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NECESSARY AND SUFFICIENT CONDITIONS FOR WEIGHTED ORLICZ CLASS INEQUALITIES FOR MAXIMAL FUNCTIONS AND SINGULAR INTEGRALS. II

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Abstract. This paper continues the investigation of weight problems in Orlicz classes for maximal functions and singular integrals defined on homogeneous type spaces considered in [1].

§ 1. Weak Type Weighted Inequalities for Singular Integrals

Our further discussion will involve singular integrals with kernels which in homogeneous type spaces are analogues of Calderon–Zygmund kernels.

It will be assumed that $k : X\times X\backslash\{(x,x) : x\in X\}\to\mathbb{R}^1$ is a measurable function satisfying the conditions

$$|k(x,y)|\le\frac{c_1}{\mu B(x,d(x,y))}\tag{1.1}$$

and there exists positive constants c_2 and b_0 such that

$$\begin{aligned}\big|k(x',y)-k(x,y)\big|+\big|k(y,x')-k(y,x)\big|\le\\ \le c_2\omega\Big(\frac{d(x,x')}{d(x,y)}\Big)\frac{1}{\mu B(x,d(x,y))}\end{aligned}\tag{1.2}$$

for arbitrary x, y and x' with the condition $d(x,y)>b_0d(x',x)$. Here $\omega:(0,1)\to\mathbb{R}^1$ is a nondecreasing function with the condition $\omega(0)=0$, $\omega(2t)\le c\omega(t)$ and

$$\int_0^1\frac{\omega(t)}{t}\,dt<\infty.$$

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Definition 1.1. A kernel k will be said to belong to the class CZ, ($k \in$ CZ), if conditions (1.1), (1.2) are fulfilled and the singular integral

$$\mathcal{K}f(x) = \lim_{\varepsilon \to 0} \mathcal{K}_\varepsilon f(x) = \lim_{\varepsilon \to 0} \int\limits_{d(x,y)>\varepsilon} k(x,y) f(y)\, d\mu$$

generates a continuous operator in $L_{p_0}(X,\mu)$ for some p_0, $1 < p_0 < \infty$.

Singular integrals with such kernels were treated in [2–6]. We set

$$\mathcal{K}^* f(x) = \sup_{\varepsilon>0} \big|\mathcal{K}_\varepsilon f(x)\big|.$$

The following theorem is well known.

Theorem B [5]. *If $k \in$ CZ, then for an arbitrary $w \in \mathcal{A}_p$ $(1 < p < \infty)$ we have the inequality*

$$\int_X \big(\mathcal{K}^* f(x)\big)^p w(x)\, d\mu \le c_3 \int_X |f(x)|^p w(x)\, d\mu, \tag{1.3}$$

where the constant c_3 is independent of f.

We shall begin our investigation of weighted problems for singular integrals of the above-mentioned kind with weak type inequalities.

Theorem 1.1. *Let $\varphi \in \Phi \cap \Delta_2$ and $k \in$ CZ. If there exists a constant $c_4 > 0$ such that*

$$\begin{aligned} \int_B \widetilde{\varphi}\left(\frac{\int_B \varphi(\lambda w_1(y)) w_2(y) d\mu}{\lambda \mu B w_1(x) w_2(x)}\right) w_2(x)\, d\mu \le \\ \le c_4 \int_B \varphi\big(\lambda w_1(x)\big) w_2(x)\, d\mu \end{aligned} \tag{1.4}$$

for an arbitrary $\lambda > 0$ and any ball B, then we have the inequality

$$\int\limits_{\{x:|\mathcal{K}f(x)|>\lambda\}} \varphi\big(\lambda w_1(x)\big) w_2(x)\, d\mu \le c_5 \int_X \varphi\big(f(x) w_1(x)\big) w_2(x)\, d\mu, \tag{1.5}$$

where the constant c_5 is independent of λ and f.

Proof. Let $\lambda > 0$ and $f : X \to \mathbb{R}^1$ be a μ-measurable function with a compact support. We set

$$\widetilde{\mathcal{M}} f(x) = \sup_{r>0} \frac{1}{\mu B(x,r)} \int_{B(x,r)} |f(y)|\, d\mu$$

to be a centered maximal function.

Let further $\Omega = \{x : \widetilde{\mathcal{M}} f(x) > \lambda\}$. One can easily verify that the set Ω is open and bounded. If $X = \Omega$, then

$$\int\limits_{\{x:|\mathcal{K}f(x)|>\lambda\}} \varphi\big(\lambda w_1(x)\big) w_2(x)\, d\mu \le \int\limits_{\{x:\widetilde{\mathcal{M}}f(x)>\lambda\}} \varphi\big(\lambda w_1(x)\big) w_2(x)\, d\mu$$

and the validity of Theorem 1.1 follows from Theorem 3.1 from [1].

Assume that $\Omega \ne X$. By virtue of Lemma 4.1 from Part I for the set Ω and the constant $C = a_1(1 + b_0 a_0)$ there exists a sequence of balls $B_j = B(x_j, r_j)$ such that

$$\Omega = \bigcup_{j=1}^{\infty} CB_j, \quad \sum_{j=1}^{\infty} \chi_{CB_j}(x) \le \eta,$$
$$\widetilde{B}_j = B(x_j, 3Ca_1 r_j) \cap (X \backslash \Omega) \ne \varnothing, \quad j = 1, 2, \dots,$$

where the constant b_0 is from the definition of k while the numbers a_0 and a_1 are from the definition of X.

Set $F = X \backslash \Omega$. Since $\widetilde{B}_j \cap F \ne \varnothing$, we have

$$|f|_{B_j} \le c|f|_{\widetilde{B}_j} \le c\lambda, \tag{1.6}$$

where the constant c is independent of λ and j.

Let

$$g(x) = f(x)\chi_F(x) + \sum_j \big(f\big)_{B_j} \chi_{B_j}(x),$$
$$\psi(x) = f(x) - g(x) = \sum_j \big(f(x) - \big(f\big)_{B_j}\big)\chi_{B_j}(x) = \sum_j \psi_j(x).$$

By virtue of

$$|\mathcal{K}f(x)| \le |\mathcal{K}g(x)| + |\mathcal{K}\psi(x)|,$$

we have

$$\int\limits_{\{x:|\mathcal{K}f(x)|>\lambda\}} \varphi\big(\lambda w_1(x)\big) w_2(x)\, d\mu \le \int\limits_{\{x:|\mathcal{K}g(x)|>\frac{\lambda}{2}\}} \varphi\big(\lambda w_1(x)\big) w_2(x)\, d\mu +$$
$$+ \int\limits_{\{x:|\mathcal{K}\psi(x)|>\frac{\lambda}{2}\}} \varphi\big(\lambda w_1(x)\big) w_2(x)\, d\mu. \tag{1.7}$$

Since by Proposition 3.5 of Part I the function $\varphi(\lambda w_1) w_2 \in \mathcal{A}_\infty$ uniformly with respect to λ, there exists p such that $\varphi(\lambda w_1) w_2 \in \mathcal{A}_p$ uniformly with respect to λ. Therefore by Theorem B and inequality (1.6) we have

$$\int\limits_{\{x:|\mathcal{K}g(x)|>\frac{\lambda}{2}\}} \varphi\big(\lambda w_1(x)\big)w_2(x)d\mu \le c\lambda^{-p}\int_X |\mathcal{K}g(x)|^p\varphi\big(\lambda w_1(x)\big)w_2(x)d\mu \le$$

$$\le c\lambda^{-p}\int_X |g(x)|^p\varphi\big(\lambda w_1(x)\big)w_2(x)d\mu \le c\lambda^{-p}\int_F |f(x)|^p\varphi\big(\lambda w_1(x)\big)w_2(x)d\mu +$$

$$+c\int_X \varphi\big(\lambda w_1(x)\big)w_2(x)\,d\mu. \tag{1.8}$$

Next, due to the fact that $|f(x)| < \lambda$ almost everywhere on F, in the sense of the μ-measure, using (2.2) from [1] and (1.8) we conclude that

$$\lambda^{-p}|f(x)|^p\varphi\big(\lambda w_1(x)\big) \le c\varphi\big(f(x)w_1(x)\big) \tag{1.9}$$

holds for almost all $x \in F$ and sufficiently large p.

Thus (1.8) and (1.9) give the estimate

$$\int\limits_{\{x:|\mathcal{K}g(x)|>\frac{\lambda}{2}\}} \varphi\big(\lambda w_1(x)\big)w_2(x)d\mu \le c\int_X \varphi\big(f(x)w_1(x)\big)w_2(x)d\mu. \tag{1.10}$$

Now we shall estimate $|\mathcal{K}\psi(x)|$ on the set F. We have

$$\mathcal{K}\psi(x) = \sum_j \int_{B_j} k(x,y)\psi_j(y)\,d\mu.$$

From the definition of ψ_j we obtain $\int_{B_j}\psi_j(x)\,d\mu = 0$. Therefore

$$\mathcal{K}\psi_j(x) = \int_{B_j} \big(k(x,y) - k(x,x_j)\big)\psi_j(y)\,d\mu. \tag{1.11}$$

Choosing balls B_j appropriately we have $x \notin B(x_j, a_1(1+b_0a_0)r_j)$ if $x \in F$. Hence for an arbitrary $x \in F$ we have $d(x_j,x) > a_1(1+b_0a_0)r_j$.

