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GEOMETRY OF POISSON STRUCTURES

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ABSTRACT. The purpose of this paper is to consider certain mechanisms of the emergence of Poisson structures on a manifold. We shall also establish some properties of the bivector field that defines a Poisson structure and investigate geometrical structures on the manifold induced by such fields. Further, we shall touch upon the dualism between bivector fields and differential 2-forms.

1. Schoten Bracket: Definition and Some Properties

1.1. Let L be any Lie algebra over the field of real numbers and F be any commutative real algebra with unity. It is assumed that L acts on F and this action has the following properties:

(a) F is an L-modulus: for each $(u, v, a, b) \in L \times L \times F \times F$ we have $[u, v]a = uva - vua$;

(b) Leibnitz' rule: $u(a \cdot b) = (ua) \cdot b + a \cdot (ub)$.

1.2. Let us consider the spaces:

$C^k(L, F) = \{\alpha : L \times \cdots \times L \longrightarrow F \mid \alpha$ is an antisymmetric and polylinear form$\}$, $k \geq 0$;

$C^0(L, F) = F$;

$C^k(L, F) = \{0\}$ for $k < 0$.

The space $C(L, F) = \sum_{k \in \mathbb{Z}} C^k(L, F)$ is an antisymmetric graded algebra with the operation of exterior multiplication (see [1]).

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1.3. We have two endomorphisms on the space $C(L,F)$:

$$(\partial_1\alpha)(u_1,\dots,u_{k+1})=\sum_{i<j}(-1)^{i+j-1}\alpha([u_i,u_j],u_1,\dots,\widehat{u}_i,\dots,\widehat{u}_j,\dots,u_{k+1}),$$

$$(\partial_2\alpha)(u_1,\dots,u_{k+1})=\sum_{i=1}^{k+1}(-1)^{i-1}u_i\alpha(u_1,\dots,\widehat{u}_i,\dots,u_{k+1}),$$

where α is an element of $C^k(L,F)$.

The endomorphism $d=\partial_2-\partial_1$ is the coboundary operator defining the cohomology algebra of L (see [1]).

1.4. It is easy to check that the operators ∂_1 and ∂_2 are antidifferentiations, i.e., for each $\alpha\in C^m(L,F)$ and $\beta\in C(L,F)$ we have

$$\partial_1(\alpha\wedge\beta)-(\partial_1\alpha\wedge\beta+(-1)^m\alpha\wedge\partial_1\beta)=0,$$
$$\partial_2(\alpha\wedge\beta)-(\partial_2\alpha\wedge\beta+(-1)^m\alpha\wedge\partial_2\beta)=0.$$

Therefore the operator d is an antidifferentiation, too.

1.5. For each $k\in\mathbb{Z}$ the space $C_k(L,F)=\mathrm{End}(F)\otimes(\wedge^kL)$, where $\mathrm{End}(F)$ is the algebra of endomorphisms of F and $\wedge^kL$ is the exterior degree of L, is a subspace of $\mathrm{Hom}(C^k(L,F),F)$: for $\varphi\otimes u\in\mathrm{End}(F)\otimes(\wedge^kL)$ and $\omega\in C^k(L,F)$, we have $(\varphi\otimes u)(\omega)=\varphi(\omega(u))$.

The multiplication in $C^*(L,F)=\sum_{k\in\mathbb{Z}}C_k(L,F)$ is defined by the equation $(\varphi\otimes u)\cdot(\psi\otimes v)=(\varphi\circ\psi)\otimes(u\wedge v)$.

1.6. Define the operators:

$$\partial^1=(\partial_1)^*,\ \partial^2=(\partial_2)^*:\mathrm{Hom}(C^k(L,F),F)\longrightarrow\mathrm{Hom}(C^{k-1}(L,F),F)$$
$$(\partial^i(\varphi))(\alpha)=\varphi(\partial_i(\alpha)),\ i=1,2,\ \varphi\in\mathrm{Hom}(C^k(L,F),F),$$
$$\alpha\in C^{k-1}(L,F),\ \ n\in\mathbb{Z}.$$

The subspace $C^*(L,F)\subset\sum_{k\in\mathbb{Z}}\mathrm{Hom}(C^k(L,F),F)$ is invariant with respect to the operators ∂^1 and ∂^2:

$$\partial^1(\varphi\otimes(u_1\wedge\dots\wedge u_m))=\varphi\otimes\sum_{i<j}(-1)^{i+j-1}[u_i,u_j]\wedge$$
$$\wedge u_1\wedge\dots\wedge\widehat{u}_i\wedge\ldots\wedge\widehat{u}_j\wedge\ldots\wedge u_m,$$
$$\partial^2(\varphi\otimes(u_1\wedge\dots\wedge u_m))=\sum_{i=1}^{m}(-1)^{i-1}(\varphi\circ u_i)\otimes$$
$$\otimes u_1\wedge\dots\wedge\widehat{u}_i\wedge\ldots\wedge u_m.$$

The operator $\partial^2-\partial^1$ will be denoted by d^*.

1.7. Let us consider the exterior algebra of $L : \wedge(L) = \sum_{k=0}^{\infty} \wedge^k L$ which is a subalgebra of $C^*(L, F)$. The space $\wedge(L)$ is an invariant subspace with respect to the action of the operator ∂^1:

$$\partial_1(u_1 \wedge \cdots \wedge u_m) = \sum_{i<j} (-1)^{i+j-1} [u_i, u_j] \wedge u_1 \wedge \cdots \wedge \widehat{u}_i \wedge \ldots \wedge \widehat{u}_j \wedge \ldots \wedge u_m.$$

1.8. Generally speaking, the operator ∂^1 is not an antidifferentiation.

Definition. We define the map (Schoten bracket [2]) $[\ ,\ ] : \wedge(L) \times \wedge(L) \longrightarrow \wedge(L)$ as follows: let $[u, v] = \partial^1(u \wedge v) - (\partial^1(u) \wedge v + (-1)^m u \wedge \partial^1(v))$ for $u \in \wedge^m L$ and $v \in \wedge(L)$.

