text{for } \alpha = 0,1; \end{cases} \tag{3.28}$$

it is obvious that

$$\sum_{x^2+y^2=2^{\alpha-2}u}(x^4-3x^2y^2) = \sum_{\substack{x^2+y^2=u \\ x\not\equiv y \pmod 2}}(x^4-3x^2y^2) \text{ for } \alpha=2,\ u\equiv 1 \pmod 4,$$

$$= 0 \quad \text{for } \alpha \geq 2,\ u \equiv 3 \pmod 4,$$

$$= \sum_{\substack{x^2+y^2=2^{\alpha-2}u \\ x\equiv y \pmod 2}}(x^4-3x^2y^2) \text{ for } \alpha\geq 3,\ u\equiv 1 \pmod 4),$$

From (3.5), (3.15), and (3.16) it follows that

$$\frac{1}{128\pi^4}\Psi_2(\tau;4,4;0,0;0,0;2,2) = \sum_{m_1,m_2=-\infty}^{\infty}(2m_1+1)^4 Q^{(2m_1+1)^2/2+(2m_2+1)^2/2} -$$

$$-3\sum_{m_1,m_2=-\infty}^{\infty}(2m_1+1)^2(2m_2+1)^2 Q^{(2m_1+1)^2/2+(2m_2+1)^2/2},$$

i.e.,

$$\nu_3(n) = \sum_{\substack{x^2+y^2=2u \\ 2\dagger x, 2\dagger y}}(x^4-3x^2y^2) = 4\sum_{\substack{x^2+y^2=2u \\ 2\dagger x, 2\dagger y, x>0, y>0}}(x^4-3x^2y^2)$$

$$\text{for } \alpha = 0,\ u \equiv 1 \pmod 4, \tag{3.29}$$

$$= 0 \quad \text{for } \alpha = 0,\ u \equiv 3 \pmod 4 \ \text{ and for } \ \alpha > 0.$$

From (3.6) and (3.17)–(3.19) it follows that

$$-\frac{1}{32\pi^3 i}\Psi_3(\tau;4,0,4,0;0,1,1,1;0,0,0,0;2,4,2,4)=$$
$$=\Bigg\{\sum_{m_1,m_3=-\infty}^{\infty}(-1)^{m_3}(2m_1+1)^2Q^{(2m_1+1)^2/2+4m_3^2}-$$
$$-8\sum_{m_1,m_3=-\infty}^{\infty}(-1)^{m_3}2m_3^2Q^{(2m_1+1)^2/2+4m_3^2}\Bigg\}\times$$
$$\times\sum_{m_2,m_4=-\infty}^{\infty}(-1)^{m_2+m_4}(2m_2+1)Q^{(2m_2+1)^2/2+4m_4^2}=$$
$$=\sum_{m_1,m_2,m_3,m_4=-\infty}^{\infty}(-1)^{m_2+m_3+m_4}((2m_1+1)^2(2m_2+1)-8(2m_2+1)m_3^2)\times$$
$$\times Q^{\frac{1}{2}\{(2m_1+1)^2+(2m_2+1)^2+8m_3^2+8m_4^2},$$

i.e.,

$$\nu_4(n)=\sum_{\substack{x^2+y^2+8(z^2+t^2)=2n\\ 2\nmid x,2\nmid y}}(-1)^{(y-1)/2+z+t}(x^2-8z^2)y=$$
$$=4\sum_{\substack{x^2+y^2+8(z^2+t^2)=2u\\ 2\nmid x,2\nmid y,x>0,y>0}}(-1)^{(y-1)/2+z+t}(x^2-8z^2)y \qquad (3.30)$$
$$\text{for } \alpha=0,\ u\equiv 1 \pmod 4,$$
$$=0 \quad \text{for } \alpha=0,\ u\equiv 3 \pmod 4 \ \text{ and for } \ \alpha>0.$$

From (3.7) and (3.20) follows

$$-\frac{1}{512\pi^3 i}\Psi_3(\tau;8,8,8,0;1,1,1,1;0,0,0,0;4,4,4,4)=$$
$$=\sum_{m_1,m_2,m_3,m_4=-\infty}^{\infty}(-1)^{\sum_{k=1}^4 m_k}\prod_{k=1}^{3}(2m_k+1)Q^{\sum_{k=1}^3(2m_k+1)^2+4m_4^2},$$

i.e.,

$$\nu_5(n)=8\sum_{\substack{x^2+y^2+z^2+4t^2=n\\ 2\nmid x,2\nmid y,2\nmid z\\ x>0,y>0,z>0}}(-1)^{(x+y+z-3)/2+t}xyz; \qquad (3.31)$$

it is easy to verify that

$$\nu_5(n)\neq 0 \quad \text{only for} \ \ \alpha=0,\ u\equiv 3 \pmod 4.$$

The above theorem follows from (3.26), formulas for $\rho(n; f)$, and (3.27)–(3.31). $\square$

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