

**ON THE NUMBER OF REPRESENTATIONS OF
INTEGERS BY SOME QUADRATIC FORMS IN TEN
VARIABLES**

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ABSTRACT. A method of finding the so-called Liouville's type formulas for the number of representations of integers by

$$a_1(x_1^2 + x_2^2) + a_2(x_3^2 + x_4^2) + a_3(x_5^2 + x_6^2) + a_4(x_7^2 + x_8^2) + a_5(x_9^2 + x_{10}^2)$$

quadratic forms is developed.

In the papers [4,5] four classes of entire modular forms of weight 5 for the congruence subgroup $\Gamma_0(4N)$ are constructed. The Fourier coefficients of these modular forms have a simple arithmetical sense. This allows one to get sometimes the so-called Liouville's type formulas for the number of representations of positive integers by positive quadratic forms in ten variables.

In the present paper we consider positive primitive quadratic forms

$$f = a_1(x_1^2 + x_2^2) + a_2(x_3^2 + x_4^2) + a_3(x_5^2 + x_6^2) + a_4(x_7^2 + x_8^2) + a_5(x_9^2 + x_{10}^2). \quad (1)$$

For the purpose of illustration we obtain a formula for the number of representations of positive integers by the form (1) for $a_1 = \dots = a_4 = 1$ $a_5 = 4$. In a similar way one can investigate as well other forms of the kind (1). As is well known, Liouville obtained in 1865 the corresponding formula for $a_1 = \dots = a_5 = 1$ only.

1. SOME KNOWN RESULTS

1.1. In this paper $N, a, d, k, n, q, r, s, \lambda$ denote positive integers; b, u, v are odd positive integers; p is a prime number; ν, l are non-negative integers; $H, c, g, h, j, m, x, y, \alpha, \beta, \gamma, \delta$ are integers; i is an imaginary unit; z, τ are

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complex variables ($\text{Im } \tau > 0$); $e(z) = \exp 2\pi iz$; $Q = e(\tau)$; $(\frac{h}{u})$ is the generalized Jacobi symbol. Further, $\sum_{h \bmod q}$ and $\sum'_{h \bmod q}$ denote respectively sums in which h runs a complete and a reduced residue system modulo q .

Let

$$S(h, q) = \sum_{j \bmod q} e\left(\frac{hj^2}{q}\right) \quad (\text{Gaussian sum}), \tag{1.1}$$

$$c(h, q) = \sum'_{j \bmod q} e\left(\frac{hj^2}{q}\right) \quad (\text{Ramanujan's sum}), \tag{1.2}$$

$$\begin{aligned} & \vartheta_{gh}(z|\tau; c, N) = \\ &= \sum_{m \equiv c \pmod{N}} (-1)^{h(m-c)/N} e\left(\frac{1}{2N}\left(m + \frac{g}{2}\right)^2 \tau\right) e\left(\left(m + \frac{g}{2}\right)z\right) \tag{1.3} \\ & \quad (\text{theta-function with characteristics } g, h), \end{aligned}$$

hence

$$\begin{aligned} \frac{\partial^n}{\partial z^n} \vartheta_{gh}(z|\tau; c, N) &= (\pi i)^n \sum_{m \equiv c \pmod{N}} (-1)^{h(m-c)/N} (2m + g)^n \times \\ & \quad \times e\left(\frac{1}{2N}\left(m + \frac{g}{2}\right)^2 \tau\right) e\left(\left(m + \frac{g}{2}\right)z\right). \tag{1.4} \end{aligned}$$

Put

$$\begin{aligned} \vartheta_{gh}(\tau; c, N) &= \vartheta_{gh}(0|\tau; c, N), \\ \vartheta_{gh}^{(n)}(\tau; c, N) &= \frac{\partial^n}{\partial z^n} \vartheta_{gh}(z|\tau; c, N) \Big|_{z=0}. \tag{1.5} \end{aligned}$$

It is known (see, for example, [3], p. 112, formulas (2.3) and (2.5)) that

$$\begin{aligned} \vartheta_{g+2j, h}(\tau; c, N) &= \vartheta_{gh}(\tau; c + j, N), \\ \vartheta_{g+2j, h}^{(n)}(\tau; c, N) &= \vartheta_{gh}^{(n)}(\tau; c + j, N), \tag{1.6} \end{aligned}$$

$$\begin{aligned} \vartheta_{gh}(\tau; c + N_j, N) &= (-1)^{hj} \vartheta_{gh}(\tau; c, N), \\ \vartheta_{gh}^{(n)}(\tau; c + N_j, N) &= (-1)^{hj} \vartheta_{gh}^{(n)}(\tau; c, N). \tag{1.7} \end{aligned}$$

From (1.3), in particular, according to the notations (1.5), it follows that

$$\vartheta_{gh}(\tau; 0, N) = \sum_{m=-\infty}^{\infty} (-1)^{hm} Q^{(2Nm+g)^2/8N}, \tag{1.8}$$

$$\vartheta_{gh}^{(n)}(\tau; 0, N) = (\pi i)^n \sum_{m=-\infty}^{\infty} (-1)^{hm} (2Nm + g)^n Q^{(2Nm+g)^2/8N}. \tag{1.9}$$

From (1.8) and (1.9) it follows that

$$\vartheta_{-g,h}(\tau; 0, N) = \vartheta_{gh}(\tau; 0, N), \quad \vartheta_{-g,h}^{(n)}(\tau; 0, N) = (-1)^n \vartheta_{gh}^{(n)}(\tau; 0, N). \quad (1.10)$$

Everywhere in this paper a denote a least common multiple of the coefficients a_k of the quadratic form (1) and $\Delta = \prod_{k=1}^5 a_k^2$ is its determinant.

Denoting by $r(n; f)$ the number of representations of n by the form (1), we get

$$\prod_{k=1}^5 \vartheta_{00}^2(\tau; 0, a_k) = 1 + \sum_{n=1}^{\infty} r(n; f) Q^n. \quad (1.11)$$

Further, put

$$\theta(\tau; f) = 1 + \sum_{n=1}^{\infty} \rho(n; f) Q^n, \quad (1.12)$$

where

$$\rho(n; f) = \frac{\pi^5}{4! \Delta^{1/2}} n^4 \sum_{q=1}^{\infty} A(q) \quad (1.13)$$

(singular series of the problem) and

$$A(q) = q^{-10} \sum'_{h \bmod q} e\left(-\frac{hn}{q}\right) \prod_{k=1}^5 S^2(a_k h, q). \quad (1.14)$$

Finally let

$$\Gamma_0(4N) = \left\{ \begin{matrix} \alpha\tau + \beta \\ \gamma\tau + \delta \end{matrix} \mid \alpha\delta - \beta\gamma = 1, \gamma \equiv 0 \pmod{4N} \right\}$$

(nonhomogeneous congruence subgroup).

1.2. For the convenience of references we quote some known results as the following lemmas.

Lemma 1. *If $(h, q) = 1$, then*

$$S(kh, kq) = kS(h, q).$$

Lemma 2 (see, for example, [6], p. 13, Lemma 6). *If $(h, q) = 1$, then*

$$\begin{aligned} S^2(h, q) &= \left(\frac{-1}{q}\right)q \text{ for } q \equiv 1 \pmod{2}, \\ &= 2i^h q \text{ for } q \equiv 0 \pmod{4}, \\ &= 0 \text{ for } q \equiv 2 \pmod{4}. \end{aligned}$$

Lemma 3 (see, for example, [6], p. 16, Lemma 8). *If $(h, q) = 1$, then*

$$S(h, u) = \left(\frac{h}{u}\right) i^{(u-1)^2/4} u^{1/2}.$$

Lemma 4 (see, for example [6], p. 177, formula 20). *Let $q = p^\lambda$ and $p^\nu \parallel h$. Then*

$$\begin{aligned} c(h, q) &= 0 \quad \text{for } \nu < \lambda - 1, \\ &= -p^{\lambda-1} \quad \text{for } \nu = \lambda - 1, \\ &= p^{\lambda-1}(p - 1) \quad \text{for } \nu > \lambda - 1. \end{aligned}$$

Lemma 5 (see, for example, [2], p. 14, Lemma 10). *Let*

$$\chi_p = 1 + A(p) + A(p)^2 + \dots . \tag{1.15}$$

Then

$$\sum_{q=1}^{\infty} A(q) = \prod_p \chi_p.$$

Lemma 6 ([1], pp. 811 and 953). *The entire modular form $F(\tau)$ of weight r for the congruence subgroup $\Gamma_0(4N)$ is identically zero, if in its expansion in the series*

$$\begin{aligned} F(\tau) &= \sum_{m=0}^{\infty} C_m Q^m, \\ C_m &= 0 \quad \text{for all } m \leq \frac{r}{3} N \prod_{p|4N} \left(1 + \frac{1}{p}\right). \end{aligned}$$

Lemma 7 ([7], p. 18, Lemma 14). *The function $\theta(\tau; f)$ is an entire modular form of weight 5 and character $\chi(\delta) = \text{sgn } \delta \left(\frac{-\Delta}{|\delta|}\right)$ for $\Gamma_0(4a)$.*

Lemma 8 ([2], p. 21, the remark to Lemma 18). *The function $\prod_{k=1}^5 \vartheta_{00}(\tau; 2, a_k)$ is an entire modular form of weight 5 and character $\chi(\delta) = \text{sgn } \delta \left(\frac{-\Delta}{|\delta|}\right)$ for $\Gamma_0(4a)$.*

Lemma 9 ([4], p. 67, Theorem 1¹ and [5], p. 193, Theorem 1). *For a given N the functions*

$$\begin{aligned} (1) \quad &\Psi_2(\tau; g_1, g_2; h_1, h_2; 0, 0; N_1, N_2) = \\ &= \frac{1}{N_1^2} \vartheta_{g_1 h_1}^{(4)}(\tau; 0, 2N_1) \vartheta_{g_2 h_2}(\tau; 0, 2N_2) + \\ &+ \frac{1}{N_2^2} \vartheta_{g_1 h_1}(\tau; 0, 2N_1) \vartheta_{g_2 h_2}^{(4)}(\tau; 0, 2N_2) - \\ &- \frac{6}{N_1 N_2} \vartheta_{g_1 h_1}''(\tau; 0, 2N_1) \vartheta_{g_2 h_2}''(\tau; 0, 2N_2), \end{aligned} \tag{1.16}$$

¹There is a misprint in the formulation of Theorem 1 [4, p. 67] which can be easily corrected by substituting 5 for 10 and vice versa.