1.9. The space $\wedge(L)$ is not an invariant subspace of $C^*(L, F)$ with respect to the action of the operator ∂^2:

$$\partial^2(1 \otimes (u_1 \wedge \cdots \wedge u_m)) = \sum_{i=1}^{m} (-1)^{i-1} u_i \otimes (u_1 \wedge \cdots \wedge \widehat{u}_i \wedge \ldots \wedge u_m).$$

However it is easy to show that for each $u \in \wedge^m L$ and $v \in \wedge(L)$ we have

$$\partial^2(u \wedge v) - (\partial^2(u) \cdot v + (-1)^m u \cdot \partial^2(v)) = 0.$$

Therefore we can define the bracket as

$$[u, v] = (d^*(u) \cdot v + (-1)^m u \cdot d^*(v)) - d^*(u \cdot v).$$

1.10. It is easy to check that for each $u \in \wedge^m L$, $v \in \wedge^n L$, $w \in \wedge^k L$, we have:

(a) $[u, v] = (-1)^{mn}[v, u]$;

(b) $[u, v \wedge w] = [u, v] \wedge w + (-1)^{mn+n} v \wedge [u, w]$;

(c) $(-1)^{mk}[[u, v], w] + (-1)^{mn}[[v, w], u] + (-1)^{nk}[[w, u], v] = 0$.

Let L be an F-modulus and assume that for each $(u, v, a, b) \in L \times L \times F \times F$ we have:

(a) $(au)b = a(ub)$;

(b) $[u, av] = (ua)v + a[u, v]$.

For each $k = 1, 2, \ldots, \infty$ let $V^k(L, F)$ denote an exterior degree of L as an F-modulus: for $a \in F$ and $\{u_1, \ldots, u_k\} \subset L$ we have $au_1 \wedge u_2 \wedge \ldots \wedge u_k = u_1 \wedge au_2 \wedge u_3 \wedge \ldots \wedge u_k$. Assume that $V^0(L, F) = F$ and $V^k(L, F) = \{0\}$ when $k < 0$.

The space $V(L, F) = \sum_{k \in \mathbb{Z}} V^k(L, F)$ is an aniticommutative graded algebra.

1.12. Let $J : \wedge(L) \longrightarrow V(L, F)/$ be the natural homomorphism which is an epimorphism onto $\sum_{k \in \mathbb{Z} \setminus \{0\}} V^k(L, F)$.

Proposition. *If elements* $\{u, u', v, v'\} \subset \wedge(L)$ *are such that* $J(u) = J(u')$ *and* $J(v) = J(v')$, *then* $J([u, v]) = J([u', v'])$.

It is easy to prove this using the formulas (b) (1.10) and (b) (1.11).

1.13. **Definition.** We define the Schoten bracket on $V(L, F)$ as follows: for $\{x, y\} \subset \sum_{k\in\mathbb{Z}\setminus\{0\}} V^k(L, F)$ the bracket $[x, y]$ is defined as $J([u, v])$ where $J(u) = x$ and $J(v) = y$. We extend the definition to the space $V(L, F)$ using equalities (b) (1.10) and (b) (1.11), namely: if $u \in V'(L, F)$ and $a \in V^0(L, F) = F$, then $[u, a] = u(a)$; for $u = u_1 \wedge \ldots \wedge u_k \in V^k(L, F)$ and $a \in F$ we use formula (b) (1.10). Finally, we recall that elements $au_1 \wedge u_2 \wedge \ldots \wedge u_k$ form the basis of $V(L, F)$.

1.14. In the special case where $F = C^\infty(M)$ is the algebra of smooth functions on a smooth manifold M, $L = V'(M)$ is the Lie algebra of smooth vector fields on the manifold M and $V^k(M)$ is the space of antisymmetric contravariant tensors of degree k ($V^k(M)$ is locally isomorphic to $\wedge^k V'(M)$). The bracket defined above coincides with the well-known Schoten bracket (see [2]).

In that case if $u \in V^m(M)$, $v \in V^n(M)$, and $\omega \in \mathrm{Hom}(V^{m+n-1}(M), C^\infty(M))$ is a differential form, then the formula defining the bracket by means of d^* (see 1.9) gives

$$\omega([u, v]) = (-1)^{mn+n}(d(i_v\omega))(u) + (-1)^m(d(i_u\omega))(v) - (d\omega)(u \wedge v),$$

where d is the well-known exterior differentiation of differential form (see [3]).

The above formula can be used as yet another definition of the Schoten bracket.

2. Poisson Bracket and a Bivector Field

2.1. Thus we have:

M is a finite-dimensional smooth manifold;

$V^0(M) = C^\infty(M)$ is the algebra of real-valued smooth functions on M;

$V^k(M)$, $k > 0$, is the space of antisymmetric contravariant tensor fields of degree k;

$V^k(M) = \{0\}$ when $k < 0$;

$V(M) = \sum_{k\in\mathbb{Z}} V^k(M)$ is the exterior algebra of polyvector fields;

$A^0(M) = C^\infty(M)$;

$A^k(M) = \{0\}$ when $k < 0$;

$A^k(M)$, $k > 0$, is the space of exterior differential forms of degree k.