By the last inequality we have

$$\begin{aligned} a_0(1+b_0a_0)r_j < d(x_j,x) \le a_1\big(d(x_j,y)+d(y,x)\big) \le \\ \le a_1r_j + a_1a_0d(x,y) \end{aligned} \tag{1.12}$$

for an arbitrary $y \in B_j$.

From (1.12) we conclude that $d(x,y) > b_0r_j$, i.e., $b_0d(x_j,y) \le d(x,y)$ for $x \in F$ and $y \in B(x_j,r_j)$.

Using the above reasoning for $x \in F$ and condition (1.2), we write the kernel in the form

$$|\mathcal{K}\psi_j(x)| \le c\sum_j |\psi_j(y)|\omega\Big(\frac{d(x_j,y)}{d(x_j,x)}\Big)\frac{1}{\mu B(x_j,d(x_j,x))}\,d\mu \le$$

$$\leq c\omega\Big(\frac{r_j}{d(x_j,x)}\Big)\,\frac{1}{\mu B(x_j,d(x_j,x))}\int_{B_j}|\psi_j(y)|\,d\mu.$$

On the other hand, by virtue of (1.6)

$$|\psi_j|_{B_j}\leq 2|f|_{B_j}\leq 2c\lambda.$$

Hence from the preceding inequality for $x\in F$ we obtain

$$|\mathcal{K}\psi_j(x)|\leq c\lambda\omega\Big(\frac{r_j}{d(x_j,x)}\Big)\,\frac{\mu B(x_j,r_j)}{\mu B(x_j,d(x_j,x))}\quad (j=1,2,\dots).$$

On summing these inequalities, we find

$$|\mathcal{K}\psi(x)|\leq c\lambda\mathcal{I}_\omega(x) \tag{1.13}$$

for $x\in F$.

Now, applying Theorem 4.1 of [1], from (1.13) we derive

$$\int\limits_{\{x\in X:|\mathcal{K}\psi(x)|>\frac{\lambda}{2}\}}\varphi\big(\lambda w_1(x)\big)\,w_2(x)d\mu\leq\int\limits_{\{x\in F:|\mathcal{K}\psi(x)|>\frac{\lambda}{2}\}}\varphi\big(\lambda w_1(x)\big)\,w_2(x)d\mu+$$
$$+\int_\Omega\varphi\big(\lambda w_1(x)\big)\,w_2(x)\,d\mu. \tag{1.14}$$

By virtue of Proposition 3.5 (see Part I) the condition of the theorem implies that $\varphi(\lambda w_1)w_2\in\mathcal{A}_\infty$ uniformly with respect to λ. Therefore, as said above, there exists $p>1$ such that $\varphi(\lambda w_1)w_2\in\mathcal{A}_p$ uniformly with respect to λ. Thus using (1.13) and Corollary 4.2 from [1] we obtain

$$\int\limits_{\{x\in F:|\mathcal{K}\psi(x)|>\frac{\lambda}{2}\}}\varphi\big(\lambda w_1(x)\big)\,w_2(x)\,d\mu\leq c\int_F\big(\mathcal{I}_\omega(x)\big)^p\varphi\big(\lambda w_1(x)\big)\,w_2(x)\,d\mu\leq$$
$$\leq c\int_\Omega\varphi\big(\lambda w_1(x)\big)\,w_2(x)\,d\mu.$$

Therefore (1.14) gives the estimate

$$\int\limits_{\{x\in F:|\mathcal{K}\psi(x)|>\frac{\lambda}{2}\}}\varphi\big(\lambda w_1(x)\big)\,w_2(x)\,d\mu\leq c\int_\Omega\varphi\big(\lambda w_1(x)\big)\,w_2(x)\,d\mu.$$

Taking into account the definition of the set Ω and Theorem 3.1 of [1], we obtain the estimate

$$\int\limits_{\{x\in F:|\mathcal{K}\psi(x)|>\frac{\lambda}{2}\}}\varphi\big(\lambda w_1(x)\big)\,w_2(x)\,d\mu\leq c\int_X\varphi\big(\lambda w_1(x)\big)\,w_2(x)\,d\mu.$$

Finally, the last inequality together with (1.7) and (1.10) imply that the theorem is valid. $\square$

Theorem 1.2. *Let $\varphi \in \Phi \cap \Delta_2$, $k \in \mathrm{CZ}$. Then from condition (1.4) it follows that there exists a constant $c_1 > 0$ such that the inequality*

$$\int\limits_{\{x \in X : \mathcal{K}^* f(x) > \lambda\}} \varphi\big(\lambda w_1(x)\big) w_2(x)\, d\mu \le c_1 \int_X \varphi\big(f(x) w_1(x)\big) w_2(x)\, d\mu \tag{1.15}$$

is fulfilled for any $\lambda > 0$ and μ-measurable function $f : X \to \mathbb{R}^1$.

Proof. This theorem, which is more general than Theorem 1.1, is proved quite similarly to the latter provided that we show that the inequality

$$\mathcal{K}^* \psi(x) \le c\lambda \mathcal{I}_\omega(x) + c\mathcal{M} f(x) \tag{1.16}$$

holds for $x \in F$.

We have

$$\mathcal{K}_\varepsilon \psi(x) = \sum_j \int\limits_{\{y \in B_j : d(x,y) > \varepsilon\}} \psi_j(y) k(x,y) d\mu = \sum_{\{j : \mathrm{dist}(x, B_j) > \varepsilon\}} \int_{B_j} \psi(y) k(x,y) d\mu +$$

$$+ \sum_{\{j : \mathrm{dist}(x, B_j) \ge \varepsilon\}} \int\limits_{\{y \in B_j : d(x,y) > \varepsilon\}} \psi_j(y) k(x,y)\, d\mu = A_\varepsilon(x) + B_\varepsilon(x).$$

In proving Theorem 1.1, it was shown that

$$\sup_{\varepsilon > 0} |A_\varepsilon(x)| \le c\lambda \mathcal{I}_\omega(x)$$

for $x \in F$.

Further for $x \in F$, $y \in B_j$ and $z \in B_j$ we have

$$d(x,y) \le a_1\big(d(x,z) + d(z,y)\big) \le a_1\big(d(x,z) + a_1(d(z,x_j) + d(x_j,y))\big) \le$$
$$\le a_1\big(d(x,z) + a_1(a_0+1) r_j\big).$$

Since z is an arbitrary point from B_j, for $\mathrm{dist}(x, B_j) \le \varepsilon$ the last inequality implies

$$d(x,y) \le a_1 \,\mathrm{dist}(x, B_j) + a_1^2 (a_0 + 1) r_j \le c_0 \varepsilon,$$

where $c_0 = a_1 + 2a_0 a_1^2 b_0^{-1}$.

Therefore, due to (1.1), for $x \in F$ we have

$$|B_\varepsilon(x)| \le \int\limits_{\varepsilon < d(x,y) < c_0\varepsilon} \frac{\sum_{j=1}^\infty |\psi_j(y)|}{\mu B(x, d(x,y))}\, d\mu \le$$

$$\le \frac{c}{\mu B(x,\varepsilon)} \int\limits_{\varepsilon < d(x,y) \le c_0 \varepsilon} |\psi(y)|\, d\mu \le c\mathcal{M}\psi(x).$$

Now repeating the arguments from the proof of Theorem 1.1 and using inequalities (1.16), (1.17) and Theorem 3.1 from [1], we arrive at (1.14). $\square$

§ 2. Criteria of Weak Type Weighted Inequalities for Singular Integrals

In this section, from the class of Calderon–Zygmund kernels we single out a subclass of kernels such that for the corresponding singular integrals we succeed in obtaining necessary and sufficient conditions ensuring the validity of weak type inequalities.

Definition 2.1. A kernel $k : X \times X \backslash \{(x,x), x \in X\} \to \mathbb{R}^1$ belongs to the class $\mathcal{S}_1$ if for an arbitrary ball $B = B(z,r)$ there is a ball $B' = B(z',r)$ such that $B \cap B' = \varnothing$, $d(z,z') \le c_1 r$ and for any $x \in B$ and $y \in B'$ we have the inequality

$$k(x,y) \ge \frac{c_2}{\mu B(x, d(x,y))}, \tag{2.1}$$

where the constant c_2 is independent of the ball B, x and y.

It is easy to see that in the above condition the ball B' can be chosen so that $\operatorname{dist}(B,B') > r$.