where

- (a) $2|g_1, 2|g_2, N_1|N, N_2|N, 4|N\left(\frac{h_1^2}{N_1} + \frac{h_2^2}{N_2}\right), 4\left|\frac{g_1^2}{4N_1} + \frac{g_2^2}{4N_2}\right,$
- (b) for all α and δ with $\alpha\delta \equiv 1 \pmod{4N}$

$$\begin{aligned} & \left(\frac{N_1 N_2}{|\delta|}\right) \Psi_2(\tau; \alpha g_1, \alpha g_2; h_1, h_2; 0, 0; N_1, N_2) = \\ & = \left(\frac{\Delta}{|\delta|}\right) \Psi_2(\tau; g_1, g_2; h_1, h_2; 0, 0; N_1, N_2), \end{aligned}$$

$$\begin{aligned} (2) \quad & \Psi_3(\tau; g_1, \dots, g_4; h_1, \dots, h_4; 0, \dots, 0; N_1, \dots, N_4) = \\ & = \left\{ \frac{1}{N_1} \vartheta''_{g_1 h_1}(\tau; 0, 2N_1) \vartheta_{g_2 h_2}(\tau; 0, 2N_2) - \right. \\ & \quad \left. - \frac{1}{N_2} \vartheta_{g_1 h_1}(\tau; 0, 2N_1) \vartheta''_{g_2 h_2}(\tau; 0, 2N_2) \right\} \times \\ & \quad \times \vartheta'_{g_3 h_3}(\tau; 0, 2N_3) \vartheta_{g_4 h_4}(\tau; 0, 2N_4) \end{aligned} \tag{1.17}$$

and

$$\begin{aligned} (3) \quad & \Psi_4(\tau; g_1, \dots, g_4; h_1, \dots, h_4; 0, \dots, 0; N_1, \dots, N_4) = \\ & = \prod_{k=1}^3 \vartheta'_{g_k h_k}(\tau; 0, 2N_k) \vartheta_{g_4 h_4}(\tau; 0, 2N_4), \end{aligned} \tag{1.18}$$

where

- (a) $2|g_k, N_k|N \ (k = 1, 2, 3, 4), 4|N \sum_{k=1}^4 \frac{h_k^2}{N_k}, 4\left|\sum_{k=1}^4 \frac{g_k^2}{4N_k}\right>,$
- (b) for all α and δ with $\alpha\delta \equiv 1 \pmod{4N}$

$$\begin{aligned} & \left(\frac{\prod_{k=1}^4 N_k}{|\delta|}\right) \Psi_j(\tau; \alpha g_1, \dots, \alpha g_4; h_1, \dots, h_4; 0, \dots, 0; N_1, \dots, N_4) = \\ & = \operatorname{sgn} \delta \left(\frac{-\Delta}{|\delta|}\right) \Psi_j(\tau; g_1, \dots, g_4; h_1, \dots, h_4; 0, \dots, 0; N_1, \dots, N_4) \quad (j = 3, 4), \end{aligned}$$

are entire modular forms of weight 5 and character $\chi(\delta) = \operatorname{sgn} \delta \left(\frac{-\Delta}{|\delta|}\right)$ for $\Gamma_0(4N)$.

2. SUMMATION OF THE SINGULAR SERIES $\rho(n; f)$

Everywhere in this section α, β, γ denote non-negative integers and m positive odd integers.

Lemma 10. *Let $n = 2^\alpha m, a_k = 2^{\gamma_k} b_k \ (k = 1, 2, \dots, 5), (b_1, \dots, b_5) = 1, b = [b_1, \dots, b_5], \gamma_5 \geq \gamma_4 \geq \gamma_3 \geq \gamma_2 \geq \gamma_1 = 0, \gamma = \sum_{k=1}^5 \gamma_k$. Then*

$$\begin{aligned} \chi_2 &= 1 + (-1)^{(b_1-m)/2} \text{ for } 0 \leq \alpha \leq \gamma_2 - 2, \\ &= 1 \text{ for } \alpha = \gamma_2 - 1, \alpha = \gamma_2 < \gamma_3, \gamma_2 = \gamma_3 \leq \alpha = \gamma_4 - 1, \\ &\quad \gamma_2 + 1 = \gamma_3 \leq \alpha = \gamma_4 - 1, \gamma_2 = \gamma_3 \leq \alpha = \gamma_4 < \gamma_5, \end{aligned}$$

$$\begin{aligned}
& \gamma_2 + 1 = \gamma_3 \leq \alpha = \gamma_4 < \gamma_5, \\
& = 1 + (-1)^{(b_1+b_2)/2} (1 - 2^{\gamma_2-\alpha} \cdot 3) \text{ for } \gamma_2 + 1 \leq \alpha < \gamma_3, \\
& = 1 + (-1)^{(\sum_{k=1}^3 b_k - m)/2} \cdot 2^{\gamma_2+\gamma_3-2\alpha-2} \text{ for } \gamma_2 = \gamma_3 \leq \alpha < \gamma_4 - 2, \\
& \quad \gamma_2 + 1 = \gamma_3 \leq \alpha \leq \gamma_4 - 2, \\
& = 1 + (-1)^{(b_1+b_2)/2} (1 - 2^{\gamma_2-\gamma_3+1}) + (-1)^{(\sum_{k=1}^3 b_k - m)/2} \cdot 2^{\gamma_2-\gamma_3-2\alpha-2} \\
& \quad \text{for } \gamma_2 + 2 \leq \gamma_3 \leq \alpha \leq \gamma_4 - 2, \\
& = 1 + (-1)^{(b_1+b_2)/2} (1 - 2^{\gamma_2-\gamma_3+1}) \text{ for } \gamma_2 + 2 \leq \gamma_3 \leq \alpha = \gamma_4 - 1, \\
& \quad \gamma_2 + 2 \leq \gamma_3 \leq \alpha = \gamma_4 < \gamma_5, \\
& = 1 + (-1)^{(b_1+b_2)/2} (1 - 2^{\gamma_2-\gamma_3+1}) + (-1)^{(\sum_{k=1}^4 b_k)/2} \times \\
& \times 2^{\gamma_2+\gamma_3-2\gamma_4} \left\{ \frac{1}{7} (1 - 2^{-3(\alpha-\gamma_4-1)}) - 2^{-3(\alpha-\gamma_4)} \right\} \\
& \quad \text{for } \gamma_4 + 1 \leq \alpha < \gamma_5 \text{ but } \gamma_4 \geq \gamma_3 \geq \gamma_2 + 2, \\
& = 1 + (-1)^{(\sum_{k=1}^4 b_k)/2} \cdot 2^{\gamma_2+\gamma_3-2\gamma_4} \left\{ \frac{1}{7} (1 - 2^{-3(\alpha-\gamma_4-1)}) - 2^{-3(\alpha-\gamma_4)} \right\} \\
& \quad \text{for } \gamma_4 + 1 \leq \alpha < \gamma_5 \text{ but } \gamma_4 \geq \gamma_3 = \gamma_2 \text{ or } \gamma_4 \geq \gamma_3 = \gamma_2 + 1, \\
& = 1 + (-1)^{(\sum_{k=1}^5 b_k - m)/2} \cdot 2^{\gamma-4\alpha-4} \text{ for } \alpha \geq \gamma_5 = \gamma_4 \geq \gamma_3 = \gamma_2, \\
& \quad \alpha \geq \gamma_5 = \gamma_4 \geq \gamma_3 = \gamma_2 + 1, \\
& = 1 + (-1)^{(b_1+b_2)/2} (1 - 2^{\gamma_2-\gamma_3+1}) + (-1)^{(\sum_{k=1}^5 b_k - m)/2} \cdot 2^{\gamma-4\alpha-4} \\
& \quad \text{for } \alpha \geq \gamma_5 = \gamma_4 \geq \gamma_3 \geq \gamma_2 + 2, \\
& = 1 + (-1)^{(b_1+b_2)/2} (1 - 2^{\gamma_2-\gamma_3+1}) + (-1)^{(\sum_{k=1}^4 b_k)/2} \times \\
& \times 2^{\gamma_2+\gamma_3-2\gamma_4} (1 - 2^{-3(\gamma_5-\gamma_4-1)})/7 + (-1)^{(\sum_{k=1}^5 b_k - m)/2} \cdot 2^{\gamma-4\alpha-4} \\
& \quad \text{for } \alpha \geq \gamma_5 > \gamma_4 \geq \gamma_3 \geq \gamma_2 + 2, \\
& = 1 + (-1)^{(\sum_{k=1}^4 b_k)/2} \cdot 2^{\gamma_2+\gamma_3-2\gamma_4} (1 - 2^{-3(\gamma_5-\gamma_4-1)})/7 + \\
& + (-1)^{(\sum_{k=1}^5 b_k - m)/2} \cdot 2^{\gamma-4\alpha-4} \text{ for } \alpha \geq \gamma_5 > \gamma_4 \geq \gamma_3 = \gamma_2, \\
& \quad \alpha \geq \gamma_5 > \gamma_4 \geq \gamma_3 = \gamma_2 + 1.
\end{aligned}$$

Proof. I. If in (1.14) we put $q = 2^\lambda$ and then instead of h introduce a new letter of summation y defined by the congruence $h \equiv by \pmod{2^\lambda}$, then we get

$$A(2^\lambda) = 2^{-10\lambda} \sum'_{y \pmod{2^\lambda}} e(-2^{\alpha-\lambda} mby) \prod_{k=1}^5 S^2(2^{\gamma_k} b_k by, 2^\lambda). \quad (2.1)$$

From (2.1), according to Lemmas 1, 2, and 4 it follows that

(1) for $\lambda = 1, \gamma_k + 1$ ($k = 2, 3, 4, 5$)

$$A(2^\lambda) = 0, \tag{2.2}$$

because $S^2(b_1by, 2) = 0, S^2(2^{\gamma_k}b_kby, 2^{\gamma_k+1}) = 0$ ($k = 2, 3, 4, 5$);

(2) for $2 \leq \lambda \leq \gamma_2$

$$\begin{aligned} A(2^\lambda) &= 2^{-10\lambda} \sum'_{y \bmod 2^\lambda} e(-2^{\alpha-\lambda}mby)(2i^{b_1by} \cdot 2^\lambda)2^{8\lambda} = \\ &= 2^{1-\lambda} \sum'_{y \bmod 2^\lambda} e\left(\frac{b_1by}{4} - \frac{2^\alpha mby}{2^\lambda}\right) = \\ &= 2^{1-\lambda} e\left(\frac{(b_1 - 2^{\alpha-\lambda+2}m)b}{4}\right) \sum_{y=0}^{2^{\lambda-1}-1} e\left(\frac{(2^{\lambda-2}b_1 - 2^\alpha m)by}{2^{\lambda-1}}\right) = \\ &= \begin{cases} e((b_1 - 2^{\alpha-\lambda+2}m)b/4) & \text{if } 2^{\lambda-1} | (2^{\lambda-2}b_1 - 2^\alpha m)b, \\ 0 & \text{if } 2^{\lambda-1} \nmid (2^{\lambda-2}b_1 - 2^\alpha m)b, \end{cases} \end{aligned}$$

i.e.,

$$A(2^\lambda) = (-1)^{b_1-m)/2} \text{ if } \lambda = \alpha + 2, \tag{2.3}$$

$$= 0 \text{ if } \lambda \neq \alpha + 2; \tag{2.3_1}$$

(3) for $\gamma_2 + 2 \leq \lambda \leq \gamma_3$

$$\begin{aligned} A(2^\lambda) &= 2^{-10\lambda} \sum'_{y \bmod 2^\lambda} e(-2^{\alpha-\lambda}mby)(2i^{b_1by} \cdot 2^\lambda)2^{2\gamma_2} \cdot 2i^{b_2by} \times \\ &\times 2^{\lambda-\gamma_2} \cdot 2^{6\lambda} = 2^{\gamma_2-2\lambda+2}(-1)^{(b_1+b_2)/2}c(2^\alpha mb, 2^\lambda), \end{aligned}$$

hence

$$A(2^\lambda) = (-1)^{(b_2+b_2)/2} \cdot 2^{\gamma_2+1-\lambda} \text{ if } \lambda < \alpha + 1, \tag{2.4}$$

$$= -(-1)^{(b_2+b_2)/2} \cdot 2^{\gamma_2-\alpha} \text{ if } \lambda = \alpha + 1, \tag{2.4_1}$$