At the same time it is clear that $A^k(M) = \mathrm{Hom}(V^k(M), C^\infty(M))$ and $V^k(M) = \mathrm{Hom}(A^k(M), C^\infty(M))$ for $k \in \mathbb{Z}$ (in the sense of homomorphisms of the $C^\infty(M)$-moduli).

2.2. An element of the space $V^2(M)$ will be called a bivector field on the manifold M.

Given any bivector field ξ, for $f, g \in C^\infty(M)$ the bracket $\{f, g\} \in C^\infty(M)$ is defined to be $(df \wedge dg)(\xi)$.

It is easy to show that the bracket defined by ξ satisfies the following conditions:

(a) antisymmetricity: $\{f, g\} = -\{g, f\}$;

(b) bilinearity: $\{f, c_1 g_1 + c_2 g_2\} = c_1\{f, g_1\} + c_2\{f, g_2\}$ for each $c_1, c_2 \in \mathbb{R}$;

(c) Leibnitz' rule: $\{f, g \cdot h\} = \{f, g\} \cdot h + \{f, h\} \cdot g$;

(d) for $f, g, h \in C^\infty(M)$ we have

$$\{\{f, g\}, h\} + \{\{h, f\}, g\} + \{\{g, h\}, f\} = \frac{1}{2}(df \wedge dg \wedge dh)([\xi, \xi])$$

where $[\ ,\ ]$ is the Schoten bracket (see 1.14).

2.3. **Proposition.** *Let $\{\ ,\ \}$ be any bracket on $C^\infty(M)$, having properties (a), (b), (c) from 2.2. There is one and only one bivector field ξ on M, defining the bracket $\{\ ,\ \}$ as describe in 2.2.*

The bracket $\{\ ,\ \}$ defines the structure of a Lie algebra on a subspace $A \subset C^\infty(M)$ when and only when for each $f, g, h \in A$ we have $(df \wedge dg)(\xi) \in A$ and $(df \wedge dg \wedge dh)([\xi, \xi]) = 0$.

2.4. We can consider ξ as a homomorphism of exterior algebras: for $f \in A^0(M)$, $\alpha, \beta \in A'(M)$ we have $\widetilde{\xi}(f) = f$, $\beta(\widetilde{\xi}(\alpha)) = (\alpha \wedge \beta)(\xi)$.

As follows from 2.3, the bracket $\{\ ,\ \}$ defines in exact terms the structure of a Lie algebra on $C^\infty(M)$ when $[\xi, \xi] = 0$.

Proposition. *If $[\xi, \xi] = 0$, then the map $\widetilde{\xi} \circ d : C^\infty(M) \longrightarrow V'(M)$ is a homomorphism of Lie algebras; $C^\infty(M)$ is a central extension of $I_m(\widetilde{\xi} \circ d)$ and $\mathbb{R} \subset \mathrm{Ker}(\widetilde{\xi} \circ d)$.*

Proof. In that case the pair $(C^\infty(M), \{\ ,\ \})$ is called the Poisson structure on M and the map $f \longmapsto \widetilde{\xi}(df) = \{f,\ \}$ is the so-called Hamiltonian map which is a homomorphism of Lie algebras (see [4]). $\square$

2.5. Let ω be any differential 2-form on the manifold M, giving rise to the homomorphism of $C^\infty(M)$-moduli: $\widetilde{\omega} : V'(M) \longrightarrow A'(M)$, $\widetilde{\omega}(X) = \omega(X,\ )$, which is an isomorphism when ω is nondegenerate. In that case the induced map denoted similarly by $\widetilde{\omega} : V^k(M) \longrightarrow A^k(M)$, $\widetilde{\omega}(u_1 \wedge \ldots \wedge u_k) = \widetilde{\omega}(u_1) \wedge \ldots \wedge \widetilde{\omega}(u_k)$, $k = 1, \ldots, \infty$, is also an isomorphism. Let $\xi_\omega \in V^2(M)$ be $\widetilde{\omega}^{-1}(\omega)$.

More clearly, let $\omega=\sum_{i=1}^n a_i\wedge b_i$, $a_i,b_i\in A'(M)$, $i=1,\dots,n$; the nondegeneracy of ω means that $\{a_i,b_i\mid i=1,\dots,n\}$ is a basis of $A'(M)$ as a $C^\infty(M)$-modulus. We introduce the following vector-fields on M: $\frac{\partial}{\partial a_i}$, $\frac{\partial}{\partial b_i}$, $i=1,\dots,n$,

$$a_k\Big(\frac{\partial}{\partial a_i}\Big)=b_k\Big(\frac{\partial}{\partial b_i}\Big)=\left[\begin{array}{ll}1, & \text{when } k=i,\\ 0, & \text{when } k\neq i,\end{array}\right.\quad k=1,\dots,n;$$

$$a_p\Big(\frac{\partial}{\partial b_q}\Big)=b_p\Big(\frac{\partial}{\partial a_q}\Big)=0,\quad p,q=1,\dots,n.$$

With this notation and keeping in mind the definition of $\widetilde{\omega}$ we have $\widetilde{\omega}\big(\frac{\partial}{\partial a_i}\big)=b_i$, $\widetilde{\omega}\big(\frac{\partial}{\partial b_i}\big)=-a_i$, $i=1,\dots,n$. Consequently, $\xi_\omega=\sum_{i=1}^n\frac{\partial}{\partial a_i}\wedge\frac{\partial}{\partial b_i}$.

2.6. **Theorem.** $\widetilde{\omega}([\xi_\omega,\xi_\omega])=-2d\omega$.