Indeed, in addition to the ball $B(z,r)$ let us consider the ball $B(z,mr)$, where $m = a_1 + a_1^2(1+a_0)$. By assumption, there exists a ball $B(z_0,mr)$ such that $B(z,mr) \cap B(z_0,mr) = \varnothing$ and (2.1) is fulfilled for arbitrary $x \in B(z,mr)$ and $y \in B(z_0,mr)$. Now for a given ball $B(z,r)$ we shall take $B(z_0,r)$ as B'. Then already $\operatorname{dist}(B,B') > r$. Indeed, for arbitrary $x \in B$ and $y \in B'$ we have

$$\begin{gathered} a_1\big(1 + a_1(1+a_0)\big)r \le d(z,z_0) \le a_1\big(d(z,x) + d(x,z_0)\big) \le \\ \le a_1\big(r + a_1\big(d(x,y) + d(y,z_0)\big)\big) \le a_1 r + a_1^2 d(x,y) + a_1^2 a_0 d(z_0,y) \le \\ \le (a_1 + a_1^2 a_0)r + a_1^2 d(x,y). \end{gathered}$$

Hence it follows that $d(x,y) \ge r$ and therefore $r < \operatorname{dist}(B,B')$.

Definition 2.2. We shall say that $k \in \mathcal{S}_2$ if for an arbitrary ball $B(z,r)$ there exists a ball $B' = B(z',r)$ such that $B \cap B' = \varnothing$, $d(z,z') < c_2 r$ and for arbitrary $y \in B$ and $x \in B'$ we have the inequality

$$k(x,y) \ge \frac{c_4}{\mu B(x, d(x,y))}, \tag{2.1$'$}$$

where the constant c_4 is independent of x and y.

It is easy to verify that if $k \in \mathcal{S}_1$, then $\widetilde{k} \in \mathcal{S}_2$, where $\widetilde{k}(x,y) = k(y,x)$.

When X is compact, condition (2.1) is to be fulfilled for balls with a sufficiently small radius.

Further assume

$$\widetilde{\mathcal{K}} f(x) = \int_X k(y,x) f(y)\, d\mu.$$

We have

Theorem 2.1. *Let $\varphi \in \Phi$ and the kernel $k \in \mathcal{S}_1 \cup \mathcal{S}_2$. If the inequalities*

$$\int\limits_{\{x:|\mathcal{K}f(x)|>\lambda\}} \varphi\big(\lambda w_1(x)\big) w_2(x)\, d\mu \le c \int_X \varphi\big(cf(x) w_1(x)\big) w_2(x)\, d\mu, \tag{2.2}$$

$$\int\limits_{\{x:|\widetilde{\mathcal{K}}f(x)|>\lambda\}} \varphi\big(\lambda w_1(x)\big) w_2(x)\, d\mu \le c \int_X \varphi\big(cf(x) w_1(x)\big) w_2(x)\, d\mu, \tag{2.2$'$}$$

where the constant c is independent of $\lambda > 0$ and f, are fulfilled, then $\varphi \in \Delta_2$, φ is quasiconvex, and the following condition holds: there exists a constant c_0 such that

$$\int_B \widetilde{\varphi}\left(\frac{\int_B \varphi(\lambda w_1(y) w_2(y) d\mu}{\lambda \mu B w_1(x) w_2(x)}\right) w_2(x)\, d\mu \le c_0 \int_B \varphi\big(\lambda w_1(x)\big) w_2(x)\, d\mu$$

for any $\lambda > 0$ and ball B.

In §1 the last inequality figured as formula (1.4). In what follows by referring to this condition we shall mean (1.4).

The proof of the theorem is based on

Proposition 2.1. *Let E be a set of positive μ-measure not containing atoms. Assume that $k \in \mathcal{S}_1 \cup \mathcal{S}_2$.*

If there exists a constant $c > 0$ such that the inequality

$$\varphi(\lambda)\mu\big\{x \in E \backslash E_1 \ : \ |\mathcal{K}f(x)| > \lambda\big\} \le c \int_{E_1} \varphi\big(cf(x)\big)\, d\mu \tag{2.3}$$

holds for any $\lambda > 0$, μ-measurable $E_1 \subseteq E$ and μ-measurable function f, $\operatorname{supp} f \subset E_1$, *then φ is quasiconvex and satisfies the condition Δ_2.*

Proof. Let first $k \in \mathcal{S}_1$. Assume that z_0 is a density point of the set E. From the property of the kernel it follows that for each $B(z_0, r)$ there exists a ball $B(z_0', r)$ such that $B(z_0, r) \cap B(z_0', r) = \varnothing$, $d(z_0, z_0') < c_1 r$, and condition (2.1) is fulfilled for $x \in B(z_0, r)$ and $y \in B(z_0', r)$.

We shall show that the number r can be chosen so small that

$$\mu B(z_0', r) \cap E \ge \frac{1}{2}\, \mu B(z_0', r).$$

First note that the condition $d(z_0, z_0') < c_1 r$ implies

$$B(z_0', r) \subset B(z_0, a_1(c_1 + 1) r).$$

Indeed, if $x \in B(z_0', r)$, then $d(z_0, x) \le a_1\big(d(z_0, z_0') + d(z_0', x)\big) \le a_1(c_1 + 1)r$. On the other hand,

$$B(z_0, a_1(c_1 + 1)r) \subset B(z_0', c_3 r),$$

where $c_3 = a_1(a_0c_1 + a_1(c_1+1))$.

Therefore

$$B(z_0',r) \subset B(z_0, a_1(c_1+1)r) \subset B(z_0', c_3r).$$

Since z_0 is a density point of E, for $\varepsilon > 0$ there exists $\delta > 0$ such that if $r < \delta$ then

$$\mu B(z_0, a_1(c_1+1)r)\backslash E \le \varepsilon \mu B(z_0, a_1(c_1+1)r).$$

Therefore

$$\begin{aligned}\mu B(z_0',r)\backslash E \le \mu B(z_0, a_1(c_1+1)r)\backslash E \le \varepsilon \mu B(z_0, a_1(c_1+1)r) \le \\ \le \varepsilon \mu B(z_0, c_3 r) \le \varepsilon b \mu B(z_0', r),\end{aligned}$$

where the constant b is from the doubling condition.

If $\varepsilon b < \frac{1}{2}$, the last inequality implies

$$\mu B(z_0',r) \cap E > (1-\varepsilon b)\mu B(z_0',r) \ge \frac{1}{2}\mu B(z_0',r).$$

Fix some ball $B(z_0,r)$ with the condition $0 < r < \delta$. Let $B(z_0',r)$ be the ball existing by virtue of the condition $k \in \mathcal{S}_1$. Now if $f \ge 0$, $\operatorname{supp} f \subset B(z_0',r) \cap E$, for any $x \in B(z_0,r)$ we obtain

$$Tf(x) = \int_X k(x,y)f(y)\,dy \ge c_2 \int_{B(z_0',r)} \frac{f(y)d\mu}{\mu B(x,d(x,y))}. \tag{2.4}$$

Moreover, if $x \in B(z_0,r)$, $y \in B(z_0',r)$ and $z \in B(x, d(x,y))$, we have

$$\begin{aligned}d(z_0',z) &\le a_1\big(d(z_0',x) + d(x,z)\big) \le a_1^2\big(d(z_0',z_0) + d(z_0,x) + a_1 d(x,y)\big) \le \\ &\le a_1^2c_1r + a_1^2 r + a_1^2\big(d(x,z_0) + d(z_0,y)\big) \le \\ &\le a_1^2(c_1+1)r + a_1^2a_0r + a_1^3\big(d(z_0,z_0') + d(z_0',y)\big) \le c_4 r.\end{aligned}$$

Thus $B(x,d(x,y)) \subset B(z_0', c_4r)$. Hence by the doubling condition we find that $\mu B(x,d(x,y)) \le c_5 \mu B(z_0',r)$. Therefore (2.4) implies that if $r < \delta$, then for $B(z_0,r)$ there exists a ball $B(z_0',r)$ such that for $x \in B(z_0,r)$ we obtain the estimate

$$Tf(x) \ge \frac{c_2}{c_5}\frac{1}{\mu B(z_0',r)}\int_{B(z_0',r)} f(y)\,d\mu. \tag{2.5}$$

Moreover, $\mu B(z_0',r) \cap E > \frac{1}{2}\mu B(z_0,r)$, $d(z_0,z_0') < c_1 r$ and $\operatorname{dist}(B(z_0',r), B(z_0,r)) > r$.

Let now $r_k = \frac{\delta}{a_1^k(c_1+1)^k}$ and $B(z_0^k, r_k)$ be a ball corresponding to the ball $B(z_0, r_k)$ in condition (2.5).

For $x \in B(z_0^k, r_k)$ we have

$$d(z_0, x) \le a_1\big(d(z_0, z_0^k) + d(z_0^k, x)\big) \le a_1(c_1+1)r_k = r_k - 1,$$

which implies $B(z_0^k, r_k) \subset B(z_0, r_{k-1})$. Also, $B(z_0, r_{k-1}) \cap B(z_0^{k-1}, r_{k-1}) = \varnothing$. Therefore $B(z_0^k, r_k) \cap B(z_0^{k-1}, r_{k-1}) = \varnothing$. Similarly, $B(z_0^j, r_j) \cap B(z_0^i, r_i) = \varnothing$, $i \ne j$. Further $B(z_0, r_k) \subseteq B(z_0, r_i)$, $i = 1, 2, \dots, k$, and $B(z_0, r_i) \cap B(z_0^i, r_i) = \varnothing$. Thus $B(z_0, r_k) \cap B(z_0^i, r_i) = \varnothing$ $(i = 1, 2, \dots, k)$.