$$= 0 \text{ if } \lambda > \alpha + 1; \tag{2.4_2}$$

(4) for $\gamma_3 + 2 \leq \lambda \leq \gamma_4$

$$\begin{aligned} A(2^\lambda) &= 2^{-10\lambda} \sum'_{y \bmod 2^\lambda} e(-2^{\alpha-\lambda}mby)(2i^{b_1by} \cdot 2^\lambda)2^{2\gamma_2} \times \\ &\times (2i^{b_2by} \cdot 2^{\lambda-\gamma_2})2^{2\gamma_3}(2i^{b_3by} \cdot 2^{\lambda-\gamma_3}) \cdot 2^{4\lambda} = \end{aligned}$$

$$\begin{aligned}
&= 2^{\gamma_2+\gamma_3-3\lambda+3} \sum'_{y \bmod 2^\lambda} e\left(\frac{(b_1+b_2+b_3)by}{4} - \frac{2^\alpha mby}{2^\lambda}\right) = \\
&= 2^{\gamma_2+\gamma_3-3\lambda+3} e\left(\frac{1}{4}((b_1+b_2+b_3)b - 2^{\alpha-\lambda+2}mb)\right) \times \\
&\times \sum_{y=0}^{2^{\lambda-1}-1} e(2^{1-\lambda}(2^{\lambda-2}(b_1+b_2+b_3) - 2^\alpha m)by),
\end{aligned}$$

hence, as in case (2)

$$A(2^\lambda) = (-1)^{(\sum_{k=1}^3 b_k - m)/2} \cdot 2^{\gamma_2+\gamma_3-2\alpha-2} \text{ if } \lambda = \alpha + 2, \quad (2.5)$$

$$= 0 \text{ if } \lambda \neq \alpha + 2; \quad (2.5_1)$$

(5) for $\gamma_4 + 2 \leq \lambda \leq \gamma_5$

$$\begin{aligned}
A(2^\lambda) &= 2^{-10\lambda} \sum'_{y \bmod 2^\lambda} e(-2^{\alpha-\lambda}mby)(2i^{b_1by} \cdot 2^\lambda)(2^{2\gamma_2} \cdot 2i^{b_2by} \cdot 2^{\lambda-\gamma_2}) \times \\
&\times (2^{2\gamma_3} \cdot 2i^{b_3by} \cdot 2^{\lambda-\gamma_3})(2^{2\gamma_4} \cdot 2i^{b_4by} \cdot 2^{\lambda-\gamma_4})2^{2\lambda} = \\
&= 2^{\gamma_2+\gamma_3+\gamma_4-4\lambda+4} (-1)^{(\sum_{k=1}^4 b_k)/2} c(2^\alpha bm, 2^\lambda),
\end{aligned}$$

hence

$$A(2^\lambda) = (-1)^{(\sum_{k=1}^4 b_k)/2} \cdot 2^{\gamma-\gamma_5-3\lambda-3} \text{ if } \lambda < \alpha + 1, \quad (2.6)$$

$$= -(-1)^{(\sum_{k=1}^4 b_k)/2} \cdot 2^{\gamma-\gamma_5-3\alpha} \text{ if } \lambda = \alpha + 1, \quad (2.6_1)$$

$$= 0 \text{ if } \lambda > \alpha + 1; \quad (2.6_2)$$

(6) for $\lambda > \gamma_5 + 1$

$$\begin{aligned}
A(2^\lambda) &= 2^{-10\lambda} \sum'_{y \bmod 2^\lambda} e(-2^{\alpha-\lambda}mby)(2i^{b_1by} \cdot 2^\lambda)2^{2\gamma_2} \times \\
&\times (2i^{b_2by} \cdot 2^{\lambda-\gamma_2})2^{2\gamma_3} (2i^{b_3by} \cdot 2^{\lambda-\gamma_3}) \times \\
&\times (2^{2\gamma_4} \cdot 2i^{b_4by} \cdot 2^{\lambda-\gamma_4})(2^{2\gamma_5} \cdot 2i^{b_5by} \cdot 2^{\lambda-\gamma_5}) = \\
&= 2^{\gamma-5\lambda+5} \sum'_{y \bmod 2^\lambda} e\left(\frac{\sum_{k=1}^5 b_k by}{4} - \frac{2^\alpha mby}{2^\lambda}\right) = \\
&= 2^{\gamma-5\lambda+5} e\left(\frac{1}{4}\left(\sum_{k=1}^5 b_k b - 2^{\alpha-\lambda+2}mb\right)\right) \times
\end{aligned}$$

$$\times \sum_{y=0}^{2^{\lambda-1}-1} e\left(2^{1-\lambda}\left(2^{\lambda-2} \sum_{k=1}^5 b_k - 2^\alpha m\right)by\right),$$

hence, as in case (2),

$$A(2^\lambda) = (-1)\left(\sum_{k=1}^5 b_k - m\right)/2 \cdot 2^{\gamma-4\alpha-4} \text{ if } \lambda = \alpha + 2, \tag{2.7}$$

$$= 0 \text{ if } \lambda \neq \alpha + 2; \tag{2.7_1}$$

II. According to (1.15) and (2.2), we have

$$\begin{aligned} \chi_2 = & 1 + \sum_{\lambda=2}^{\gamma_2} A(2^\lambda) + \sum_{\lambda=\gamma_2+2}^{\gamma_3} A(2^\lambda) + \sum_{\lambda=\gamma_3+2}^{\gamma_4} A(2^\lambda) + \\ & + \sum_{\lambda=\gamma_4+2}^{\gamma_5} A(2^\lambda) + \sum_{\lambda=\gamma_5+2}^{\infty} A(2^\lambda). \end{aligned} \tag{2.8}$$

Consider the following cases:

(1) Let $0 \leq \alpha \leq \gamma_2 - 2$. Then from (2.8), (2.3), (2.3₁), (2.4₂), (2.5₁), (2.6₂), and (2.7₁) we get

$$\chi_2 = 1 + \sum_{\lambda=2}^{\gamma_2} A(2^\lambda) = 1 + (-1)^{(b_1-m)/2}.$$

(2) Let $\alpha = \gamma_2 - 1$ or $\alpha = \gamma_2 < \gamma_3$. Then from (2.8), (2.3₁), (2.4₂), (2.5₁), (2.6₂), and (2.7₁) we get

$$\chi_2 = 1.$$

(3) Let $\gamma_2 + 1 \leq \alpha < \gamma_3$. Then from (2.8), (2.3₁), (2.4), (2.4₁), (2.4₂), (2.5₁), (2.6₂), and (2.7₁) we get

$$\chi_2 = 1 + \sum_{\lambda=\gamma_2+2}^{\gamma_3} A(2^\lambda) = 1 + \sum_{\lambda=\gamma_2+2}^{\alpha} (-1)^{(b_1+b_2)/2} \cdot 2^{\gamma_2+1-\lambda} - (-1)^{(b_1+b_2)/2} \cdot 2^{\gamma_2-\alpha}.$$

(4) Let $\gamma_2 = \gamma_3 \leq \alpha \leq \gamma_4 - 2$ or $\gamma_2 + 1 = \gamma_3 \leq \alpha \leq \gamma_4 - 2$. Then from (2.8), (2.3₁), (2.5), (2.5₁), (2.6₂), and (2.7₁) we get

$$\chi_2 = 1 + \sum_{\lambda=\gamma_3+2}^{\gamma_4} A(2^\lambda) = 1 + (-1)\left(\sum_{k=1}^3 b_k - m\right)/2 \cdot 2^{\gamma_2+\gamma_3-2\alpha-2}.$$

(5) Let $\gamma_2 + 2 \leq \gamma_3 \leq \alpha \leq \gamma_4 - 2$. Then from (2.8), (2.3₁), (2.4), (2.5), (2.5₁), (2.6₂), and (2.7₁) we get

$$\begin{aligned} \chi_2 &= 1 + \sum_{\lambda=\gamma_2+2}^{\gamma_3} A(2^\lambda) + \sum_{\lambda=\gamma_3+2}^{\gamma_4} A(2^\lambda) = \\ &= 1 + \sum_{\lambda=\gamma_2+2}^{\gamma_3} (-1)^{(b_1+b_2)/2} \cdot 2^{\gamma_2+1-\lambda} + (-1)^{(\sum_{k=1}^3 b_k - m)/2} \cdot 2^{\gamma_2+\gamma_3-2\alpha-2}. \end{aligned}$$

(6) Let $\gamma_2 = \gamma_3 \leq \alpha = \gamma_4 - 1$ or $\gamma_2 + 1 = \gamma_3 \leq \alpha = \gamma_4 - 1$ or $\gamma_2 = \gamma_3 \leq \alpha = \gamma_4 < \gamma_5$ or $\gamma_2 + 1 = \gamma_3 \leq \alpha = \gamma_4 < \gamma_5$. Then from (2.8), (2.3₁), (2.5₁), (2.6₂), and (2.7₁) we get

$$\chi_2 = 1.$$

(7) Let $\gamma_2 + 2 \leq \gamma_3 \leq \alpha = \gamma_4 - 1$ or $\gamma_2 + 2 \leq \gamma_3 \leq \alpha = \gamma_4 < \gamma_5$. Then from (2.8), (2.3₁), (2.4), (2.4₂), (2.5₁), (2.6₁), and (2.7₁) we get

$$\chi_2 = 1 + \sum_{\lambda=\gamma_2+2}^{\gamma_3} A(2^\lambda) = 1 + \sum_{\lambda=\gamma_2+2}^{\gamma_3} (-1)^{(b_1+b_2)/2} \cdot 2^{\gamma_2+1-\lambda}.$$

8) Let $\gamma_4 + 1 \leq \alpha < \gamma_5$, but $\gamma_4 \geq \gamma_3 \geq \gamma_2 + 2$. Then from (2.8), (2.3₁), (2.4), (2.5₁), (2.6), (2.6₁), and (2.7₁) we get

$$\begin{aligned} \chi_2 &= 1 + \sum_{\lambda=\gamma_2+2}^{\gamma_3} A(2^\lambda) + \sum_{\lambda=\gamma_4+2}^{\gamma_5} A(2^\lambda) = \\ &= 1 + \sum_{\lambda=\gamma_2+2}^{\gamma_3} (-1)^{(b_1+b_2)/2} \cdot 2^{\gamma_2+1-\lambda} + \\ &+ \sum_{\lambda=\gamma_4+2}^{\alpha} (-1)^{(\sum_{k=1}^4 b_k)/2} \cdot 2^{\gamma-\gamma_5-3\lambda+3} - (-1)^{(\sum_{k=1}^4 b_k)/2} \cdot 2^{\gamma-\gamma_5-3\alpha}. \end{aligned}$$

(9) Let $\gamma_4 + 1 \leq \alpha < \gamma_5$, but $\gamma_4 \geq \gamma_3 = \gamma_2$ or $\gamma_4 \geq \gamma_3 = \gamma_2 + 1$. Then from (2.8), (2.3₁), (2.5₁), (2.6), (2.6₁), and (2.7₁) we get

$$\begin{aligned} \chi_2 &= 1 + \sum_{\lambda=\gamma_4+2}^{\gamma_5} A(2^\lambda) + \sum_{\lambda=\gamma_5+2}^{\infty} A(2^\lambda) = \\ &= 1 + \sum_{\lambda=\gamma_4+2}^{\alpha} (-1)^{(\sum_{k=1}^4 b_k)/2} \cdot 2^{\gamma-\gamma_5-3\lambda+3} - (-1)^{(\sum_{k=1}^4 b_k)/2} \cdot 2^{\gamma-\gamma_5-3\alpha}. \end{aligned}$$