Proof. Using property (b) from 1.10 and the bilinearity of the Schoten bracket, we obtain

$$[\xi_\omega,\xi_\omega]=\Big[\sum_{i=1}^n\frac{\partial}{\partial a_i}\wedge\frac{\partial}{\partial b_i},\sum_{i=1}^n\frac{\partial}{\partial a_k}\wedge\frac{\partial}{\partial b_k}\Big]=$$

$$=\sum_{i,k}\Big[\frac{\partial}{\partial a_i}\wedge\frac{\partial}{\partial b_i},\frac{\partial}{\partial a_k}\wedge\frac{\partial}{\partial b_k}\Big]=\sum_{i,k}\Big(-\Big[\frac{\partial}{\partial a_i},\frac{\partial}{\partial a_k}\Big]\wedge$$

$$\wedge\frac{\partial}{\partial b_i}\wedge\frac{\partial}{\partial b_k}+\Big[\frac{\partial}{\partial a_i},\frac{\partial}{\partial b_k}\Big]\wedge\frac{\partial}{\partial b_i}\wedge\frac{\partial}{\partial a_k}+$$

$$+\Big[\frac{\partial}{\partial b_i},\frac{\partial}{\partial a_k}\Big]\wedge\frac{\partial}{\partial a_i}\wedge\frac{\partial}{\partial b_k}-\Big[\frac{\partial}{\partial b_i},\frac{\partial}{\partial b_k}\Big]\wedge\frac{\partial}{\partial a_i}\wedge\frac{\partial}{\partial a_k}.$$

By the definition of $\widetilde{\omega}$ (see 2.5) we have

$$\widetilde{\omega}([\xi_\omega,\xi_\omega])=\sum_{i,m,k}\Big(b_m\Big(\Big[\frac{\partial}{\partial a_i},\frac{\partial}{\partial a_k}\Big]\Big)\cdot a_m\wedge a_i\wedge a_k-$$

$$-a_m\Big(\Big[\frac{\partial}{\partial a_i},\frac{\partial}{\partial a_k}\Big]\Big)\cdot b_m\wedge a_i\wedge a_k+b_m\Big(\Big[\frac{\partial}{\partial a_i},\frac{\partial}{\partial b_k}\Big]\Big)\cdot a_m\wedge a_i\wedge b_k-$$

$$-a_m\Big(\Big[\frac{\partial}{\partial a_i},\frac{\partial}{\partial b_k}\Big]\Big)\cdot b_m\wedge a_i\wedge b_k+b_m\Big(\Big[\frac{\partial}{\partial b_i},\frac{\partial}{\partial a_k}\Big]\Big)\cdot a_m\wedge b_i\wedge a_k-$$

$$-a_m\Big(\Big[\frac{\partial}{\partial b_i},\frac{\partial}{\partial a_k}\Big]\Big)\cdot b_m\wedge b_i\wedge a_k+b_m\Big(\Big[\frac{\partial}{\partial b_i},\frac{\partial}{\partial b_k}\Big]\Big)\cdot a_m\wedge b_i\wedge b_k-$$

$$-a_m\Big(\Big[\frac{\partial}{\partial b_i},\frac{\partial}{\partial b_k}\Big]\Big)\cdot b_m\wedge b_i\wedge b_k\equiv\Omega.$$

It is obvious that $d\omega=\sum_{i=1}^n(da_i\wedge b_i-a_i\wedge db_i)$.

The monomials $u'_{mik} = \frac{\partial}{\partial a_m} \wedge \frac{\partial}{\partial a_i} \wedge \frac{\partial}{\partial a_k}$, $u^2_{mik} = \frac{\partial}{\partial b_m} \wedge \frac{\partial}{\partial a_i} \wedge \frac{\partial}{\partial a_k}$, $u^3_{mik} = \frac{\partial}{\partial b_m} \wedge \frac{\partial}{\partial b_i} \wedge \frac{\partial}{\partial a_k}$, $u^4_{mik} = \frac{\partial}{\partial b_m} \wedge \frac{\partial}{\partial b_i} \wedge \frac{\partial}{\partial b_k}$, $\{m,i,k\} \subset \{1,\dots,n\}$ form the basis of $V^3(M)$ as a $C^\infty(M)$-modulus and it is easy to check that $\Omega(u^j_{mik}) = -2(d\omega)(u^j_{mik})$ for each $j \in \{1,2,3,4\}$ and $\{m,i,k\} \subset \{1,\dots,n\}$.

We have therefore ascertained that $\Omega = -2d\omega$. $\square$

2.7. Let (M,ω) be a symplectic manifold (see [3], [5]). For $f \in C^\infty(M)$ we define the vector field X_f by the formula $df = \omega(\ , X_f)$. It is a well-known fact (see [3], [5]) that ω defines a Poisson structure on M: for $f,g \in C^\infty(M)$ we have $\{f,g\} = \omega(X_f, X_g)$. It is easy to show that the corresponding bivector field is ξ_ω, i.e., $(df \wedge dg)(\xi_\omega) = \omega(X_f, X_g)$.

As follows from 2.6, the equality $d\omega = 0$ is equivalent to $[\xi_\omega, \xi_\omega] = 0$.

2.8. **Lemma.** *If $\omega \in A^2(M)$, $\alpha, \beta \in A'(M)$ and $X, Y \in V^2(M)$, then we have $(\omega \wedge \alpha \wedge \beta)(X \wedge Y) = \omega(X) \cdot (\alpha \wedge \beta)(Y) + \omega(Y) \cdot (\alpha \wedge \beta)(X) - \omega(\widetilde{X}(\alpha), \widetilde{Y}(\beta)) + \omega(\widetilde{X}(\beta), \widetilde{Y}(\alpha))$.*

Proof. It is sufficient to prove the lemma for the case $\omega = \varphi \wedge \psi$ where $\varphi, \psi \in A'(M)$.