We set

$$f(x) = \frac{\lambda}{c}\chi_{\overset{k}{\underset{i=1}{\cup}} B(z_0^i, r_i) \cap E},$$

where the constant is from (2.3) and k is chosen so that $\frac{kc_2}{2cc_5} > 2$. Then by (2.5) for $x \in B(z_0, r_k)$ we obtain

$$\begin{aligned} Tf(x) &= \int_X k(x,y) f(y)\, d\mu = \sum_{i=1}^{k} \int_{B(z_0^i, r_i)} k(x,y) f(y)\, d\mu \ge \\ &\ge \frac{\lambda c_2}{cc_5} \sum_{i=1}^{k} \frac{\mu B(z_0^i, r_i) \cap E}{\mu B(z_0^i, r_i)} \ge \frac{\lambda c_2}{2cc_5} > 2\lambda. \end{aligned} \tag{2.6}$$

Now substituting $E_1 = \overset{k}{\underset{i=1}{\cup}} B(z_0^i, r_i) \cap E$ in (2.3) and taking into account that by virtue of the above reasoning $B(z_0, r_k) \cap E \subset E \backslash E_1$, by (2.6) we have

$$\varphi(2\lambda)\mu B(z_0, r_k) \cap E \le c\varphi(\lambda) \sum_{i=1}^{k} \mu B(z_0^i, r_i) \cap E$$

which implies the fulfillment of the condition Δ_2.

Next we want to prove that the function φ is quasiconvex. Let $z_0 \in E$ be a density point of this set. As shown above, there exists a ball $B(z_0', r_0)$ such that $\operatorname{dist}(B(z_0, r_0), B(z_0', r_0)) > r_0$,

$$\mu B(z_0, r_0) \cap E \ge \frac{1}{2}\, \mu B(z_0, r_0),$$

$$\mu B\Big(z_0', \frac{r_0}{2a_1}\Big) \cap E > \frac{1}{2}\, \mu B\Big(z_0', \frac{r_0}{2a_1}\Big),$$

and for arbitrary $x \in B(z_0, r_0)$, $f \ge 0$, $\operatorname{supp} f \subset B(z_0', r_0)$ we have

$$Tf(x) \ge \frac{c_2}{c_5} \frac{1}{\mu B(z_0', r_0)} \int_{B(z_0', r_0)} f(y)\, d\mu. \tag{2.7}$$

Let $z_1 \in B(z_0', \frac{r}{2a_1}) \cap E$ be a density point of the set E.

One can readily verify that

$$B\Big(z_1, \frac{r_0}{2a_1}\Big) \subset B(z_0', r_0) \subset B\Big(z_1, (\frac{a_0}{2} + a_1) r_0\Big).$$

Let r_1 be a positive number so small that $r_1 < \frac{r_0}{2a_1}$ and for any $r \le r_1$ we have

$$\mu B(z_1, r) \cap E \ge \frac{1}{2}\, \mu B(z_0, r).$$

Let further $0 \le t_1 < t_2 < \infty$ and

$$r_2 = \sup\Big\{ r \ :\ \mu B(z_1, r) \le \frac{t_1}{t_2}\, \mu B(z_1, r_1) \Big\}.$$

Then

$$\begin{aligned} \mu B(z_1, r_2) &\le b\mu B\Big(z_1, \frac{r_2}{2}\Big) \le b\, \frac{t_1}{t_2}\, \mu B(z_1, r_1) \le \\ &\le b\mu B(z_1, 2r_2) \le b^2 \mu B(z_1, r_2). \end{aligned} \tag{2.8}$$

We write

$$f(x) = \frac{2c_5 b\mu B\left(z_1, \left(\frac{a_0}{2} + a_1\right) r_0\right)}{c_2 \mu B(z_1, r_1)} \cdot t_2 \chi_{B(z_1, r_2) \cap E}.$$

For $x \in B(z_0, r_0)$ inequalities (2.7) and (2.8) give

$$\begin{aligned} Tf(x) &\ge \frac{c_2}{c_5 \mu B\left(z_1, \left(\frac{a_0}{2} + a_1\right) r_0\right)} \int_{B(x, r_2)} f(x)\, d\mu = \\ &= \frac{2bt_2 \mu B(z_1, r_2) \cap E}{\mu B(z_1, r_1)} > t_1. \end{aligned} \tag{2.9}$$

Set

$$c_6 = \frac{2c_5 b\mu B\left(z_1, \left(\frac{a_0}{2} + a_1\right) r_0\right)}{c_2 \mu B(z_1, r_1)}.$$

Taking into account that $B(z_1, r_1) \subset B(z_0, r_0)$, $B(z_0', r_0) \cap B(z_0, r_0) = \varnothing$, we obtain $B(z_1, r_1) \cap B(z_0, r_0) = \varnothing$. On the other hand, by the definition of the number r_2 we have $r_2 < r_1$ and therefore $B(z_1, r_2) \cap B(z_0, r_0) = \varnothing$. Now putting $E_1 = B(z_1, r_2) \cap E$ in (2.3), by (2.9) we obtain

$$\varphi(t_1)\mu B(z_0, r_0) \cap E \le c\varphi(cc_6 t_2)\mu B(z_1, r_2) \cap E \le c\varphi(cc_6 t_2) b\, \frac{t_1}{t_2}\, \mu B(z_1, r_1)$$

which implies that the function $\frac{\varphi(t)}{t}$ quasiincreases and thus by Lemma 2.1 from [1] φ is quasiconvex.

The case with $k \in \mathcal{S}_2$ is proved similarly and hence omitted. $\square$

Proof of Theorem 2.1. By Proposition 2.1 either of conditions (2.2) and (2.2′) guarantees the quasiconvexity of φ and the fulfillment of the condition $\varphi \in \Delta_2$. Indeed, choose a number such that $E = \{x : k^{-1} \le w_1(x),\ k^{-1} \le w_2(x)\}$ has a positive measure. Then for any set $E_1 \subset E$, $\mu E_1 > 0$, and function f, $\operatorname{supp} f \subset E_1$, say from (2.2), we find that (2.3) is fulfilled and therefore by Proposition 2.1 φ is quasiconvex and $\varphi \in \Delta_2$.

It remains to show that condition (1.4) holds. Under the condition of the theorem for an arbitrary ball $B = B(z_0, r)$ there exists a ball $B' = B(z_0', r)$ such that $d(z_0, z_0') < c_1 r$ and for $x \in B(z_0, r)$ and $y \in B(z_0', r)$ (2.1) holds. Therefore if $g \geq 0$ and $\operatorname{supp} g \subset B(z_0', r)$, then for $x \in B(z_0, r)$ we have

$$\mathcal{K}g(x) \geq \frac{c}{\mu B'} \int_{B'} g(y)\, d\mu.$$

For such functions (2.2) gives the estimate

$$\int_B \varphi\big(g_{B'} w_1(x)\big) w_2(x)\, d\mu \leq c \int_{B'} \varphi\big(cg(x) w_1(x)\big) w_2(x)\, d\mu. \tag{2.10}$$

On the other hand, for $\widetilde{g} \geq 0$, $\operatorname{supp} \widetilde{g} \subset B(z_0, r)$, and $x \in B(z_0', r)$ we have

$$\widetilde{\mathcal{K}}\widetilde{g}(x) \geq \frac{c}{\mu B} \int_B \widetilde{g}(y)\, d\mu.$$

Therefore (2.2$'$) implies

$$\int_B \varphi\big(\widetilde{g}_B w_1(x)\big) w_2(x)\, d\mu \leq c \int_{B'} \varphi\big(c\widetilde{g}(x) w_1(x)\big) w_2(x)\, d\mu. \tag{2.11}$$

Let now $f \geq 0$ be an arbitrary locally summable function. Substituting $g = \chi_{B'} f_B$ in (2.10), we obtain

$$\int_B \varphi\big(f_B w_1(x)\big) w_2(x)\, d\mu \leq c \int_{B'} \varphi\big(f_B w_1(x)\big) w_2(x)\, d\mu. \tag{2.12}$$

Substituting the function $f\chi_B$ in (2.11) gives

$$\int_{B'} \varphi\big(f_B w_1(x)\big) w_2(x)\, d\mu \leq c \int_B \varphi\big(f(x) w_1(x)\big) w_2(x)\, d\mu. \tag{2.13}$$

From (2.12) and (2.13) we conclude that the inequality

$$\int_B \varphi\big(f_B w_1(x)\big) w_2(x)\, d\mu \leq c \int_B \varphi\big(f(x) w_1(x)\big) w_2(x)\, d\mu$$

holds for any locally summable function f, $\operatorname{supp} f \subset B$.

By virtue of Theorem 3.1 from [1] the last inequality implies that (1.4) is valid. $\square$

In the concrete cases the necessary and sufficient conditions for weak type weighted inequalities for singular integrals acquire a rather transparent form. Namely, by virtue of Theorem 3.2 of Part I, from Theorems 1.1 and 2.1 we immediately conclude that the statements below are valid.