(10) Let $\alpha \geq \gamma_5 = \gamma_4 \geq \gamma_3 = \gamma_2$ or $\alpha \geq \gamma_5 = \gamma_4 \geq \gamma_3 = \gamma_2 + 1$. Then from (2.8), (2.3₁), (2.5₁), and (2.7₁) we get

$$\chi_2 = 1 + \sum_{\lambda=\gamma_5+2}^{\infty} A(2^\lambda) = 1 + (-1)^{(\sum_{k=1}^5 b_k - m)/2} \cdot 2^{\gamma-4\alpha-4}.$$

(11) Let $\alpha \geq \gamma_5 = \gamma_4 \geq \gamma_3 \geq \gamma_2 + 2$. Then from (2.8), (2.3₁), (2.4), (2.5₁), (2.7), and (2.7₁) we get

$$\begin{aligned} \chi_2 &= 1 + \sum_{\lambda=\gamma_2+2}^{\gamma_3} A(2^\lambda) + \sum_{\lambda=\gamma_5+2}^{\infty} A(2^\lambda) = \\ &= 1 + \sum_{\lambda=\gamma_2+2}^{\gamma_3} (-1)^{(b_1+b_2)/2} \cdot 2^{\gamma_2+1-\lambda} + (-1)^{(\sum_{k=1}^5 b_k - m)/2} \cdot 2^{\gamma-4\alpha-4}. \end{aligned}$$

(12) Let $\alpha \geq \gamma_5 > \gamma_4 \geq \gamma_3 \geq \gamma_2 + 2$. Then from (2.8), (2.3₁), (2.4), (2.5₁), (2.6), (2.7), and (2.7₁) we get

$$\begin{aligned} \chi_2 &= 1 + \sum_{\lambda=\gamma_2+2}^{\gamma_3} A(2^\lambda) + \sum_{\lambda=\gamma_4+2}^{\gamma_5} A(2^\lambda) + \\ &+ \sum_{\lambda=\gamma_5+2}^{\infty} A(2^\lambda) = 1 + \sum_{\lambda=\gamma_2+2}^{\gamma_3} (-1)^{(b_1+b_2)/2} \cdot 2^{\gamma_2+1-\lambda} + \\ &+ \sum_{\lambda=\gamma_4+2}^{\gamma_5} (-1)^{(\sum_{k=1}^4 b_k)/2} \cdot 2^{\gamma-\gamma_5-3\lambda+3} + (-1)^{(\sum_{k=1}^5 b_k - m)/2} \cdot 2^{\gamma-4\alpha-4}. \end{aligned}$$

(13) Let $\alpha \geq \gamma_5 > \gamma_4 \geq \gamma_3 = \gamma_2$ or $\alpha \geq \gamma_5 > \gamma_4 \geq \gamma_3 = \gamma_2 + 1$. Then from (2.8), (2.3₁), (2.5₁), (2.6), (2.7), and (2.7₁) we get

$$\begin{aligned} \chi_2 &= 1 + \sum_{\lambda=\gamma_4+2}^{\gamma_5} A(2^\lambda) + \sum_{\lambda=\gamma_5+2}^{\infty} A(2^\lambda) = \\ &= 1 + \sum_{\lambda=\gamma_4+2}^{\gamma_5} (-1)^{(\sum_{k=1}^4 b_k)/2} \cdot 2^{\sum_{k=2}^4 \gamma_k - 3\lambda + 3} + (-1)^{(\sum_{k=1}^5 b_k - m)/2} \cdot 2^{\gamma-4\alpha-4}. \end{aligned}$$

Calculating the sums in the right-hand sides of the above given equalities, we complete the proof of the lemma. \square

Lemma 11. *Let $p > 2$, $p^\beta \parallel n$, $p^{\ell_k} \parallel a_k$ ($k=1, \dots, 5$), $\max \ell_k = \bar{\ell}$, $\min \ell_k = \underline{\ell} = 0$, $\bar{\ell} \geq \ell''' \geq \ell'' \geq \ell' \geq \underline{\ell} = 0$, $\ell = \sum_{k=1}^5 \ell_k = \bar{\ell} + \ell''' + \ell'' + \ell' + \underline{\ell}$, $\eta(\ell') = 1$ if $2 \nmid \ell'$ and $\eta(\ell') = 0$ if $2 \mid \ell'$. Then*

$$\chi_p = (1 - p^{-1})(\beta + 1) \text{ for } \ell' \geq \beta + 1, p \equiv 1 \pmod{4},$$

$$\begin{aligned}
&= (1 + p^{-1}) \text{ for } \ell' \geq \beta + 1, p \equiv 3 \pmod{4}, 2|\beta, \\
&= 0 \text{ for } \ell' \geq \beta + 1, p \equiv 3 \pmod{4}, 2 \nmid \beta; \\
&= (1 - p^{-1})\ell' + (1 + p^{-1})(1 - p^{-(\beta - \ell' + 1)}) \\
&\quad \text{for } \ell' \leq \beta < \ell'', p \equiv 1 \pmod{4}, \\
&= (1 + p^{-1})\eta(\ell') - (-1)^{\ell'}(1 + p)p^{-(\beta - \ell' + 2)} \\
&\quad \text{for } \ell' \leq \beta < \ell'', p \equiv 3 \pmod{4}, \\
&= (1 - p^{-1})\ell' + (1 + p^{-1}) - p^{-(\ell'' - \ell')} \{ (1 + p^{-2(\beta - \ell'' - 1)})(p + 1)^{-1} + \\
&\quad + p^{-2(\beta - \ell'' - 3)} \} \text{ for } \ell'' \leq \beta < \ell''', p \equiv 1 \pmod{4}, \\
&= (1 + p^{-1})\eta(\ell') - (-1)^{\ell'} p^{-(\ell'' - \ell' + 1)} \{ (1 + (p - 1)(p^2 + 1))^{-1} \times \\
&\quad \times (1 - (-1)^{\beta + \ell''} p^{-2(\beta - \ell'')}) - (-1)^{\beta + \ell''} p^{-2(\beta - \ell'' + 1)} \} \\
&\quad \text{for } \ell'' \leq \beta < \ell''', p \equiv 3 \pmod{4}, \\
&= (1 - p^{-1})\ell' + (1 + p^{-1}) - p^{-(\ell'' - \ell')} \{ (1 + p^{-2(\ell''' - \ell'' - 1)})(p + 1)^{-1} - \\
&\quad - p^{-2(\ell''' - \ell'' - 1)}(1 - p^{-3(\beta - \ell''')}) (p^2 + p + 1)^{-1} \} - p^{-3(\beta + 1) + \ell - \bar{\ell} - 1} \\
&\quad \text{for } \ell''' \leq \beta < \bar{\ell}, p \equiv 1 \pmod{4}, \\
&= (1 + p^{-1})\eta(\ell') - (-1)^{\ell'} p^{-(\ell'' - \ell' + 1)} \{ (1 + (p - 1)(p^2 + 1))^{-1} \times \\
&\quad \times (1 - (-1)^{\ell''' + \ell''} p^{-2(\beta - \ell''')}) \} + (-1)^{\ell - \bar{\ell}} p^{\ell - \bar{\ell} - 3\ell''' - 1} \times \\
&\quad \times \{ (1 - p^{-3(\beta - \ell''')}) (p^2 + p + 1)^{-1} - p^{-3(\beta - \ell''' + 1)} \} \\
&\quad \text{for } \ell''' \leq \beta < \bar{\ell}, p \equiv 3 \pmod{4}, \\
&= (1 - p^{-1})\ell' + (1 + p^{-1}) - p^{-(\ell'' - \ell')} \{ (1 + p^{-2(\ell''' - \ell'' - 1)})(p + 1)^{-1} - \\
&\quad - p^{-2(\ell''' - \ell'' - 1)}(1 - p^{-3(\bar{\ell} - \ell''')}) (p^2 + p + 1)^{-1} \} + \\
&\quad + p^{\ell - 4\bar{\ell} - 1} \{ (1 - p^{-4(\beta - \bar{\ell})}) (p^3 + p^2 + p + 1)^{-1} - p^{-4(\beta - \bar{\ell} + 1)} \} \\
&\quad \text{for } \beta \geq \bar{\ell}, p \equiv 1 \pmod{4}, \\
&= (1 + p^{-1})\eta(\ell') - (-1)^{\ell'} p^{-(\ell'' - \ell' + 1)} \{ (1 + (p - 1)(p^2 + 1))^{-1} \times \\
&\quad \times (1 - (-1)^{\ell''' + \ell''} p^{-2(\ell''' - \ell''')}) \} + (-1)^{\ell - \bar{\ell}} p^{\ell - \bar{\ell} - 3\ell''' - 1} \times \\
&\quad \times \{ (1 - p^{-3(\bar{\ell} - \ell''')}) (p^2 + p + 1)^{-1} - p^{-3(\bar{\ell} - \ell''')} (1 - (-1)^{\bar{\ell} + \beta} p^{-4(\beta - \bar{\ell})}) \times \\
&\quad \times (p - 1)(p^4 + 1)^{-1} \} + (-1)^{\beta + \ell} p^{\ell - 4\beta - 5} \text{ for } \beta \geq \bar{\ell}, p \equiv 3 \pmod{4}.
\end{aligned}$$

Proof. I. Let $2 \nmid q$ and $q = (q, a_k)q_k$ ($k = 1, \dots, 5$). Then from (1.14) and Lemmas 1 and 2 it follows that

$$\begin{aligned} A(q) &= q^{-10} \sum'_{h \bmod q} e\left(-\frac{hn}{q}\right) \prod_{k=1}^5 (q, a_k)^2 S^2\left(\frac{a_k}{(q, a_k)} h, q_k\right) = \\ &= q^{-10} \sum'_{h \bmod q} e\left(-\frac{hn}{q}\right) \prod_{k=1}^5 (q, a_k)^2 \left(\frac{-1}{q_k}\right) q_k = \\ &= q^{-5} \sum'_{h \bmod q} e\left(-\frac{hn}{q}\right) \prod_{k=1}^5 (q, a_k) \left(\frac{-1}{q_k}\right), \end{aligned}$$

where, putting $q = p^\lambda$ and taking into account that $(a_1, \dots, a_5) = 1$, we get

$$\begin{aligned} A(p^\lambda) &= p^{-5\lambda} \sum'_{h \bmod p^\lambda} e\left(-\frac{hn}{p^\lambda}\right) \prod_{k=1}^5 (p^\lambda, a_k) \left(\frac{-1}{p^\lambda / (p^\lambda, a_k)}\right) = \\ &= \left(\frac{-1}{p}\right)^{\lambda + \min(\lambda, \bar{\ell}) + \min(\lambda, \ell''') + \min(\lambda, \ell'') + \min(\lambda, \ell')} \times \\ &\times p^{\min(\lambda, \bar{\ell}) + \min(\lambda, \ell''') + \min(\lambda, \ell'') + \min(\lambda, \ell')} p^{-5\lambda} c(n, p^\lambda). \end{aligned} \tag{2.9}$$

It follows from (2.9) and Lemma 4 that

(1) for $\lambda \leq \ell'$

$$A(p^\lambda) = \left(\frac{-1}{p}\right)^\lambda (1 - p^{-1}) \text{ if } \lambda < \beta + 1, \tag{2.10}$$

$$= -\left(\frac{-1}{p}\right)^{\beta+1} p^{-1} \text{ if } \lambda = \beta + 1; \tag{2.10_1}$$