So, using the definition of the exterior product of differential forms (see [3]), we obtain

$$\begin{gathered}(\varphi \wedge \psi \wedge \alpha \wedge \beta)(X \wedge Y) = (\varphi \wedge \psi)(X) \cdot (\alpha \wedge \beta)(Y) + \\ +(\varphi \wedge \alpha)(X) \cdot (\beta \wedge \psi)(Y) + (\varphi \wedge \beta)(X) \cdot (\psi \wedge \alpha)(Y) + \\ +(\psi \wedge \alpha)(X) \cdot (\varphi \wedge \beta)(Y) + (\psi \wedge \beta)(X) \cdot (\alpha \wedge \varphi)(Y) + \\ +(\alpha \wedge \beta)(X) \cdot (\varphi \wedge \psi)(Y) = \omega(X) \cdot (\alpha \wedge \beta)(Y) + \\ +\omega(Y) \cdot (\alpha \wedge \beta)(X) - \omega(\widetilde{X}(\alpha), \widetilde{Y}(\beta)) + \omega(\widetilde{X}(\beta), \widetilde{Y}(\alpha)). \quad \square\end{gathered}$$

2.9. A submodulus $W \subset V'(M)$ is said to be an involutory differential system if for each pair $X, Y \in W$ we have $[X,Y] \in W$ (see [6]).

Theorem. *If $\widetilde{\xi} : A'(M) \longrightarrow V'(M)$ is the homomorphism corresponding to the bivetor field ξ (see 2.4), then the differential system $I_m\widetilde{\xi}$ is involutory in exact terms when $[\xi,\xi] \in I_m\widetilde{\xi} \wedge I_m\widetilde{\xi} \wedge I_m\widetilde{\xi}$.*

Proof. We can use any local coordinate system $\{x_1,\dots,x_n\}$. So, we want to show that for each pair $\{i,j\} \subset \{1,\dots,n\}$ the vector field $[\widetilde{\xi}(dx_i), \widetilde{\xi}(dx_j)]$ is an element of $I_m\widetilde{\xi}$ or, which is the same thing, that $\sigma([\widetilde{\xi}(dx_i), \widetilde{\xi}(dx_j)]) = 0$ for each $\sigma \in (I_m\widetilde{\xi})^\perp \subset A'(M)$.

By the definition of the Schoten bracket (see 1.14) we obtain $(d\sigma \wedge dx_i \wedge dx_j)(\xi \wedge \xi) = 2(d\sigma)(\xi) \cdot (dx_i \wedge dx_j)(\xi) - (\sigma \wedge dx_i \wedge dx_j)([\xi,\xi])$. Using Lemma 2.8, we have $(d\sigma \wedge dx_i \wedge dx_j)(\xi \wedge \xi) = 2(d\sigma)(\xi) \cdot (dx_i \wedge dx_j)(\xi) -$

$2(d\sigma)(\widetilde{\xi}(dx_i),\widetilde{\xi}(dx_j))$. Thus $(\sigma \wedge dx_i \wedge dx_j)([\xi,\xi]) = 2(d\sigma)(\widetilde{\xi}(dx_i),\widetilde{\xi}(dx_j))$. Clearly, $(d\sigma)(\widetilde{\xi}(dx_i),\widetilde{\xi}(dx_j)) = -\sigma([\widetilde{\xi}(dx_i),\widetilde{\xi}(dx_j)])$. Keeping in mind these identities, we obtain $(\sigma \wedge dx_i \wedge dx_j)([\xi,\xi]) = -2\sigma([\widetilde{\xi}(dx_i),\widetilde{\xi}(dx_j)])$. □

2.10. **Definition.** An integer $2k \geq 0$ is said to be a rank of the bivector field ξ at a point $a \in M$ if $(\wedge^k \xi_a) \neq 0$ and $\wedge^{k+1}\xi_a = 0$.

Let $e = \{e_1,\dots,e_n\}$ be a basis of $T_a(M)$ and $e' = \{e^1,\dots,e^n\}$ be the corresponding dual basis of $T_a^*(M)$. As is known (see [3]), a basis e can be chosen so that $\xi_a = e_1 \wedge e_2 + \cdots + e_{2k-1} \wedge e_{2k}$. From the definition of $\widetilde{\xi}$ (see 2.4) it follows that $\{e^1,\dots,e^{2k}\}$ is a basis of $I_m\widetilde{\xi}_a$. Also, it is clear that $\wedge^k\xi_a = e_1 \wedge \ldots \wedge e_{2k}$ and $\wedge^{k+1}\xi_a = 0$. We have therefore ascertained that $\dim(I_m\widetilde{\xi}_a) = rank\ \xi_a$.

2.11. If the rank $\xi = const$ and $[\xi,\xi] \in \wedge^3 I_m\widetilde{\xi}$, then Theorem 2.9 and Frobenius' theorem imply that the differential system $I_m\widetilde{\xi}$ is integrable (see [3]), i.e., for each point $a \in M$ there is a submanifold $N \subset M$ such that $a \in N$ and for each $X \in N$ we have $I_m\widetilde{\xi}_x = T_x(N)$. It is clear that $dim\ N = rank\ \xi$.

2.12. **Proposition.** *If* $[\xi,\xi] = 0$, *then the differential system* $I_m\widetilde{\xi}$ *is integrable.*

The proof follows from Hermann's generalization of Frobenius' theorem (see [7]) and the fact that for each function $f \in C^\infty(M)$ the one-parameter group corresponding to $\widetilde{\xi}(df)$ preserves ξ. Consequently, the rank $\widetilde{\xi}$ is invariant under the action of this group.