Theorem 2.2. *Let $\varphi \in \Phi$ and the kernel $k \in \mathrm{CZ} \cap \mathcal{S}_1 \cup \mathcal{S}_2$. Then the following statements are equivalent:*

(i) *φ is quasiconvex, $\varphi \in \Delta_2$, and $w \in \mathcal{A}_{p(\varphi)}$;*

(ii) *there exists a positive constant $c_1 > 0$ such that for any $\lambda > 0$ and μ-measurable function we have*

$$\varphi(\lambda) w\big\{x \ : \ |\mathcal{K} f(x)| > \lambda\big\} \leq c_1 \int_X \varphi\big(cf(x)\big) w(x)\, d\mu,$$

$$\varphi(\lambda) w\big\{x \ : \ |\widetilde{\mathcal{K}} f(x)| > \lambda\big\} \leq c_1 \int_X \varphi\big(cf(x)\big) w(x)\, d\mu.$$

Theorem 2.3. *Let $\varphi \in \Phi$, $k \in \mathrm{CZ} \cap \mathcal{S}_1 \cup \mathcal{S}_2$. Then the following statements are equivalent:*

(i) *φ is quasiconvex, $\varphi \in \Delta_2$, $w^{p(\varphi)} \in \mathcal{A}_{p(\varphi)}$, $w^{-p(\widetilde{\varphi})} \in \mathcal{A}_{p(\widetilde{\varphi})}$;*

(ii) *we have the inequalities*

$$\int\limits_{\{x:|\mathcal{K} f(x)|>\lambda\}} \varphi\big(\lambda w(x)\big)\, d\mu \leq c \int_X \varphi\big(cf(x) w(x)\big)\, d\mu,$$

$$\int\limits_{\{x:|\widetilde{\mathcal{K}} f(x)|>\lambda\}} \varphi\big(\lambda w(x)\big)\, d\mu \leq c \int_X \varphi\big(cf(x) w(x)\big)\, d\mu,$$

where the constant c is independent of f and $\lambda > 0$.

Theorem 2.4. *Let $\varphi \in \Phi$ and $k \in \mathrm{CZ} \cap \mathcal{S}_1 \cup \mathcal{S}_2$. Then the following conditions are equivalent:*

(i) *φ is quasiconvex, $\varphi \in \Delta_2$, and $w \in \mathcal{A}_{p(\widetilde{\varphi})}$;*

(ii) *there exists a constant $c > 0$ such that for any $\lambda > 0$ and μ-measurable function $f : X \to \mathbb{R}^1$ we have the inequalities*

$$\int\limits_{\{x:|\mathcal{K} f(x)|>\lambda\}} \varphi\Big(\frac{\lambda}{w(x)}\Big) w(x)\, d\mu \leq c \int_X \varphi\Big(c\,\frac{f(x)}{w(x)}\Big) w(x)\, d\mu,$$

$$\int\limits_{\{x:|\widetilde{\mathcal{K}} f(x)|>\lambda\}} \varphi\Big(\frac{\lambda}{w(x)}\Big) w(x)\, d\mu \leq c \int_X \varphi\Big(c\,\frac{f(x)}{w(x)}\Big) w(x)\, d\mu.$$

§ 3. Criteria for Strong Type Weighted Inequalities for Maximal Functions and Singular Integrals

Using the previous results as well as the general interpolation theorem to be given below, in this section we are able to obtain a solution of the problem, to give a full description of classes of the function φ and weights w ensuring the validity of strong type weighted inequalities for maximal functions and singular integrals defined on homogeneous type spaces.

For maximal functions we have

Theorem 3.1. *Let $\varphi \in \Phi$. Then the following statements are equivalent:*

(i) *φ^α is quasiconvex for some α, $0<\alpha<1$, $w^{p(\varphi)} \in \mathcal{A}_{p(\varphi)}$, and $w^{-p(\widetilde{\varphi})} \in \mathcal{A}_{p(\widetilde{\varphi})}$;*

(ii) *there exists $c_1>0$ such that the inequality*

$$\int_X \varphi\big(\mathcal{M}f(x)w(x)\big)\,d\mu \le c_1 \int_X \varphi\big(c_1 f(x)w(x)\big)\,d\mu \tag{3.1}$$

is fulfilled for any locally μ-summable function $f : X \to \mathbb{R}^1$.

For singular integrals the solution of one-weighted problems is given by the statements as follows.

Theorem 3.2. *Let $\varphi \in \Phi$ and $k \in \mathrm{CZ}$. If $\varphi \in \Delta_2$, φ^α is quasiconvex for some α, $0<\alpha<1$, and $w \in \mathcal{A}_{p(\varphi)}$, then there is $c_2>0$ such that the following inequalities hold:*

$$\int_X \varphi\big(\mathcal{K}f(x)\big)w(x)\,d\mu \le c_2 \int_X \varphi\big(f(x)\big)w(x)\,d\mu, \tag{3.2}$$

$$\int_X \widetilde{\varphi}\Big(\frac{\mathcal{K}f(x)}{w(x)}\Big)w(x)\,d\mu \le c_3 \int_X \widetilde{\varphi}\Big(\frac{f(x)}{w(x)}\Big)w(x)\,d\mu. \tag{3.3}$$

Similar statements hold for the operator $\widetilde{\mathcal{K}}$ as well.

Theorem 3.3. *Let $\varphi \in \Phi$ and $k \in \mathrm{CZ} \cap \mathcal{S}_1 \cup \mathcal{S}_2$. Then the following conditions are equivalent:*

(i) *the inequality* (3.2) *is fulfilled for $\mathcal{K}$ and $\widetilde{\mathcal{K}}$;*

(ii) *the inequality* (3.3) *is fulfilled for $\mathcal{K}$ and $\widetilde{\mathcal{K}}$;*

(iii) *$\varphi \in \Delta_2$, φ^α is quasiconvex for some α, $0<\alpha<1$, and $w \in \mathcal{A}_{p(\varphi)}$.*

Theorem 3.4. *If $\varphi \in \Phi \cap \Delta_2$ and $k \in \mathrm{CZ}$, φ^α is quasiconvex for some α, $0<\alpha<1$, $w^{p(\varphi)} \in \mathcal{A}_{p(\varphi)}$, and $w^{-p(\widetilde{\varphi})} \in \mathcal{A}_{p(\widetilde{\varphi})}$, then there exists a constant $c_4>0$ such that the following inequalities are fulfilled:*

$$\int_X \varphi\big(\mathcal{K}f(x)w(x)\big)\,d\mu \le c_4 \int_X \varphi\big(f(x)w(x)\big)\,d\mu, \tag{3.4}$$

$$\int_X \varphi\big(\widetilde{\mathcal{K}}f(x)w(x)\big)\,d\mu \le c_4 \int_X \varphi\big(f(x)w(x)\big)\,d\mu. \tag{3.5}$$

Theorem 3.5. *Let $\varphi \in \Phi$ and $k \in \mathrm{CZ} \cap \mathcal{S}_1 \cup \mathcal{S}_2$. Then the following conditions are equivalent:*

(i) *$\varphi \in \Delta_2$, φ^α is quasiconvex for some α, $0<\alpha<1$, $w^{p(\varphi)} \in \mathcal{A}_{p(\varphi)}$ and $w^{-p(\widetilde{\varphi})} \in \mathcal{A}_{p(\widetilde{\varphi})}$;*

ii) *inequalities* (3.4) *and* (3.5) *hold.*

To prove the above-formulated theorems we next give the interpolation theorem.

Let (M,S,ν) be a space with measure. Let further the subadditive operator T be the one mapping the space D of functions measurable on (M,S,ν) into the space of functions measurable and defined on another space (M_1,S_1,ν_1) with measure.

Theorem 3.6. *Let $\varphi\in\Phi$ and be quasiconvex. Let further $1\le r<p(\varphi)\le p'(\widetilde{\varphi})<s<\infty$ and for the case $s=\infty$ assume that $p'(\widetilde{\varphi})\le\infty$.*

Let there exist constants c_1 and c_2 such that for any $\lambda>0$ and $f\in L^r\dot{+}L^s$ we have the inequalities

$$\int\limits_{\{x\in M_1:|Tf(x)|>\lambda\}} d\nu_1\le c_1\lambda^{-r}\int_M |f(x)|^r\,d\nu, \tag{3.6}$$

$$\int\limits_{\{x\in M_1:|Tf(x)|>\lambda\}} d\nu_1\le c_2\lambda^{-s}\int_M |f(x)|^s\,d\nu \tag{3.7}$$

and for $s=\infty$ we have

$$\|Tf\|_{L^\infty}\le c_2\|f\|_{L^\infty}.$$

Then there exists a constant $c_3>0$ such that the following inequality holds:

$$\int_{M_1}\varphi\big(Tf(x)\big)\,d\nu_1\le c_3\int_M\varphi\big(f(x)\big)\,d\nu. \tag{3.8}$$