(2) for $\ell' < \lambda \leq \ell''$

$$A(p^\lambda) = \left(\frac{-1}{p}\right)^{\ell'} p^{-(\lambda - \ell')} (1 - p^{-1}) \text{ if } \lambda < \beta + 1, \tag{2.11}$$

$$= -\left(\frac{-1}{p}\right)^{\ell'} p^{\ell' - \beta - 2} \text{ if } \lambda = \beta + 1; \tag{2.11_1}$$

(3) for $\ell'' < \lambda \leq \ell'''$

$$A(p^\lambda) = \left(\frac{-1}{p}\right)^{\lambda + \ell'' + \ell'} p^{-2\lambda + \ell'' \ell'} (1 - p^{-1}) \text{ if } \lambda < \beta + 1, \tag{2.12}$$

$$= -\left(\frac{-1}{p}\right)^{\beta + 1 + \ell'' + \ell'} p^{-2\beta + \ell'' + \ell' - 3} \text{ if } \lambda = \beta + 1; \tag{2.12_1}$$

(4) for $\ell''' < \lambda \leq \bar{\ell}$

$$A(p^\lambda) = \left(\frac{-1}{p}\right)^{\ell-\bar{\ell}} p^{\ell-\bar{\ell}-3\lambda}(1-p^{-1}) \text{ if } \lambda < \beta+1, \quad (2.13)$$

$$= -\left(\frac{-1}{p}\right)^{\ell-\bar{\ell}} p^{\ell-\bar{\ell}} \text{ if } \lambda = \beta+1; \quad (2.13_1)$$

(5) for $\lambda > \bar{\ell}$

$$A(p^\lambda) = \left(\frac{-1}{p}\right)^{\lambda+\ell} p^{\ell-4\lambda}(1-p^{-1}) \text{ if } \lambda < \beta+1, \quad (2.14)$$

$$= -\left(\frac{-1}{p}\right)^{\beta+\ell+1} p^{\ell-4\beta-5} \text{ if } \lambda = \beta+1. \quad (2.14_1)$$

In all the above-mentioned cases

$$A(p^\lambda) = 0 \text{ if } \lambda > \beta+1. \quad (2.15)$$

II. According to (1.15) we have

$$\begin{aligned} \chi_p = 1 + \sum_{\lambda=1}^{\ell'} A(p^\lambda) + \sum_{\lambda=\ell'+1}^{\ell''} A(p^\lambda) + \sum_{\lambda=\ell''+1}^{\ell'''} A(p^\lambda) + \\ + \sum_{\lambda=\ell'''+1}^{\bar{\ell}} A(p^\lambda) + \sum_{\lambda=\bar{\ell}+1}^{\infty} A(p^\lambda). \end{aligned} \quad (2.16)$$

Consider the following cases:

(1) Let $\ell' \geq \beta+1$. Then from (2.16), (2.10), (2.10₁), and (2.15) we get

$$\chi_p = 1 + \sum_{\lambda=1}^{\beta} \left(\frac{-1}{p}\right)^\lambda (1-p^{-1}) - \left(\frac{-1}{p}\right)^{\beta+1} p^{-1}.$$

(2) Let $\ell' \leq \beta < \ell''$. Then from (2.16), (2.10) (2.11), (2.11₁), and (2.15) we get

$$\begin{aligned} \chi_p = 1 + \sum_{\lambda=1}^{\ell'} \left(\frac{-1}{p}\right)^\lambda (1-p^{-1}) + \sum_{\lambda=\ell'+1}^{\beta} \left(\frac{-1}{p}\right)^{\ell'} p^{-(\lambda-\ell')}(1-p^{-1}) - \\ - \left(\frac{-1}{p}\right)^{\ell'} p^{-(\beta-\ell'+1)}. \end{aligned}$$

(3) Let $\ell'' \leq \beta < \ell'''$. Then from (2.16), (2.10), (2.11), (2.12), (2.12₁), and (2.15) we get

$$\begin{aligned} \chi_p = & 1 + \sum_{\lambda=1}^{\ell'} \left(\frac{-1}{p}\right)^\lambda (1-p^{-1}) + \sum_{\lambda=\ell'+1}^{\ell''} \left(\frac{-1}{p}\right)^{\ell'} p^{-(\lambda-\ell')} (1-p^{-1}) + \\ & + \sum_{\lambda=\ell''+1}^{\beta} \left(\frac{-1}{p}\right)^{\lambda+\ell''+\ell'} p^{-2\lambda+\ell''+\ell'} (1-p^{-1}) - \left(\frac{-1}{p}\right)^{\beta+1+\ell''+\ell'} p^{-2\beta+\ell''+\ell'+3}. \end{aligned}$$

(4) Let $\ell''' \leq \beta < \bar{\ell}$. Then from (2.16), (2.10), (2.11), (2.12), (2.13), (2.13₁), and (2.15) we get

$$\begin{aligned} \chi_p = & 1 + \sum_{\lambda=1}^{\ell'} \left(\frac{-1}{p}\right)^\lambda (1-p^{-1}) + \sum_{\lambda=\ell'+1}^{\ell''} \left(\frac{-1}{p}\right)^{\ell'} p^{-(\lambda-\ell')} (1-p^{-1}) + \\ & + \sum_{\lambda=\ell''+1}^{\ell'''} \left(\frac{-1}{p}\right)^{\lambda+\ell''+\ell'} p^{-2\lambda+\ell''+\ell'} (1-p^{-1}) + \\ & + \sum_{\lambda=\ell'''+1}^{\beta} \left(\frac{-1}{p}\right)^{\ell-\bar{\ell}} p^{\ell-\bar{\ell}-3\lambda} (1-p^{-1}) - \left(\frac{-1}{p}\right)^{\ell-\bar{\ell}} p^{\ell-\bar{\ell}-3\beta-4}. \end{aligned}$$

(5) Let $\beta \geq \bar{\ell}$. Then from (2.16), (2.10), (2.11), (2.12), (2.13), (2.14), (2.14₁), and (2.15) we get

$$\begin{aligned} \chi_p = & 1 + \sum_{\lambda=1}^{\ell'} \left(\frac{-1}{p}\right)^\lambda (1-p^{-1}) + \sum_{\lambda=\ell'+1}^{\ell''} \left(\frac{-1}{p}\right)^{\ell'} (1-p^{-1}) p^{-(\lambda-\ell')} + \\ & + \sum_{\lambda=\ell''+1}^{\ell'''} \left(\frac{-1}{p}\right)^{\lambda+\ell''+\ell'} p^{\ell''+\ell'-2\lambda} (1-p^{-1}) + \sum_{\lambda=\ell'''+1}^{\bar{\ell}} \left(\frac{-1}{p}\right)^{\ell-\bar{\ell}} p^{\ell-\bar{\ell}-3\lambda} (1-p^{-1}) + \\ & + \sum_{\lambda=\bar{\ell}+1}^{\beta} \left(\frac{-1}{p}\right)^{\lambda+\ell} p^{\ell-4\lambda} (1-p^{-1}) - \left(\frac{-1}{p}\right)^{\beta+\ell+1} p^{\ell-4\beta-5}. \quad (2.17) \end{aligned}$$

Calculating the sums in the right-hand sides of the above-mentioned equalities, we complete the proof of the lemma. \square

Theorem 1. Let $n = 2^\alpha m = 2^\alpha uv$, $u = \prod_{\substack{p|n \\ p \nmid 2\Delta}} p^\beta$, $v = \prod_{p|\Delta, p>2} p^\beta$. Then

$$\rho(n; f) = \frac{2^{4\alpha+6} v^4}{5\Delta^{1/2}} \chi_2 \prod_{\substack{p|\Delta \\ p>2}} \chi_p \prod_{p|\Delta, p>2} \left(1 - \left(\frac{-1}{p}\right) p^{-5}\right)^{-1} \sum_{d_1 d_2 = u} \left(\frac{-1}{d_1}\right) d_2^4,$$

where the values of χ_2 and χ_p are given in Lemmas 10 and 11.

Proof. Let $p > 2$ $p^\beta \|n$, $p \nmid \Delta$ (i.e., $\ell = 0$). In (2.17) putting $\ell = 0$, we get

$$\begin{aligned} \chi_p &= 1 + \sum_{\lambda=1}^{\beta} \left(\frac{-1}{p}\right)^\lambda p^{-4\lambda} (1 - p^{-1}) - \left(\frac{-1}{p}\right)^{\beta+1} p^{-4\beta-5} = \\ &= 1 + \sum_{\lambda=1}^{\beta} \left(\frac{-1}{p}\right)^\lambda p^{-4\lambda} - \left(\frac{-1}{p}\right) p^{-5} \sum_{\lambda=1}^{\beta} \left(\frac{-1}{p}\right)^{\lambda-1} p^{-4(\lambda-1)} - \\ &- \left(\frac{-1}{p}\right)^{\beta+1} p^{-4\beta-5} = \sum_{\lambda=0}^{\beta} \left(\frac{-1}{p}\right)^\lambda p^{-4\lambda} - \left(\frac{-1}{p}\right) p^{-5} \sum_{\lambda=0}^{\beta} \left(\frac{-1}{p}\right)^\lambda p^{-4\lambda} = \\ &= \left(1 - \left(\frac{-1}{p}\right) p^{-5}\right) \sum_{\lambda=0}^{\beta} \left(\frac{-1}{p}\right)^\lambda p^{-4\lambda} = \\ &= \left(1 - \left(\frac{-1}{p}\right) p^{-5}\right) \sum_{d|p^\beta} \left(\frac{-1}{d}\right) p^{-4}. \end{aligned} \quad (2.18)$$

For $p \nmid \Delta n$, i.e., for $\beta = 0$, from (2.18) we get

$$\chi_p = 1 - \left(\frac{-1}{p}\right) p^{-5}. \quad (2.19)$$

Thus it follows from Lemma 5, (2.18), and (2.19) that

$$\begin{aligned} \sum_{q=1}^{\infty} A(q) &= \chi_2 \prod_{\substack{p|\Delta \\ p>2}} \chi_p \prod_{\substack{p|n \\ p \nmid 2\Delta}} \chi_p \prod_{\substack{p \nmid \Delta n \\ p>2}} \chi_p = \\ &= \chi_2 \prod_{\substack{p|\Delta \\ p>2}} \chi_p \prod_{\substack{p|n \\ p \nmid 2\Delta}} \left\{ \left(1 - \left(\frac{-1}{p}\right) p^{-5}\right) \sum_{d|p^\beta} \left(\frac{-1}{d}\right) d^{-4} \right\} \prod_{\substack{p \nmid \Delta n \\ p>2}} \left(1 - \left(\frac{-1}{p}\right) p^{-5}\right) = \\ &= \chi_2 \prod_{\substack{p|\Delta \\ p>2}} \chi_p \prod_{p>2} \left(1 - \left(\frac{-1}{p}\right) p^{-5}\right) \prod_{\substack{p|\Delta \\ p>2}} \left(1 - \left(\frac{-1}{p}\right) p^{-5}\right)^{-1} \sum_{d|u} \left(\frac{-1}{d}\right) d^4 = \\ &= \chi_2 \prod_{\substack{p|\Delta \\ p>2}} \chi_p \mathcal{L}^{-1}(5, -1) \prod_{\substack{p|\Delta \\ p>2}} \left(1 - \left(\frac{-1}{p}\right) p^{-5}\right)^{-1} \frac{1}{u^4} \sum_{d_1 d_2 = u} \left(\frac{-1}{d_1}\right) d_2^4, \end{aligned} \quad (2.20)$$

where $\mathcal{L}(5, -1)$ is the Dirichlet \mathcal{L} -function and it is well known that

$$\mathcal{L}(5, -1) = \frac{5\pi^4}{2^6 \cdot 4!}. \quad (2.21)$$

Thus the lemma follows from (1.13), (2.20), and (2.21). \square

3. FORMULAS FOR $r(n; f)$

As an example let us consider the quadratic form

$$f = x_1^2 + \cdots + x_8^2 + 4(x_9^2 + x_{10}^2)$$

in which

$$a_1 = \cdots a_4 = 1, \quad a_5 = 4, \quad b_1 = \cdots = b_5 = 1, \\ \gamma_5 = 2, \quad \gamma_4 = \gamma_3 = \gamma_2 = \gamma_1 = 0, \quad \gamma = 2, \quad a = 4, \quad \Delta = 2^4.$$