2.13. **Definition.** The bivector field ξ is said to be nondegenerate at a point $a \in M$ if the rank $\xi_a = \dim\ M$. It is said to be nondegenerate on the manifold M if it is nondegenerate at each point of M.

2.14. If ξ is nondegenerate on M, then $\widetilde{\xi}$ is an isomorphism defining the differential 2-form $\omega = \widetilde{\xi}^{-1}(\xi)$, which is a symplectic form exactly when $[\xi,\xi] = 0$.

The Poisson bracket defined by ξ coincides with that defined by ω.

As mentioned in 2.12, if $[\xi,\xi] = 0$, then ξ defines the foliation on M perhaps with fibers of different dimensions. Let N be any fiber from this foliation and ξ_N be the restriction of ξ on the manifold N. It is easy to check that

(a) $\xi_N \in V^2(N)$;

(b) ξ_N is nondegenerate on N.

Consequently,

(c) N is a symplectic manifold with the differential 2-form $\omega_N = \widetilde{\xi}_N^{-1}(\xi_N)$.

3. Some Cohomology Properties of Bivector Fields

3.1. Let ξ be a bivector field on the manifold M. Setting $u = \xi$ in equality (b) of 1.10, we obtain

$$[\xi, v \wedge w] = [\xi, v] \wedge w + (-1)^n v \wedge [\xi, w]$$

which implies that the endomorphism

$$[\xi, \ ] : V(M) \longrightarrow V(M)$$

is an antidifferentiation of degree 1:

$$[\xi, V^m(M)] \subset V^{m+1}(M), \quad m \in \mathbb{Z}.$$

Let $[\xi, \xi] = 0$. Then by (c) from 1.10 we obtain $[\xi, [\xi, X]] = 0$ for each $X \in V(M)$. So the endomorphism $[\xi, \ ]$ can be regarded as a coboundary operator defining some cohomology algebra $H_\xi(M)$.

To investigate bivector fields from this standpoint we have to prove some propositions.

3.2. **Lemma.** *If ξ is a bivector field with $[\xi, \xi] = 0$, then for each closed 1-form α we have $[\xi, \widetilde{\xi}(\alpha)] = 0$.*

Proof. Using the local coordinate system $x_1, \dots, x_m$, the formula from 1.14, and the definition of $\widetilde{\xi}$ (see 2.4), we find that for each $i, j = 1, \dots, n$ we have $(dx_i \wedge dx_j)([\xi, \widetilde{\xi}(\alpha)]) = -(d((\alpha \wedge dx_i)(\xi)((\cdot dx_j - (\alpha \wedge dx_j)(\xi) \cdot dx_i))(\xi) + \alpha \wedge d((dx_i \wedge dx_j)(\xi)) = -(d((dx_i \wedge dx_j)(\xi) \cdot \alpha - (dx_i \wedge \alpha)(\xi) \cdot dx_j + (dx_j \wedge \alpha)(\xi) \cdot dx_i))(\xi) = -\frac{1}{2}(dx_i \wedge dx_j \wedge \alpha)([\xi, \xi]) = 0$. Consequently, $[\xi, \widetilde{\xi}(\alpha)] = 0$. $\square$

3.3. **Theorem.** *If ξ is a bivector field with $[\xi, \xi] = 0$, then the diagram*

$$\begin{array}{ccc} A(M) & \xrightarrow{d} & A(M) \\ {\scriptstyle \widetilde{\xi}}\downarrow & & \downarrow{\scriptstyle \widetilde{\xi}} \\ V(M) & \xrightarrow{[\xi, \ ]} & V(M) \end{array}$$

is commutative.

Proof. So, the aim is to show that for each form ω we have $\widetilde{\xi}(d\omega) = [\xi, \widetilde{\xi}(\omega)]$. It is suffiecient to show this for $\omega = f \cdot dx_1 \wedge \ldots \wedge dx_m$, where $f, x_1, \dots, x_m$ are smooth functions on M:

$$\begin{aligned} \widetilde{\xi}(\omega) &= f \cdot \widetilde{\xi}(dx_1) \wedge \ldots \wedge \widetilde{\xi}(dx_m); \\ [\xi, \widetilde{\xi}(\omega)] &= [\xi, f \cdot \widetilde{\xi}(dx_1) \wedge \ldots \wedge \widetilde{\xi}(dx_m)] = \\ &= [\xi, f \cdot \widetilde{\xi}(dx_1)] \wedge \widetilde{\xi}(dx_2) \wedge \ldots \wedge \widetilde{\xi}(dx_m) \pm \\ &\pm f \cdot \widetilde{\xi}(dx_1) \wedge [\xi, \widetilde{\xi}(dx_2) \wedge \ldots \wedge \widetilde{\xi}(dx_m)] = \end{aligned}$$

$$= f \cdot [\xi, \widetilde{\xi}(dx_1)] \wedge \widetilde{\xi}(dx_2) \wedge \ldots \wedge \widetilde{\xi}(dx_m) +$$
$$+\widetilde{\xi}(df) \wedge \widetilde{\xi}(dx_1) \wedge \ldots \wedge \widetilde{\xi}(dx_m) \pm$$
$$\pm f \cdot \widetilde{\xi}(dx_1)] \wedge [\xi, \widetilde{\xi}(dx_2) \wedge \ldots \wedge \widetilde{\xi}(dx_m)].$$

The preceding lemma and formula (b) from 1.10 give

$$[\xi, \widetilde{\xi}(dx_i)] = [\xi, \widetilde{\xi}(dx_1) \wedge \ldots \wedge \widetilde{\xi}(dx_m)] = 0.$$