Proof. The theorem is proved by the standard arguments. Let $s<\infty$. By virtue of the definition of $p(\varphi)$ there exists p_1, $r<p_1<p(\varphi)$, such that $t^{-p_1}\varphi(t)$ almost increases. Similarly, there exists $\varepsilon>0$ such that $t^{-(p(\widetilde{\varphi})-ve)}\widetilde{\varphi}(t)$ almost increases, which by virtue of Lemma 2.5 from [1] means that $t^{-(p(\widetilde{\varphi})-\varepsilon)'}\varphi(t)$ almost decreases. Therefore there exists p_2 such that $\widetilde{p}'(\widetilde{\varphi})<p_2<s$ so that the function $t^{-p_2}\varphi(t)$ will almost decrease. Based on these facts, we readily conclude that

$$\int_0^t\frac{d\varphi(\tau)}{\tau^r}\le c\,\frac{\varphi(t)}{t^r}\quad\text{and}\quad\int_t^\infty\frac{d\varphi(\tau)}{\tau^s}\le c\,\frac{\varphi(t)}{t^s}. \tag{3.9}$$

For each $\lambda>0$ we put

$$f_1(x)=\begin{cases}f(x) & \text{if } |f(x)|\ge\frac{\lambda}{2c_2},\\ 0 & \text{if } |f(x)|<\frac{\lambda}{2c_2},\end{cases}\qquad f_2(x)=\begin{cases}f(x) & \text{if } |f(x)|\le\frac{\lambda}{2c_2},\\ 0 & \text{if } |f(x)|>\frac{\lambda}{2c_2}.\end{cases}$$

Further we have

$$\int_{M_1}\varphi\big(Tf(x)\big)\,d\nu_1=\int_0^\infty\nu_1\big\{x\in M_1:|Tf(x)|>\lambda\big\}\,d\varphi(\lambda)\le$$

$$\leq c\Big(\int_0^\infty \nu_1\{x\in M : |Tf_1(x)| > \frac{\lambda}{2c_2}\}\,d\varphi(\lambda)+$$
$$+\int_0^\infty \nu_1\{x\in M_1 : |Tf_2(x)| > \frac{\lambda}{2c_2}\}\,d\varphi(\lambda)\Big) = I_1+I_2.$$

On account of (3.6) and the first inequality of (3.9) we have

$$I_1 \leq c\int_0^\infty \frac{2^r c_1}{\lambda^r}\Big(\int_M |f_1(x)|^r d\nu\Big)d\varphi(\lambda) =$$
$$= c\int_0^\infty \frac{2^r c_1}{\lambda^r}\Big(\int\limits_{\{x:2c_2|f(x)|>\lambda\}} |f(x)|^r d\nu\Big)d\varphi(\lambda) =$$
$$= cc_1\int_M |f(x)|^r\Big(\int_0^{2c_2|f(x)|}\frac{d\varphi(\lambda}{\lambda^r}\Big)\,d\nu \leq c_3\int_M \varphi\big(f(x)\big)\,d\nu.$$

Similarly, (3.7) and the second inequality of (3.9) imply

$$I_2 \leq c_3\int_M \varphi\big(f(x)\big)\,d\nu$$

and as a result we have (3.8).

If $s=\infty$, then $\|f\|_\infty < \frac{\lambda}{2c_2}$ and therefore $I_2 = 0$. $\square$

Proof of Theorem 3.1. Let us show that the implication (i)⇒(ii) holds. From the condition $w^{p(\varphi)} \in \mathcal{A}_{p(\varphi)}$ it follows that $w^{p(\varphi)-\varepsilon} \in \mathcal{A}_{p(\varphi)-\varepsilon}$. If $p(\widetilde{\varphi}) > 1$, then the condition $w^{-p(\widetilde{\varphi})} \in \mathcal{A}_{p(\widetilde{\varphi})}$ implies $w^{-(p(\widetilde{\varphi})-\varepsilon)} \in \mathcal{A}_{p(\widetilde{\varphi})-\varepsilon}$ for $\varepsilon > 0$. Therefore $w^{(p(\widetilde{\varphi})-\varepsilon)'} \in \mathcal{A}_{(p(\widetilde{\varphi})-\varepsilon)'}$.

Consider the operator

$$Tf = w\mathcal{M}\Big(\frac{f}{w}\Big).$$

Then due to Proposition 3.2 from [1] we obtain the inequalities

$$\int_X |Tf(x)|^{p(\varphi)-\varepsilon}d\mu \leq c_1\int_X |f(x)|^{p(\varphi)-\varepsilon}d\mu,$$
$$\int_X |Tf(x)|^{(p(\widetilde{\varphi})-\varepsilon)'}d\mu \leq c_2\int_X |f(x)|^{(p(\widetilde{\varphi})-\varepsilon)'}d\mu.$$

For $p(\widetilde{\varphi}) = 1$ the function w^{-1} belongs to the class $\mathcal{A}_1$ and it is clear that

$$\|Tf\|_\infty \leq c_2\|f\|_\infty.$$

Since $p(\varphi)-\varepsilon < p(\varphi) < p'(\varphi) < (p(\widetilde{\varphi})-\varepsilon)'$, by Theorem 3.6 the above inequalities imply that (3.1) is valid.

As to the implication (ii)⇒(i), note that by virtue of the second part of Theorem 3.2 from [1] we find from the condition of the theorem that φ is quasiconvex, $w^{p(\varphi)} \in \mathcal{A}_{p(\varphi)}$, and $w^{-p(\widetilde{\varphi})} \in \mathcal{A}_{p(\widetilde{\varphi})}$.

To show that condition (ii) implies the quasiconvexity of φ^α for some α, $0<\alpha<1$, we have to prove the following propositions.

Proposition 3.1. *Let $k \in \mathcal{S}_1 \cup \mathcal{S}_2$, $\varphi \in \Phi$, and $\mu E > 0$. If we have the inequality*

$$\int_E \varphi\big(\mathcal{K}f(x)\big)\,d\mu \le c \int_E \varphi\big(cf(x)\big)\,d\mu, \quad \operatorname{supp} f \subset E, \tag{3.10}$$

with constant c independent of f, then φ^α is qusiconvex for some α, $0 < \alpha < 1$, i.e., $\widetilde{\varphi} \in \Delta_2$.

Proof. (3.10) implies (2.3) and hence φ is quasiconvex and satisfies the condition Δ_2. Therefore by (2.10) from [1] there exists $c_1 > 0$ such that

$$\varphi\Big(c\,\frac{\widetilde{\varphi}(t)}{t}\Big) \le c_1 \widetilde{\varphi}(t), \quad t > 0. \tag{3.11}$$

Assume now that E_1 is any μ-measurable subset of E and $\operatorname{supp} f \subset E_1$. Applying the Young inequality, (3.11) and (3.10), we obtain the chain of inequalities

$$\begin{aligned}
&\int_{E\setminus E_1} \widetilde{\varphi}\big(\widetilde{\mathcal{K}}f(x)\big)d\mu = \int_{E\setminus E_1} \frac{\widetilde{\varphi}(\widetilde{\mathcal{K}}f(x))}{\widetilde{\mathcal{K}}f(x)}\,\widetilde{\mathcal{K}}f(x)\,d\mu = \\
&= \int_{E\setminus E_1} \mathcal{K}\Big(\frac{\widetilde{\varphi}(\widetilde{\mathcal{K}}f(x))}{\widetilde{\mathcal{K}}f(x)}\Big)(x) f(x)\,d\mu \le \\
&\le \frac{1}{2c_1 c} \int_{E\setminus E_1} \varphi\left(\mathcal{K}\Big(\frac{\widetilde{\varphi}(\widetilde{\mathcal{K}}f(x))}{\widetilde{\mathcal{K}}f(x)}\chi_{E\setminus E_1}\Big)(x)\right) d\mu + \frac{1}{2cc_1}\int_{E_1} \widetilde{\varphi}\big(2cc_1 f(x)\big) \le \\
&\le \frac{1}{2c_1} \int_{E\setminus E_1} \varphi\left(c\frac{\widetilde{\varphi}(\widetilde{\mathcal{K}}f(x))}{\widetilde{\mathcal{K}}f(x)}\right) d\mu + \frac{1}{2cc_1}\int_{E_1} \widetilde{\varphi}\big(2cc_1 f(x)\big)\,d\mu \le \\
&\le \frac{1}{2}\int_{E\setminus E_1} \widetilde{\varphi}\big(\widetilde{\mathcal{K}}f(x)\big)\,d\mu + \frac{1}{2cc_1}\int_{E_1} \widetilde{\varphi}\big(2cc_1 f(x)\big)\,d\mu,
\end{aligned}$$

which allow us to conclude that

$$\int_{E\setminus E_1} \widetilde{\varphi}\big(\widetilde{\mathcal{K}}f(x)\big)\,d\mu \le \frac{1}{cc_1}\int_{E_1} \widetilde{\varphi}\big(2cc_1 f(x)\big)\,d\mu.$$

Since E_1 is an arbitrary measurable subset of E, from the last inequality we find by Proposition 2.1 that $\widetilde{\varphi} \in \Delta_2$, i.e., φ^α is quasiconvex for some α, $0 < \alpha < 1$. $\square$

Proposition 3.2. *Let k satisfy condition* (2.1) *or* (2.1′), $\varphi \in \Phi$. *If for some weight functions w_i $(i = 1,2,3,4)$ we have the inequality*

$$\int_X \varphi\big(\mathcal{K}f(x)w_1(x)\big)w_2(x)\,d\mu \le c\int_X \varphi\big(cf(x)w_3(x)\big)w_4(x)\,d\mu$$

with the constant c independent of f, then $\widetilde{\varphi} \in \Delta_2$.