Lemma 12. *The function*

$$\begin{aligned} \psi(\tau; f) = & \vartheta_{00}^8(\tau; 0, 2) \vartheta_{00}^2(\tau; 0, 8) - \theta(\tau; f) + \frac{42}{5 \cdot 512\pi^4} \Psi_2(\tau; 8, 8; 0, 0; 0, 0; 4, 4) - \\ & - \frac{544}{5 \cdot 8192\pi^4} \Psi_2(\tau; 0, 0; 0, 0; 0, 0; 4, 4) + \frac{3}{5 \cdot 128\pi^4} \Psi_2(\tau; 4, 4; 0, 0; 0, 0; 2, 2) + \\ & + \frac{2}{32\pi^3 i} \Psi_3(\tau; 4, 0, 4, 0; 0, 1, 1, 1; 0, 0, 0, 0; 2, 4, 2, 4) + \\ & + \frac{24}{512\pi^3 i} \Psi_4(\tau; 8, 8, 8, 0; 1, 1, 1, 1; 0, 0, 0, 0; 4, 4, 4, 4) \end{aligned} \quad (3.1)$$

is an entire modular form of weight 5 and character $\chi(\delta) = \text{sgn } \delta(\frac{-1}{|\delta|})$ for $\Gamma_0(16)$.

Proof. According to Lemmas 7 and 8, the first two summands in (3.1) are entire modular forms of weight 5 and character $\chi(\delta) = \text{sgn } \delta(\frac{-1}{|\delta|})$ for $\Gamma_0(16)$.

Put $N = 4$. Then it is obvious that the following five summands in (3.1) satisfy the condition (a) from Lemma 9.

If $\alpha\delta \equiv 1 \pmod{16}$, then $\alpha\delta \equiv 1 \pmod{4}$, i.e.,

$$\alpha \equiv \pm 1 \pmod{4} \quad \text{and respectively} \quad \delta \equiv \pm 1 \pmod{4}. \quad (3.2)$$

In the third, fourth, and fifth summands from (3.1) we have

$$\left(\frac{N_1 N_2}{|\delta|}\right) = 1 \quad \text{and} \quad \left(\frac{\Delta}{|\delta|}\right) = 1. \quad (3.3)$$

By (1.16), (1.6), (1.7), and (1.10) we have

$$\begin{aligned} \Psi_2(\tau; 8\alpha, 8\alpha; 0, 0; 0, 0; 4, 4) &= \frac{1}{8} \vartheta_{8\alpha,0}^{(4)}(\tau; 0, 8) \vartheta_{8\alpha,0}(\tau; 0, 8) - \frac{3}{8} \vartheta_{8\alpha,0}''^2(\tau; 0, 8) = \\ &= \frac{1}{8} \vartheta_{\pm 8+8(\alpha \mp 1),0}^{(4)}(\tau; 0, 8) \vartheta_{\pm 8+8(\alpha \mp 1),0}(\tau; 0, 8) - \frac{3}{8} \vartheta_{\pm 8+8(\alpha \mp 1),0}''^2(\tau; 0, 8) = \\ &= \frac{1}{8} \vartheta_{\pm 8,0}^{(4)}(\tau; 4(\alpha \mp 1), 8) \vartheta_{\pm 8,0}(\tau; 4(\alpha \mp 1), 8) - \frac{3}{8} \vartheta_{\pm 8,0}''^2(\tau; 4(\alpha \mp 1), 8) = \\ &= \frac{1}{8} \vartheta_{\pm 8,0}^{(4)}(\tau; 0, 8) \vartheta_{\pm 8,0}(\tau; 0, 8) - \frac{3}{8} \vartheta_{\pm 8,0}''^2(\tau; 0, 8) = \frac{1}{8} \vartheta_{80}^{(4)}(\tau; 0, 8) \times \end{aligned}$$

$$\times \vartheta_{80}(\tau; 0, 8) - \frac{3}{8} \vartheta_{80}''^2(\tau; 0, 8) = \Psi_2(\tau; 8, 8; 0, 0; 0, 0; 4, 4) \quad (3.4)$$

and similarly

$$\begin{aligned} \Psi_2(\tau; 4\alpha, 4\alpha; 0, 0; 0, 0; 2, 2) &= \frac{1}{2} \vartheta_{4\alpha, 0}^{(4)}(\tau; 0, 4) \vartheta_{4\alpha, 0}(\tau; 0, 4) - \frac{3}{2} \vartheta_{4\alpha, 0}''^2(\tau; 0, 4) = \\ &= \frac{1}{2} \vartheta_{40}^{(4)}(\tau; 0, 4) \vartheta_{40}(\tau; 0, 4) - \frac{3}{2} \vartheta_{40}''^2(\tau; 0, 4) = \\ &= \Psi_2(\tau; 4, 4; 0, 0; 0, 0; 2, 2). \end{aligned} \quad (3.5)$$

Hence, according to (3.2)–(3.5) the functions $\Psi_2(\tau; 8, 8; 0, 0; 0, 0; 4, 4)$ and $\Psi_2(\tau; 4, 4; 0, 0; 0, 0; 2, 2)$ satisfy also the condition (b) from Lemma 9. Thus they are entire modular forms of weight 5 and character $\chi(\delta) = \operatorname{sgn} \delta \left(\frac{-1}{|\delta|} \right)$ for $\Gamma_0(16)$.

Analogously, by (1.17), (1.18), (1.6), (1.7), and (1.10) we obtain

$$\begin{aligned} \Psi_3(\tau; 4\alpha, 0, 4\alpha, 0; 0, 1, 1, 1; 0, 0, 0, 0; 2, 4, 2, 4) &= \left\{ \frac{1}{2} \vartheta_{4\alpha, 0}''(\tau; 0, 4) \vartheta_{01}(\tau; 0, 8) - \right. \\ &\quad \left. - \frac{1}{4} \vartheta_{4\alpha, 0}(\tau; 0, 4) \vartheta_{01}''(\tau; 0, 8) \right\} \vartheta_{4\alpha, 1}''(\tau; 0, 4) \vartheta_{01}(\tau; 0, 8) = \\ &= \pm \left\{ \frac{1}{2} \vartheta_{40}''(\tau; 0, 4) \vartheta_{01}(\tau; 0, 8) - \frac{1}{4} \vartheta_{40}(\tau; 0, 4) \vartheta_{01}''(\tau; 0, 8) \right\} \vartheta_{41}''(\tau; 0, 4) \times \\ &\quad \times \vartheta_{01}(\tau; 0, 8) = \pm \Psi_3(\tau; 4, 0, 4, 0; 0, 1, 1, 1; 0, 0, 0, 0; 2, 4, 2, 4), \quad (3.6) \\ &\quad \Psi_4(\tau; 8\alpha, 8\alpha, 8\alpha, 0; 1, 1, 1, 1; 0, 0, 0, 0; 4, 4, 4, 4) = \\ &= \vartheta_{8\alpha, 1}'^3(\tau; 0, 8) \vartheta_{01}(\tau; 0, 8) = \pm \vartheta_{81}'^3(\tau; 0, 8) \vartheta_{01}(\tau; 0, 8) = \\ &= \pm \Psi_4(\tau; 8, 8, 8, 0; 1, 1, 1, 1; 0, 0, 0, 0; 4, 4, 4, 4). \end{aligned} \quad (3.7)$$

In ((3.6) and (3.7) we have “+” if $\alpha \equiv 1 \pmod{4}$ and “−” if $\alpha \equiv -1 \pmod{4}$). Hence the functions $\Psi_3(\tau; 4, 0, 4, 0; 0, 1, 1, 1; 0, 0, 0, 0; 2, 4, 2, 4)$ and $\Psi_4(\tau; 8, 8, 8, 0; 1, 1, 1, 1; 0, 0, 0, 0; 4, 4, 4, 4)$ also satisfy the condition (b) from Lemma 9, since

$$\left(\frac{\prod_{k=1}^4 N_k}{|\delta|} \right) = 1 \quad \text{and} \quad \operatorname{sgn} \delta \left(\frac{-1}{|\delta|} \right) = \begin{cases} 1 & \text{if } \delta \equiv 1 \pmod{4}, \\ -1 & \text{if } \delta \equiv -1 \pmod{4}. \end{cases}$$

Thus, they are entire modular forms of weight 5 and character $\chi(\delta) = \operatorname{sgn} \delta \left(\frac{-1}{|\delta|} \right)$ for $\Gamma_0(16)$. \square

Theorem 2.

$$\begin{aligned} \vartheta_{00}^8(\tau; 0, 2) \vartheta_{00}^2(\tau; 0, 8) &= \theta(\tau; f) - \frac{42}{5 \cdot 512\pi^4} \Psi_2(\tau; 8, 8; 0, 0; 0, 0; 4, 4) + \\ &+ \frac{544}{5 \cdot 8192\pi^4} \Psi_2(\tau; 0, 0; 0, 0; 0, 0; 4, 4) - \frac{3}{5 \cdot 128\pi^4} \Psi_2(\tau; 4, 4; 0, 0; 0, 0; 2, 2) - \end{aligned}$$

$$\begin{aligned}
 &-\frac{2}{32\pi^3i}\Psi_3(\tau; 4, 0, 4, 0; 0, 1, 1, 1; 0, 0, 0, 0; 2, 4, 2, 4) - \\
 &-\frac{24}{512\pi^3i}\Psi_4(\tau; 8, 8, 8, 0; 1, 1, 1, 1; 0, 0, 0, 0; 4, 4, 4, 4). \tag{3.8}
 \end{aligned}$$

Proof. According to Lemma 6, the function $\psi(r; f)$ will be identically zero if all coefficients by Q^n ($n \leq 10$) in its expansion by powers of Q are zero.