Eventually, $[\xi, \widetilde{\xi}(\omega)] = \widetilde{\xi}(df) \wedge \widetilde{\xi}(dx_1) \wedge \ldots \wedge \widetilde{\xi}(dx_m) = \widetilde{(}\xi(d\omega)$. $\square$

3.4. To say otherwise, we have the following homomorphism of cochain complexes:

$$\begin{array}{ccccccc}
\mathbb{R} & \longrightarrow & A^0(M) = C^\infty(M) & \xrightarrow{d} & A'(M) & \xrightarrow{d} & \cdots \\
{\scriptstyle Id}\big\downarrow & & {\scriptstyle \widetilde{\xi}=Id}\big\downarrow & & {\scriptstyle \widetilde{\xi}}\big\downarrow & & \\
\mathbb{R} & \longrightarrow & V^0(M) = C^\infty(M) & \xrightarrow{[\xi,\,]} & V'(M) & \xrightarrow{[\xi,\,]} & \cdots
\end{array}$$

where the top complex is that of De-Rham.

The above homomorphism defines the homomorphism between the De-Rham cohomology algebra $H(M, \mathbb{R})$ and the cohomology algebra $H_\xi(M)$, which will also be denoted by $\widetilde{\xi}$.

3.5. **Example.** Let $M = T^*(X)$ where X is any smooth manifold. As known, there is a canonical symplectic form ω on M (see [3], [4], [5]), defining the Poisson structure on $C^\infty(M)$. Consider the corresponding bivector field $\xi_\omega = \widetilde{\omega}^{-1}(\omega)$ (see 2.5, 2.7). It is clear that $\widetilde{\xi}_\omega(\omega) = \xi_\omega$. Since $\omega = d\lambda$, where λ is the Liouville form (see [3]), by the theorem from 3.3 we obtain $\xi_\omega = \widetilde{\xi}(d\lambda) = [\xi_\omega, \widetilde{\xi}_\omega(\lambda)]$.

It is easy to show that the vector field $\widetilde{\xi}_\omega(\lambda)$ is the vector field corresponding to the one-parameter group $\varphi_t(u) = e^{-t} \cdot u$, $t \in \mathbb{R}$, $u \in T^*(X)$. Otherwise, $\widetilde{\xi}_\omega(\lambda)|_u = -u$.

3.6. **Example.** Let L be a finite-dimensional real vector space and $s : L \wedge L \longrightarrow L$ be any linear map. We have the bivector field ξ on the manifold $M = L^*$ defined by means of s. Clearly, $T^*(M) = L^* \times L$ and for each point $a \in L^*$ we have $\wedge^2 T^*_a(M) = L \wedge L$. Now we define ξ as follows: let $\alpha(\xi_a) = a(s(\alpha))$ for $a \in L^*$ and $\alpha \in \wedge^2 T^*_\alpha(M)$.

3.7. **Theorem.** *The equality $[\xi, \xi] = 0$ for the above-defined bivector field holds if and only if the linear map s defines the structure of a Lie algebra on L, i.e., we have*

$$s(s(u \wedge v) \wedge w) + s(s(w \wedge u) \wedge v) + s(s(v \wedge w) \wedge u) = 0$$

for each $u, v, w \in L$.

Proof. Let $\{u, v, w\} \subset L$ and $\omega = u \wedge v \wedge w$ be an element of $V^3(L^*)$. Clearly, $d\omega = 0$ and for $p \in L^*$ we have $(i_\xi \omega)|_p = ((u \wedge v)(\xi) \cdot w + (w \wedge u)(\xi) \cdot v + (v \wedge w)(\xi) \cdot u)|_p = p(s(u \wedge v)) \cdot w + p(s(w \wedge u)) \cdot v + p(s(v \wedge w)) \cdot u$. As one can see, the form $i_\xi \omega$ depends linearly on p and therefore $d(i_\xi \omega) = s(u \wedge v) \wedge w + s(w \wedge u) \wedge v + s(v \wedge w) \wedge u$.

Using the formula from 1.14, we obtain $\omega([\xi, \xi])|_p = 2d(i_\xi \omega)(\xi)|_p = p(s(s(u \wedge v) \wedge w + s(s(w \wedge u)) \wedge v + s(s(v \wedge w)) \wedge u))$, $p \in L^*$.

Thus $[\xi, \xi] = 0$ exactly when $\omega([\xi, \xi])|_p = 0$ for each $\omega = u \wedge v \wedge w$ and $p \in L^*$; otherwise, $p(s(s(u \wedge v) \wedge w) + s(s(w \wedge u) \wedge v) + s(s(v \wedge w) \wedge u)) = 0$ for each $p \in L^*$, which is the same as $s(s(u \wedge v) \wedge w + s(s(w \wedge u)) \wedge v + s(s(v \wedge w)) \wedge u) = 0$. $\square$

3.8. We have ascertained that ξ defines the Poisson structure on $C^\infty(L^*)$ if and only if the bracket $[u, v] = s(u \wedge v)$ defines the structure of a Lie algebra on L.

Clearly, L is a subspace of $C^\infty(L^*)$. Moreover, L is a Lie subalgebra of the Poisson algebra $C^\infty(M)$ and the bracket $[\ ,\ ]$ coincides with the Poisson bracket $\{\ ,\ \}$ on L: for $u, v \in L$ we have $\{u, v\}(p) = (u, v)(\xi)|_p = p([u, v])$, $p \in L^*$. Finally, we find that the element $[u, v]$ as a linear function on L^* coincides with $\{u, v\}$.

3.9. Let us consider the exterior algebra $\wedge(L^*) = \sum_{k \in \mathbb{Z}} \wedge^k L^*$. Clearly, $\wedge(L^*)$ is a subalgebra of the exterior algebra $V(L^*)$.