Proof. It is sufficient to choose the number m such that the set $E = \{x \in X : m^{-1} \le w_1(x),\ m^{-1} \le w_2(x),\ w_3(x) \le m,\ w_4(x) \le m\}$ has a positive measure and to apply Proposition 3.1 for the set E. $\square$

Proof of Theorem 3.2. From the condition $w \in \mathcal{A}_{p(\varphi)}$ it follows that $w \in \mathcal{A}_{p(\varphi)-\varepsilon}$ for some ε, and since $p(\varphi) \le p'(\widetilde{\varphi})$ we have $w \in \mathcal{A}_{p'(\widetilde{\varphi})+\varepsilon}$. Next, applying interpolation Theorem 3.6 for the operator $T = \mathcal{K}$, we obtain (3.2).

Now we shall prove (3.3). Note that the condition $w \in \mathcal{A}_{p(\varphi)-\varepsilon}$ implies $w^{1-(p(\varphi)-\varepsilon)'} \in \mathcal{A}_{(p(\varphi)-\varepsilon)'}$. Since $p(\varphi) < p'(\widetilde{\varphi}) \le (p(\widetilde{\varphi})-\varepsilon)'$, we have $w \in \mathcal{A}_{(p(\widetilde{\varphi})-\varepsilon)'}$. Hence it follows that $w^{1-(p(\widetilde{\varphi})-\varepsilon)} \in \mathcal{A}_{p(\widetilde{\varphi})-\varepsilon}$. Consider the operator

$$Tf = \frac{1}{w}\,\widetilde{\mathcal{K}}(fw).$$

By Theorem B we have the inequalities

$$\int_X |Tf(x)|^{p(\widetilde{\varphi})-\varepsilon}w(x)\,d\mu \le c_1\int_X |f(x)|^{p(\widetilde{\varphi})-\varepsilon}w(x)\,d\mu,$$
$$\int_X |Tf(x)|^{(p(\varphi)-\varepsilon)'}w(x)\,d\mu \le c_2\int_X |f(x)|^{(p(\varphi)-\varepsilon)'}w(x)\,d\mu.$$

On the other hand, since $p(\widetilde{\varphi})-\varepsilon < p(\widetilde{\varphi}) < (p(\varphi)-\varepsilon)'$ we conclude by Theorem 3.2 that inequality (3.3) holds. $\square$

Proof of Theorem 3.3. The validity of this theorem follows from the preceding theorem, Theorems 2.2, 2.3 and Proposition 3.2. $\square$

Proof of Theorem 3.4. This theorem is proved similarly to the proof of Theorem 3.1 using Theorem B. $\square$

Proof of Theorem 3.5. The validity of this theorem follows from the preceding theorem, Theorem 2.3, and Proposition 3.2. $\square$

Remark 1. In establishing the criteria for weak and strong type inequalities, we virtually used the following property of the operator: for any ball B there exists a ball B' of the same volume such that the distance between these balls does not exceed $c_1 \operatorname{rad} B$. For any $f \ge 0$, $\operatorname{supp} f \subseteq B'$, the inequality

$$\mathcal{K}f(x) \ge \frac{c}{\nu B'}\int_{B'} f(y)\,d\mu$$

holds for any $x \in B'$.

§ 4. Applications to Classical Singular Integrals

First we shall consider the Hilbert transform

$$Hf(x) = \int_{-\infty}^{\infty} \frac{f(y)}{x-y}\,dy.$$

Given $X = \mathbb{R}^1$, μ is the Lebesgue measure and d is the Euclidean metric, we conclude from Theorem 3.3 that the statements below are valid.

Theorem 4.1. *Let $\varphi \in \Phi$. In order that there exist a constant $c_1 > 0$ such that either of the inequalities*

$$\int_{-\infty}^{\infty} \varphi\big(Hf(x)\big)w(x)\,dx \le c_1 \int_{-\infty}^{\infty} \varphi\big(f(x)\big)w(x)\,dx, \tag{4.1}$$

$$\int_{-\infty}^{\infty} \widetilde{\varphi}\Big(\frac{Hf(x)}{w(x)}\Big)w(x)\,dx \le c_1 \int_{-\infty}^{\infty} \widetilde{\varphi}\Big(\frac{f(x)}{w(x)}\Big)w(x)\,dx \tag{4.2}$$

is fulfilled, it is necessary and sufficient that $\varphi \in \Delta_2$, φ^α be quasiconvex for some α, $0 < \alpha < 1$, and $w \in \mathcal{A}_{p(\varphi)}$.

The validity of our next statement follows from Theorem 3.1.

Theorem 4.2. *Let $\varphi \in \Phi$. In order that there exist a constant $c_2 > 0$ such that the inequality*

$$\int_{-\infty}^{\infty} \varphi\big(Hf(x)w(x)\big)\,dx \le c_2 \int_{-\infty}^{\infty} \varphi\big(f(x)w(x)\big)\,dx \tag{4.3}$$

is fulfilled, it is necessary and sufficient that $\varphi \in \Delta_2$, φ^α be quasiconvex for some α, $0 < \alpha < 1$, $w^{p(\varphi)} \in \mathcal{A}_{p(\varphi)}$, and $w^{-p(\widetilde{\varphi})} \in \mathcal{A}_{p(\widetilde{\varphi})}$.

Let now $X = [0,\infty)$, $d(x,y) = |x^2 - y^2|$, the measure $d\mu = x\,dx$. It can be easily seen that intervals will be balls in X and $\mu B(x,r) \sim r$.

The kernel

$$k(x,y) = \frac{1}{x^2 - y^2}$$

satisfies conditions (1.1), (1.2), and (2.1).

For odd and even functions the Hilbert transforms are respectively written as

$$H_o f(x) = \frac{2}{\pi}\int_0^{\infty} \frac{yf(y)}{x^2-y^2}\,dy,$$

$$H_e f(x) = \frac{2}{\pi}\int_0^{\infty} \frac{xf(y)}{x^2-y^2}\,dy.$$

Obviously, $H_o f = \mathcal{K} f$ and $H_e f(x) = x\mathcal{K}\big(\frac{f}{\cdot}\big)(x)$. Also note that the function $w(x) = |x| \in \mathcal{A}_2$ and therefore the operator $\mathcal{K}$ is bounded in $L^2(X, x\,dx)$. Now we are able to apply Theorems 3.3 and 3.5 and to conclude that the statements below are valid.

Theorem 4.3. *Let $\varphi \in \Phi$. Then the following conditions are equivalent:*
(i) *$\varphi \in \Delta_2$, φ^α is qusiconvex for some α, $0 < \alpha < 1$, and*

$$\sup\Big(\frac{1}{b^2-a^2}\int_a^b w(x)\,dx\Big)\Big(\frac{1}{b^2-a^2}\int_a^b w^{1-p'(\varphi)}x^{p'(\varphi)}dx\Big)^{p(\varphi)-1}, \quad (4.4)$$

where the exact upper bound is taken with respect to all intervals $[a,b] \subset [0,\infty)$;
(ii) *there exists $c > 0$ such that*

$$\int_0^\infty \varphi\big(H_o f(x)\big)w(x)\,dx \le c\int_0^\infty \varphi\big(f(x)\big)w(x)\,dx; \quad (4.5)$$

(iii) *there exists $c > 0$ such that*

$$\int_0^\infty \widetilde{\varphi}\Big(\frac{xH_o f(x)}{w(x)}\Big)w(x)\,dx \le c\int_0^\infty \widetilde{\varphi}\Big(\frac{xf(x)}{w(x)}\Big)w(x)\,dx. \quad (4.6)$$

On the other hand, since

$$H_e f(x) = xH_o\big(\frac{f}{y}\big)(x),$$

by Theorem 4.2 we conclude that the following theorem is valid.

Theorem 4.4. *Let $\varphi \in \Phi$. Then the following two conditions are equivalent:*
(i) *the inequality*

$$\int_0^\infty \varphi\Big(\frac{H_e f(x)}{w(x)}\Big)w(x)\,dx \le c\int_0^\infty \varphi\Big(\frac{f(x)}{w(x)}\Big)w(x)\,dx \quad (4.7)$$

holds;
(ii) *$\varphi \in \Delta_2$, φ^α is quasiconvex for some α, $0 < \alpha < 1$, and we have*

$$\sup\Big(\frac{1}{b^2-a^2}\int_a^b w(x)dx\Big)\Big(\frac{1}{b^2-a^2}\int_a^b w^{1-p'(\widetilde{\varphi})}x^{p'(\widetilde{\varphi})}dx\Big)^{p(\widetilde{\varphi})-1} < \infty, \quad (4.8)$$

where the exact upper bound is taken with respect to all intervals $[a,b] \subset [0,\infty)$.

For Calderon–Zygmund singular integrals in $\mathbb{R}^n$ we conclude by Theorem 3.1 that the following statement is true.