I. In Theorem 1 and Lemma 10 put $n = 2^\alpha u$, $m = u$, $v = 1$, $\Delta = 2^4$. Then

$$\begin{aligned}
 \rho(n; f) &= \frac{16}{5} \sum_{d_1 d_2 = u} \left(\frac{-1}{d_1}\right) d_2^4 \text{ for } \alpha = 0, \\
 &= \frac{224}{5} \sum_{d_1 d_2 = u} \left(\frac{-1}{d_1}\right) d_2^4 \text{ for } \alpha = 1, \\
 &= \frac{4}{5} \left(2^{4\alpha-1} \cdot 9 + \left(\frac{-1}{u}\right)\right) \sum_{d_1 d_2 = u} \left(\frac{-1}{d_1}\right) d_2^4 \text{ for } \alpha \geq 2.
 \end{aligned}$$

Calculating the values of $\rho(n; f)$ by the above formulas for all $n \leq 10$, we get

$$\begin{aligned}
 \theta(\tau; f) &= 1 + \frac{16}{5}Q + \frac{224}{5}Q^2 + 256Q^3 + \frac{4612}{5}Q^4 + \frac{10016}{5}Q^5 + 3584Q^6 + \\
 &+ 7680Q^7 + \frac{73732}{5}Q^8 + \frac{103696}{5}Q^9 + \frac{140224}{5}Q^{10} + \dots \tag{3.9}
 \end{aligned}$$

It follows from (1.8) that

$$\begin{aligned}
 \vartheta_{00}^8(\tau; 0, 2)\vartheta_{00}^2(\tau; 0, 8) &= 1 + 16Q + 112Q^2 + 448Q^3 + 1140Q^4 + 2080Q^5 + \\
 &+ 3584Q^6 + 7296Q^7 + 13876Q^8 + 20240Q^9 + 27104Q^{10} + \dots \tag{3.10}
 \end{aligned}$$

II. From (1.8) and (1.9) we have

$$\begin{aligned}
 (1) \quad &\frac{1}{8}\vartheta_{80}^{(4)}(\tau; 0, 8)\vartheta_{80}(\tau; 0, 8) = \frac{1}{8}4096\pi^4 \sum_{m_1=-\infty}^{\infty} (2m_1 + 1)^4 Q^{(2m_1+1)^2} \times \\
 &\times \sum_{m_2=-\infty}^{\infty} Q^{(2m_2+1)^2} = 512\pi^4 \cdot 4Q^2(1 + 81Q^8 + 625Q^{24} + \dots) \times \\
 &\times (1 + Q^8 + Q^{24} + \dots) = 512\pi^4(4Q^2 + 328Q^{10} + \dots), \tag{3.11} \\
 &\frac{3}{8}\vartheta_{80}''(\tau; 0, 8) = \frac{3}{8}4096\pi^4 \sum_{m_1=-\infty}^{\infty} (2m_1 + 1)^4 Q^{(2m_1+1)^2} \times
 \end{aligned}$$

$$\begin{aligned} & \times \sum_{m_2=-\infty}^{\infty} (2m_2+1)^4 Q^{(2m_2+1)^2} = 3 \cdot 512\pi^4 \cdot 4Q^2 \times \\ & \times (1+9Q^8+25Q^{24}+\dots)^2 = 512\pi^4(12Q^2+216Q^{10}+\dots); \quad (3.12) \end{aligned}$$

$$\begin{aligned} (2) \quad \frac{1}{8} \vartheta_{00}^{(4)}(\tau; 0, 8) \vartheta_{00}(\tau; 0, 8) &= \frac{1}{8} \cdot 2^{16} \pi^4 \sum_{m_1=-\infty}^{\infty} m_1^4 Q^{4m_1^2} \sum_{m_2=-\infty}^{\infty} Q^{4m_2^2} = \\ &= 8192\pi^4(2Q^4+4Q^8+32Q^{16}+\dots), \quad (3.13) \end{aligned}$$

$$\begin{aligned} \frac{3}{8} \vartheta_{\infty}^{\prime\prime 2}(\tau; 0, 8) &= \frac{3}{8} \cdot 2^{16} \pi^4 \sum_{m_1=-\infty}^{\infty} m_1^2 Q^{4m_1^2} \sum_{m_2=-\infty}^{\infty} m_2^2 Q^{4m_2^2} = \\ &= 8192\pi^4(12Q^8+96Q^{20}+\dots); \quad (3.14) \end{aligned}$$

$$\begin{aligned} (3) \quad \frac{1}{2} \vartheta_{40}^{(4)}(\tau; 0, 4) \vartheta_{40}(\tau; 0, 4) &= \frac{1}{2} \cdot 256\pi^4 \sum_{m_1=-\infty}^{\infty} (2m_1+1)^4 Q^{(2m_1+1)^2/2} \times \\ & \times \sum_{m_2=-\infty}^{\infty} Q^{(2m_2+1)^2/2} = \\ &= 128\pi^4(4Q+328Q^5+324Q^9+2504Q^{13}+\dots), \quad (3.15) \end{aligned}$$

$$\begin{aligned} \frac{3}{2} \vartheta_{40}^{\prime\prime 2}(\tau; 0, 4) &= \frac{3}{2} \cdot 256\pi^4 \sum_{m_1=-\infty}^{\infty} (2m_1+1)^2 Q^{(2m_1+1)^2/2} \times \\ & \times \sum_{m_2=-\infty}^{\infty} (2m_2+1)^2 Q^{(2m_2+1)^2/2} = \\ &= 128\pi^4(12Q+216Q^5+972Q^9+600Q^{13}+\dots); \quad (3.16) \end{aligned}$$

$$\begin{aligned} (4) \quad \frac{1}{2} \vartheta_{40}^{\prime\prime}(\tau; 0, 4) \vartheta_{01}(\tau; 0, 8) &= \frac{1}{2} \cdot 16\pi^2 \sum_{m_1=-\infty}^{\infty} (2m_1+1)^2 Q^{(2m_1+1)^2/2} \times \\ & \times \sum_{m_3=-\infty}^{\infty} (-1)^{m_3} Q^{4m_3^2} = -8\pi^2 \cdot 2Q^{1/2}(1+9Q^4+25Q^{12}+\dots)(1-2Q^4+ \\ & +2Q^{16}-\dots) = -8\pi^2 \cdot 2Q^{1/2}(1+7Q^4-18Q^8+25Q^{12}+\dots), \quad (3.17) \end{aligned}$$

$$\begin{aligned} \frac{1}{4} \vartheta_{40}(\tau; 0, 4) \vartheta_{01}^{\prime\prime}(\tau; 0, 8) &= -\frac{1}{4} \sum_{m_1=-\infty}^{\infty} Q^{(2m_1+1)^2/2} \cdot 256\pi^2 \times \\ & \times \sum_{m_3=-\infty}^{\infty} (-1)^{m_3} m_3^2 Q^{4m_3^2} = -64\pi^2 \cdot 2Q^{1/2}(1+Q^4+Q^{12}+\dots)(-2Q^4+ \\ & +8Q^{16}-\dots) = -64\pi^2 \cdot 2Q^{1/2}(-2Q^4-2Q^8+6Q^{16}+\dots), \quad (3.18) \end{aligned}$$

$$\begin{aligned} \vartheta'_{41}(\tau; 0, 4)\vartheta_{01}(\tau; 0, 8) &= -4\pi i \sum_{m_2=-\infty}^{\infty} (-1)^{m_2} Q^{(2m_2+1)^2/2} \times \\ &\times \sum_{m_4=-\infty}^{\infty} (-1)^{m_4} Q^{4m_4^2} = 4\pi i \cdot 2Q^{1/2}(1 - 3Q^4 + 5Q^{12} - \dots)(1 - 2Q^4 + \\ &+ 2Q^{16} - \dots) = 4\pi i \cdot 2Q^{1/2}(1 - 5Q^4 + 6Q^8 + 5Q^{12} - \dots), \end{aligned} \quad (3.19)$$

$$\begin{aligned} (5) \vartheta'_{81}{}^3(\tau; 0, 8)\vartheta_{01}(\tau; 0, 8) &= -512\pi^3 i \sum_{m_1, m_2, m_3=-\infty}^{\infty} (-1)^{m_1+m_2+m_3} (2m_1+1) \times \\ &\times (2m_2+1)(2m_3+1) Q^{(2m_1+1)^2+(2m_2+1)^2+(2m_3+1)^2} \sum_{m_4=-\infty}^{\infty} (-1)^{m_4} Q^{4m_4^2} = \\ &= -512\pi^3 i (8Q^3 - 16Q^7 - 72Q^{11} + \dots). \end{aligned} \quad (3.20)$$

III. From (3.4), (3.11), and (3.12) it follows that

$$\frac{1}{512\pi^4} \Psi_2(\tau; 8, 8; 0, 0; 0, 0; 4, 4) = -8Q^2 + 112Q^{10} + \dots \quad (3.21)$$

From (1.16), (3.13), and (3.14) it follows that

$$\frac{1}{8192\pi^4} \Psi_2(\tau; 0, 0; 0, 0; 0, 0; 4, 4) = 2Q^4 - 8Q^8 + 31Q^{16} + \dots \quad (3.22)$$

From (3.5), (3.15), and (3.16) it follows that

$$\frac{1}{128\pi^4} \Psi_2(\tau; 4, 4; 0, 0; 0, 0; 2, 2) = -8Q + 112Q^5 + 1904Q^{13} - \dots \quad (3.23)$$

From (3.6), (3.17)–(3.19) it follows that

$$\begin{aligned} -\frac{1}{32\pi^3 i} \Psi_3(\tau; 4, 0, 4, 0; 0, 1, 1, 1; 0, 0, 0, 0; 2, 4, 2, 4) &= \\ &= 4Q + 72Q^5 - 444Q^9 + 712Q^{13} + \dots \end{aligned} \quad (3.24)$$

From (3.7) and (3.20) follows

$$\begin{aligned} -\frac{1}{512\pi^3 i} \Psi_4(\tau; 8, 8, 8, 0; 1, 1, 1, 1; 0, 0, 0, 0; 4, 4, 4, 4) &= \\ &= 8Q^3 - 16Q^7 - 72Q^{11} + \dots \end{aligned} \quad (3.25)$$

According to (3.1), (3.9), (3.10), and (3.21)–(3.25) it is easy to verify that all coefficients by Q^n ($n \leq 10$) in the expansion of the function $\psi(\tau; f)$ by powers of Q are zero. \square

Theorem 2a. *Let $n = 2^\alpha u$. Then*

$$\begin{aligned}
r(n; f) &= \frac{16}{5} \sum_{d_1 d_2 = u} \left(\frac{-1}{d}\right) d_2^4 - \frac{12}{5} \sum_{\substack{x^2 + y^2 = 2u \\ 2 \nmid x, 2 \nmid y, x > 0, y > 0}} (x^4 - 3x^2 y^2) + \\
&+ 8 \sum_{\substack{x^2 + y^2 + 8(z^2 + t^2) = 2u \\ 2 \nmid x, 2 \nmid y, x > 0, y > 0}} (-1)^{(y-1)/2 + z + t} (x^2 - 8z^2)y \\
&\quad \text{for } \alpha = 0, u \equiv 1 \pmod{4}, \\
&= \frac{16}{5} \sum_{d_1 d_2 = u} \left(\frac{-1}{d}\right) d_2^4 + 192 \sum_{\substack{x^2 + y^2 + z^2 + 4t^2 = u \\ 2 \nmid x, 2 \nmid y, 2 \nmid z, \\ x > 0, y > 0, z > 0}} (-1)^{(x+y+z-3)/1+t} xyz \\
&\quad \text{for } \alpha = 0, u \equiv 3 \pmod{4}, \\
&= \frac{224}{5} \sum_{d_1 d_2 = u} \left(\frac{-1}{d}\right) d_2^4 - \frac{168}{5} \sum_{\substack{x^2 + y^2 = 2u \\ 2 \nmid x, 2 \nmid y, x > 0, y > 0}} (x^4 - 3x^2 y^2) \\
&\quad \text{for } \alpha = 0, u \equiv 1 \pmod{4}, \\
&= \frac{224}{5} \sum_{d_1 d_2 = u} \left(\frac{-1}{d}\right) d_2^4 \text{ for } \alpha = 1, u \equiv 3 \pmod{4}, \\
&= \frac{4612}{5} \sum_{d_1 d_2 = u} \left(\frac{-1}{d}\right) d_2^4 + \frac{1088}{5} \sum_{\substack{x^2 + y^2 = u \\ 2 \nmid x, x > 0, y = 0}} (x^4 - 3x^2 y^2) + \\
&+ \frac{2176}{5} \sum_{\substack{x^2 + y^2 = u \\ x \not\equiv y \pmod{2} \\ x > 0, y > 0}} (x^4 - 3x^2 y^2) \text{ for } \alpha = 2, u \equiv 1 \pmod{8}, \\
&= \frac{4612}{5} \sum_{d_1 d_2 = u} \left(\frac{-1}{d}\right) d_2^4 + \frac{2176}{5} \sum_{\substack{x^2 + y^2 = u \\ x \not\equiv y \pmod{2} \\ x > 0, y > 0}} (x^4 - 3x^2 y^2) \\
&\quad \text{for } \alpha = 2, u \equiv 5 \pmod{8}, \\
&= \frac{4}{5} (2^{4\alpha-1} \cdot 9 - 1) \sum_{d_1 d_2 = u} \left(\frac{-1}{d}\right) d_2^4 \text{ for } \alpha \geq 2, u \equiv 3 \pmod{4}, \\
&= \frac{73732}{5} \sum_{d_1 d_2 = u} \left(\frac{-1}{d}\right) d_2^4 + \frac{2176}{5} \sum_{\substack{x^2 + y^2 = 2u \\ 2 \nmid x, 2 \nmid y, x > 0, y > 0}} (x^4 - 3x^2 y^2) \\
&\quad \text{for } \alpha = 3, u \equiv 1 \pmod{4},
\end{aligned}$$