Theorem. *The subalgebra* $\wedge(L^*)$ *in* $V(L^*)$ *is an invariant subspace of the operator* $[\xi,\ ]$, *and* $[\xi,\ ] : \wedge(L^*) \longrightarrow \wedge(L^*)$ *is the Chevalley–Eilenberg operator* (*see the operator* ∂_1 *in* 1.3) *defining the cohomology of the Lie algebra L with coefficients in* $\mathbb{R}$.

Proof. Let $\alpha \in \wedge^k L^*$. Then $[\xi, \alpha] \in \wedge^{k+1} L^*$. We must prove that for $u_1 \wedge \ldots \wedge u_{k+1} \in \wedge^{k+1} L \subset A^{k+1}(L^*)$ we have $(u_1 \wedge \ldots \wedge u_{k_1})([\xi, \alpha]) = \sum_{i<j} (-1)^{i+j-1} \alpha([u_i, u_j], u_1, \ldots, u_{k+1})$ (recall that for $X \in \wedge^m L^* \subset V^m(L^*)$ and $\lambda \in \wedge^m L \subset A^m(L^*)$ we have $\lambda(X) = X(\lambda)$) : $(u_1 \wedge \ldots \wedge u_{k+1})([\xi, \alpha]) = (-1)^k (d(i_\alpha(u_1 \wedge \ldots \wedge u_{k+1})))(\xi) + (d(i_\xi(u_1 \wedge \ldots \wedge u_{k+1})))(\alpha) - (d(u_1 \wedge \ldots \wedge u_{k+1}))(\xi \wedge \alpha)$. Clearly,

$$d(i_\alpha(u_1 \wedge \ldots \wedge u_{k+1})) = d(u_1 \wedge \ldots \wedge u_{k+1}) = 0;$$

$$i_\xi(u_1 \wedge \ldots \wedge u_{k+1})|_p = \sum_{i<j} (-1)^{i+j-1} p([u_i, u_j]) \cdot u_1 \wedge \ldots \wedge u_{k+1}$$

$p \in L^*$ and $d(i_\xi(u_1 \wedge \ldots \wedge u_{k+1})) = \sum_{i<j} (-1)^{i+j-1} [u_i, u_j]) \wedge u_1 \wedge \ldots \wedge \widehat{u}_i \wedge \ldots \wedge \widehat{u}_j \wedge \ldots \wedge u_{k+1}$.

Therefore we obtain

$$(u_1 \wedge \ldots \wedge u_{k+1})([\xi, \alpha]) = [\xi, \alpha](u_1, \ldots, u_{k+1}) =$$

$$= \alpha\Big(\sum_{i<j}(-1)^{i+j-1}[u_i,u_j]\wedge u_1\wedge\ldots\wedge\widehat{u}_i\wedge\ldots\wedge\widehat{u}_j\wedge\ldots\wedge u_{k+1}\Big). \quad \square$$

3.10. Let $\bar{A}(M)$ be a sheaf of local differential forms on M and $\bar{V}(M)$ be a sheaf of local polyvector fields on M. Since the diagram in 3.3 is commutative, the diagram of morphisms of sheaves

$$\begin{array}{ccc} \bar{A}(M) & \xrightarrow{d} & \bar{A}(M) \\ {\scriptstyle\widetilde{\xi}}\big\downarrow & & \big\downarrow{\scriptstyle\widetilde{\xi}} \\ \bar{V}(M) & \xrightarrow{[\xi,\ ]} & \bar{V}(M) \end{array}$$

will also be commutative. Therefore we can talk about the sheave $\bar{I}m\widetilde{\xi}$, with the coboundary operator $[\xi,\ ] : \bar{I}_m\widetilde{\xi} \longrightarrow \bar{I}_m\widetilde{\xi}$. On the global sections of $\bar{I}_m\widetilde{\xi}$ the operator $[\xi,\ ]$ defines some cohomology algebra which will be denoted by $h_\xi(M)$. The homomorphism $\widetilde{\xi}$ induces a homomorphism from $H(M,\mathbb{R})$ into $h_\xi(M)$. The element $\xi \in V^2(M)$ defines some cohomology class $[\xi] \in h_\xi(M)$.

3.11. Let N be any integral manifold of the differential system $Im\widetilde{\xi}$. Then the restriction map $I_m\widetilde{\xi} \ni X \longrightarrow X_N \in V(N)$ induces a homomorphism from $h_\xi(M)$ into $H_{\xi_N}(N)$. Since the bivector field ξ_N is nondegenerate, there is an isomorphism $\widetilde{\xi}_N : H(N,\mathbb{R}) \longrightarrow H_{\xi_N}(N)$ and therefore we have a homomorphism from $h_\xi(M)$ into $H(N,\mathbb{R})$. Finally, we find that for each N which is an integral manifold of $Im\widetilde{\xi}$ there is a homomorphism

$$r_N : h_\xi(M) \longrightarrow H(N,\mathbb{R}).$$

3.12. Let us return to 3.6, 3.7, 3.8. As was proved, the canonical bivector field ξ on L^*, where L is a Lie algebra, is such that $[\xi,\xi]=0$. Therefore ξ defines the foliation in L^*. One can show that if L is a Lie algebra of the connected Lie group G, then the orbits of the Ad^*G-representation (see [1], [4]) are just the fibers of the foliation defined by ξ, while for each fiber N the symplectic form $\xi_N^{-1}(\xi_N)$ is just the Souriau–Kostant form on the orbits of the coadjoint representation.

If the cohomology class $[\xi] \in h_\xi(M)$ is zero, then, as follows from 3.11, each orbit satisfies the Souriau–Kostant prequantization condition (see [8]).

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