Theorem 4.5. *Let*

$$\mathcal{K}f(x) = \int_{\mathbb{R}^n} k(x-y)f(y)\,dy,$$

where the kernel k satisfies the conditions:

$$\|\widehat{k}\|_\infty < c, \quad |k(x)| \le c|x|^{-n}$$

$$|k(x) - k(x-y)| \le \frac{c\omega\left(\frac{|y|}{|x|}\right)}{|x|^n} \quad \text{for } |y| < \frac{|x|}{2},$$

where ω is an increasing function satisfying the condition

$$\int_0^1 \frac{\omega(s)}{s}\,ds < \infty.$$

If $\varphi \in \Delta_2$, φ^α is quasiconvex for some α, $0 < \alpha < 1$, $w \in \mathcal{A}_{p(\varphi)}$, then we have the inequalities

$$\int_{\mathbb{R}^n} \varphi\big(\mathcal{K}f(x)\big)w(x)\,dx \le c\int_{\mathbb{R}^n} \varphi\big(f(x)\big)w(x)\,dx, \tag{4.9}$$

$$\int_{\mathbb{R}^n} \widetilde{\varphi}\Big(\frac{\mathcal{K}f(x)}{w(x)}\Big)w(x)\,dx \le c\int_{\mathbb{R}^n} \widetilde{\varphi}\Big(\frac{f(x)}{w(x)}\Big)w(x)\,dx \tag{4.10}$$

with the constant c independent of f.

Further, if $\varphi \in \Delta_2$, φ^α is quasiconvex for some α, $0 < \alpha < 1$, $w^{p(\varphi)} \in \mathcal{A}_{p(\varphi)}$, $w^{-p(\widetilde{\varphi})} \in \mathcal{A}_{p(\widetilde{\varphi})}$, then we have the inequality

$$\int_{-\infty}^{\infty} \varphi\big(\mathcal{K}f(x)w(x)\big)\,dx \le c\int_{-\infty}^{\infty} \varphi\big(f(x)w(x)\big)\,dx. \tag{4.11}$$

Theorem 4.6. *If*

$$k(x) = \frac{\Omega(x)}{|x|^n},$$

where $\Omega : \mathbb{R}^n \to \mathbb{R}^1$ is a homogeneous function of degree zero not vanishing identically on the sphere S^{n-1} and satisfying the Lipschitz condition of order α, $0 < \alpha \le 1$, on the sphere S^{n-1},

$$\int_{S^{n-1}} \Omega(x')\,dx' = 0.$$

Then the simultaneous fulfillment of inequality (4.9) for $\mathcal{K}$ and $\widetilde{\mathcal{K}}$ is equivalent to that of the combination of conditions: $\varphi \in \Delta_2$, φ^α is quasiconvex for some α, $0 < \alpha < 1$, and $w \in \mathcal{A}_{p(\varphi)}$.

An analogous statement holds for (4.10).

Theorem 4.7. *Under the conditions of Theorem* 4.6 *for the kernel* k *the simultaneous fulfillment of* (4.11) *for* $\mathcal{K}$ *and* $\widetilde{\mathcal{K}}$ *is equivalent to that of the combination of conditions:* $\varphi \in \Delta_2$, φ^α *is quasiconvex for some* α, $0 < \alpha < 1$, $w^{p(\varphi)} \in \mathcal{A}_{p(\varphi)}$ *and* $w^{-p(\widetilde{\varphi})} \in \mathcal{A}_{p(\widetilde{\varphi})}$.

In particular, the above-formulated theorems are valid for Riesz transforms

$$R_j f(x) = \int_{\mathbb{R}^n} \frac{x_j - y_j}{|x-y|^{n+1}} f(y)\, dy,$$

$j = 1, \dots, n$, where $x = (x_1, \dots, x_n)$.

Let now $\Gamma \subset \mathbb{C}$ be a connected rectifiable curve and ν be an arc-length measure on Γ. By definition, Γ is regular if

$$\nu\big(\Gamma \cap B(z,r)\big) \le cr$$

for every $z \in \mathbb{C}$ and $r > 0$.

For r less than half the diameter of Γ we have the reverse inequality

$$\nu\big(\Gamma \cap B(z,r)\big) \ge r$$

for all $z \in \Gamma$. When equipped with ν and the Euclidean metric, a regular curve becomes a homogeneous type space. Then $k(z_1, w) = (z-w)^{-1}$ is the standard kernel and the Cauchy integral

$$S_\Gamma f(t) = \int_\Gamma \frac{f(t)}{t-\tau}\, d\nu$$

is defined as the Calderon–Zygmund singular operator on a homogeneous type space.

G. David [7] has shown that the operator S_Γ is bounded in $L^p(\Gamma, \nu)$ iff Γ is regular.

Theorem 4.8. *Let* Γ *be a regular curve. The following conditions are equivalent:*

(i) $\varphi \in \Delta_2$, φ^α *is quasiconvex for some* α, $0 < \alpha < 1$, *and*

$$\sup \Big(\frac{1}{r} \int\limits_{B(z,r)\cap\Gamma} w(x)\, dx\Big)\Big(\frac{1}{r} \int\limits_{B(z,r)\cap\Gamma} w^{1-p'(\varphi)} d\nu\Big)^{p(\varphi)-1} < \infty, \tag{4.12}$$

where the supremum is taken over all balls;

(ii) *there exists a positive constant* $c > 0$ *such that*

$$\int_\Gamma \varphi\big(S_\Gamma f(t)\big) w(t)\, d\nu \le c \int_\Gamma \varphi\big(f(t)\big) w(t)\, d\nu \tag{4.13}$$

with the constant independent of f.

An analogue of Theorem 4.2 also holds for the operator S_Γ.

Proof of Theorem 4.8. The implication (4.12)⇒(4.13) follows from David's result, Theorem 3 from [8], and Theorem 3.2 above.

On the other hand, for an arbitrary point $z \in \Gamma$ fix a ball $B(z_0, r)$ of sufficiently small radius r. Let a point z_0' be chosen such that $|z_0' - z_0| = 3r$. For any $z \in B(z_0, r)$ and $z' \in B(z_0', r)$ we have $|z - z'| > r$ and, as one can easily see, $\mathrm{Re}\,(z - z')$ or $\mathrm{Im}\,(z - z')$ preserve their sign and are greater than r. Therefore condition (2.1) is fulfilled for the kernel $\mathrm{Re}\,\frac{1}{z-z'}$ or $\mathrm{Im}\,\frac{1}{z-z'}$. Since the inequality

$$|S_\Gamma f(t)| \geq |\mathrm{Re}\,(S_\Gamma f)(t)| + |\mathrm{Im}\,(S_\Gamma f)(t)| \geq \frac{c}{r} \int\limits_{B(z',r)\cap\Gamma} f(\tau)\, d\nu$$

is fulfilled for $f \geq 0$, $\mathrm{supp}\, f \subset B(z_0', r) \cap \Gamma$, and $t \in B(z_0, r) \cap \Gamma$, the validity of the implication (ii)⇒(i) follows from Theorem 3.4 and Remark 3.1. □

§ 5. Appendix

The above-described methods of investigation make it possible to extend the results of §§2, 3, and 4 to vector-valued functions as well.

Let X be a homogeneous type space. Consider the kernels $k(x, y)$ defined on $X \times X$ with values in $\mathcal{L}(A, B)$ which is a space of all bounded operators from the Banach space A to the Banach space B. Introduce the operator norm $|\cdot| = |\cdot|_{\mathcal{L}(A,B)}$. It is assumed that $|k(x, \cdot)|$ is locally integrable apart from x and the standard conditions (1.1) and (1.2) with the norm $|\cdot|$ for Calderon–Zygmund kernels are satisfied. Moreover, the operator

$$\mathcal{K} f(x) = \int_X k(x, y) f(y)\, d\mu$$

will be assumed to be bounded from $L_A^{p_0}(X, \mu)$ into $L_B^{p_0}(X, \mu)$ for some p_0, $1 < p_0 < \infty$.

To illustrate the above we give

Theorem 5.1. *Let* $\varphi \in \Phi \cap \Delta_2$, φ^α *be quasiconvex for some* α, $0 < \alpha < 1$. *Then the inequality*

$$\int_X \varphi\big(|\mathcal{K} f(x)|_B\big) w(x)\, d\mu \leq c \int_X \varphi\big(|f(x)|_A\big) w(x)\, d\mu$$

is fulfilled.

An analogue of Theorem 3.4 also holds.

Let us further consider the case where sufficient conditions convert to criteria. Assume that $A = B$ is a Banach space with an unconditional base $(b_j)_{j\in\mathbb{N}}$. Then a B-valued measurable function $f(x)$ is the same as a sequence of measurable functions $(f_j(x))_{j\in\mathbb{N}}$ such that $\sum_j f_j(x) b_j \in B$ for every $x \in X$. If B is reflexive, then the dual base $(b_j^*)_{j\in\mathbb{N}}$ is an unconditional base of B^*.

Define

$$Tf(x) = \sum_j kf_j(x)b_j$$

If the kernel $k : X \times X \to \mathbb{R}^1$, then by condition (2.1) (or (2.1′)) we have criteria of weighted inequalities similar to Theorems 3.3 and 3.1. It is obvious that conversions of theorems of type 5.1 are obtained from the corresponding theorems for scalar-valued functions.

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