$$\begin{aligned}
 &= \frac{4}{5}(2^{4\alpha-1} \cdot 9+1) \sum_{d_1 d_2=u} \left(\frac{-1}{d}\right) d_2^4 + \frac{2176}{5} \sum_{\substack{x^2+y^2=2^{\alpha-2}u \\ 2|x, 2|y, x>0, y>0}} (x^4 - 3x^2y^2) \\
 &\quad \text{for } 2 \nmid \alpha, \alpha > 3, u \equiv 1 \pmod{4}, \\
 &= \frac{4}{5}(2^{4\alpha-1} \cdot 9+1) \sum_{d_1 d_2=u} \left(\frac{-1}{d}\right) d_2^4 + \frac{1088}{5} \sum_{\substack{x^2+y^2=2^{\alpha-2}u \\ 2|x, x>0, y=0}} (x^4 - 3x^2y^2) + \\
 &+ \frac{2176}{5} \sum_{\substack{x^2+y^2=2^{\alpha-2}u \\ 2|x, 2|y, x>0, y>0}} (x^4 - 3x^2y^2) \text{ for } 2|\alpha, \alpha > 3, u \equiv 1 \pmod{8}, \\
 &= \frac{4}{5}(2^{4\alpha-1} \cdot 9+1) \sum_{d_1 d_2=u} \left(\frac{-1}{d}\right) d_2^4 + \frac{2176}{5} \sum_{\substack{x^2+y^2=2^{\alpha-2}u \\ 2|x, 2|y, x>0, y>0}} (x^4 - 3x^2y^2) \\
 &\quad \text{for } 2|\alpha, \alpha > 3, u \equiv 5 \pmod{8}.
 \end{aligned}$$

Proof. Equating the coefficients by Q^n in both sides of (3.8), we obtain

$$\begin{aligned}
 r(n; f) &= \rho(n; f) - \frac{42}{5}\nu_1(n) + \frac{544}{5}\nu_2(n) - \\
 &\quad - \frac{3}{5}\nu_3(n) + 2\nu_4(n) + 24\nu_5(n), \tag{3.26}
 \end{aligned}$$

where $\nu_k(n)$ ($k = 1, 2, 3, 4, 5$) denote respectively the coefficients by Q^n in the expansions of the functions $\frac{1}{512\pi^4}\Psi_2(\tau; 8, 8; 0, 0; 0, 0; 4, 4)$, $\frac{1}{8192\pi^4}\Psi_2(\tau; 0, 0; 0, 0; 0, 0; 4, 4)$, $\frac{1}{128\pi^4}\Psi_2(\tau; 4, 4; 0, 0; 0, 0; 2, 2)$, $-\frac{1}{32\pi^3i}\Psi_3(\tau; 4, 0, 4, 0; 0, 1, 1, 1; 0, 0, 0, 0; 2, 4, 2, 4)$, $-\frac{1}{512\pi^3i}\Psi_4(\tau; 8, 8, 8, 0; 1, 1, 1, 1; 0, 0, 0, 0; 4, 4, 4, 4)$ by powers of Q .

From (3.4), (3.11), and (3.12) it follows that

$$\begin{aligned}
 \frac{1}{512\pi^4}\Psi_2(\tau; 8, 8; 0, 0; 0, 0; 4, 4) &= \sum_{m_1, m_2=-\infty}^{\infty} (2m_1 + 1)^4 Q^{(2m_1+1)^2+(2m_2+1)^2} - \\
 -3 \sum_{m_1, m_2=-\infty}^{\infty} (2m_1 + 1)^2(2m_2 + 1)^2 Q^{(2m_1+1)^2+(2m_2+1)^2},
 \end{aligned}$$

i.e.,

$$\nu_1(n) = 4 \sum_{\substack{x^2+y^2=n \\ 2 \nmid x, 2 \nmid y, x>0, y>0}} (x^4 - 3x^2y^2); \tag{3.27}$$

it is not difficult to verify that

$$\nu_1(n) \neq 0 \text{ for } \alpha = 1, u \equiv 1 \pmod{4} \text{ only.}$$

From (1.16), (3.13), and (3.14) follows

$$\begin{aligned} \frac{1}{8192\pi^4} \Psi_2(\tau; 0, 0; 0, 0; 0, 0; 4, 4) &= \sum_{m_1, m_2 = -\infty}^{\infty} m_1^4 Q^{4m_1 + 4m_2^2} - \\ &- 3 \sum_{m_1, m_2 = -\infty}^{\infty} m_1^2 m_2^2 Q^{4m_1^2 + 4m_2^2}, \end{aligned}$$

i.e.,

$$\begin{aligned} \nu_2(n) &= 4 \sum_{4(x^2+y^2)=n} (x^4 - 3x^2y^2) = \\ &= \begin{cases} \sum_{x^2+y^2=2^{\alpha-2}u} (x^4 - 3x^2y^2) & \text{for } \alpha \geq 2, \\ 0 & \text{for } \alpha = 0, 1; \end{cases} \end{aligned} \tag{3.28}$$

it is obvious that

$$\begin{aligned} \sum_{x^2+y^2=2^{\alpha-2}u} (x^4 - 3x^2y^2) &= \sum_{\substack{x^2+y^2=u \\ x \not\equiv y \pmod{2}}} (x^4 - 3x^2y^2) \text{ for } \alpha = 2, u \equiv 1 \pmod{4}, \\ &= 0 \text{ for } \alpha \geq 2, u \equiv 3 \pmod{4}, \\ &= \sum_{\substack{x^2+y^2=2^{\alpha-2}u \\ x \equiv y \pmod{2}}} (x^4 - 3x^2y^2) \text{ for } \alpha \geq 3, u \equiv 1 \pmod{4}, \end{aligned}$$

From (3.5), (3.15), and (3.16) it follows that

$$\begin{aligned} \frac{1}{128\pi^4} \Psi_2(\tau; 4, 4; 0, 0; 0, 0; 2, 2) &= \sum_{m_1, m_2 = -\infty}^{\infty} (2m_1 + 1)^4 Q^{(2m_1+1)^2/2 + (2m_2+1)^2/2} - \\ &- 3 \sum_{m_1, m_2 = -\infty}^{\infty} (2m_1 + 1)^2 (2m_2 + 1)^2 Q^{(2m_1+1)^2/2 + (2m_2+1)^2/2}, \end{aligned}$$

i.e.,

$$\begin{aligned} \nu_3(n) &= \sum_{\substack{x^2+y^2=2u \\ 2 \nmid x, 2 \nmid y}} (x^4 - 3x^2y^2) = 4 \sum_{\substack{x^2+y^2=2u \\ 2 \nmid x, 2 \nmid y, x > 0, y > 0}} (x^4 - 3x^2y^2) \\ &\text{for } \alpha = 0, u \equiv 1 \pmod{4}, \\ &= 0 \text{ for } \alpha = 0, u \equiv 3 \pmod{4} \text{ and for } \alpha > 0. \end{aligned} \tag{3.29}$$

From (3.6) and (3.17)–(3.19) it follows that

$$\begin{aligned} & -\frac{1}{32\pi^3 i} \Psi_3(\tau; 4, 0, 4, 0; 0, 1, 1, 1; 0, 0, 0, 0; 2, 4, 2, 4) = \\ & = \left\{ \sum_{m_1, m_3=-\infty}^{\infty} (-1)^{m_3} (2m_1 + 1)^2 Q^{(2m_1+1)^2/2+4m_3^2} - \right. \\ & \quad \left. - 8 \sum_{m_1, m_3=-\infty}^{\infty} (-1)^{m_3} 2m_3^2 Q^{(2m_1+1)^2/2+4m_3^2} \right\} \times \\ & \quad \times \sum_{m_2, m_4=-\infty}^{\infty} (-1)^{m_2+m_4} (2m_2 + 1) Q^{(2m_2+1)^2/2+4m_4^2} = \\ & = \sum_{m_1, m_2, m_3, m_4=-\infty}^{\infty} (-1)^{m_2+m_3+m_4} ((2m_1+1)^2(2m_2+1) - 8(2m_2+1)m_3^2) \times \\ & \quad \times Q^{\frac{1}{2}\{(2m_1+1)^2+(2m_2+1)^2+8m_3^2+8m_4^2\}}, \end{aligned}$$

i.e.,

$$\begin{aligned} \nu_4(n) &= \sum_{\substack{x^2+y^2+8(z^2+t^2)=2n \\ 2 \nmid x, 2 \nmid y}} (-1)^{(y-1)/2+z+t} (x^2 - 8z^2)y = \\ &= 4 \sum_{\substack{x^2+y^2+8(z^2+t^2)=2u \\ 2 \nmid x, 2 \nmid y, x>0, y>0}} (-1)^{(y-1)/2+z+t} (x^2 - 8z^2)y \quad (3.30) \\ & \quad \text{for } \alpha = 0, u \equiv 1 \pmod{4}, \\ & \quad = 0 \text{ for } \alpha = 0, u \equiv 3 \pmod{4} \text{ and for } \alpha > 0. \end{aligned}$$

From (3.7) and (3.20) follows

$$\begin{aligned} & -\frac{1}{512\pi^3 i} \Psi_3(\tau; 8, 8, 8, 0; 1, 1, 1, 1; 0, 0, 0, 0; 4, 4, 4, 4) = \\ & = \sum_{m_1, m_2, m_3, m_4=-\infty}^{\infty} (-1)^{\sum_{k=1}^4 m_k} \prod_{k=1}^3 (2m_k + 1) Q^{\sum_{k=1}^3 (2m_k+1)^2+4m_4^2}, \end{aligned}$$

i.e.,

$$\nu_5(n) = 8 \sum_{\substack{x^2+y^2+z^2+4t^2=n \\ 2 \nmid x, 2 \nmid y, 2 \nmid z \\ x>0, y>0, z>0}} (-1)^{(x+y+z-3)/2+t} xyz; \quad (3.31)$$

it is easy to verify that

$$\nu_5(n) \neq 0 \text{ only for } \alpha = 0, u \equiv 3 \pmod{4}.$$

The above theorem follows from (3.26), formulas for $\rho(n; f)$, and (3.27)–(3.31). \square

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