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SOME NEGATIVE RESULTS CONCERNING THE PROCESS OF FEJÉR TYPE TRIGONOMETRIC CONVOLUTION

G. ONIANI

ABSTRACT. The process of Fejér type trigonometric convolution and its discrete analogue have equivalent uniform residues. The situation changes under pointwise comparison. In this direction negative results have been obtained by different authors. One result of such a type is given in the present paper. In particular, a counter-example is constructed for which both comparisons diverge on the set of complete measure. The smoothness of the counter-example as well as some other problems are investigated.

1. Let $C_{2\pi}$ denote a Banach space of 2π-periodic functions continuous on $\mathbb{R}$, with sup-norm $\|\cdot\|_c$.

Let for every $n \in \mathbb{N}$

$$\chi_n(x) := \sum_{k=-n}^{n} \rho_{k,n} e^{ikx}, \tag{1.1}$$

where $\rho_{-k,n} = \rho_{k,n}$, $\rho_{0,n} = 1$.

F_n denotes an operator of trigonometric convolution

$$F_n f(x) := \frac{1}{2\pi} \int_0^{2\pi} f(u)\chi_n(x-u)du \tag{1.2}$$

and J_n is its discrete analogue

$$J_n f(x) := \frac{1}{2n+1} \sum_{j=0}^{2n} f(u_{j,n})\chi_n(x-u_{j,n}), \tag{1.3}$$

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where $u_{j,n} = 2\pi j \;/\; (2n+1)$, $0 \le j \le 2n$.

The following theorem on the operators $\{F_n\}$ and $\{J_n\}$ is valid (see [2]-[5]).

Theorem 1.1. *Let $\{\chi_n\}$ be a sequence of even polynomial kernels* (1.1) *satisfying the conditions*

$$\|\chi_n\|_1 = O(1) \quad (n \to \infty), \tag{1.4}$$

$$1 - \rho_{k,n} = o_k(1) \quad (k \in \mathbb{N}, \quad n \to \infty). \tag{1.5}$$

Then for every $f \in C_{2\pi}$ we have

$$\|F_n f - f\|_c = o(1) \quad \textit{and} \quad \|J_n f - f\|_c = o(1) \quad (n \to \infty), \tag{1.6}$$

$$\begin{gathered} \|J_n f - f\|_c = O(\|F_n f - f\|_c), \\ \|F_n f - f\|_c = O(\|J_n f - f\|_c) \ \ (n \in \mathbb{N}). \end{gathered} \tag{1.7}$$

Note also that (1.6) is equivalent to the pair of conditions (1.4) and (1.5), i.e., the sequences (processes) of the operators $\{F_n\}$ and $\{J_n\}$ ensure uniform approximation of continuous functions (such sequences of operators are called Fejér) and their residues are equivalent with respect to the norm in $C_{2\pi}$.

Pointwise comparison changes the situation. In this case residues of the operators $\{F_n\}$ and $\{J_n\}$ may turn out to be nonequivalent on the set of complete measure. In particular, the following theorems are valid (see resp. [2] and [1]):

Theorem 1.2. *Let $\{\chi_n\}$ be a sequence of even polynomial kernels satisfying the conditions* (1.4) *and*

$$1 - \rho_{k,n} = O_k\left(\frac{1}{n}\right) \quad (k \in \mathbb{N}, \quad n \to \infty). \tag{1.8}$$

Let Ω denote a class of functions continuous on $[0,\infty)$ with the following properties:

$$0 = \omega(0) < \omega(s) \le \omega(s+t) \le \omega(s) + \omega(t) \quad (s,t > 0)$$
$$\textit{and} \quad \lim_{t \to 0} \omega(t) \;/\; t = \infty$$

and let $f \in \mathrm{Lip}_1\,\omega$ denote that $|f(x) - f(y)| \le c[\omega(|x-y|)]$, $x, y \in [0, 2\pi]$ ($c > 0$ is a constant).

Then for every $\omega \in \Omega$ there is a counter-example $f_\omega \in \mathrm{Lip}_1\,\omega$ such that for $n \to \infty$

$$|J_n f_\omega(x) - f_\omega(x)| \ne O(|F_n f_\omega(x) - f_\omega(x)|) \quad \textit{for a.e.} \quad x \in \mathbb{R}. \tag{1.9}$$

Theorem 1.3. *Let $\{\chi_n\}$ be a sequence of even polynomial kernels* (1.1) *satisfying the conditions* (1.4) *and* (1.8). *Then there is a counter-example $f \in C_{2\pi}$ such that as $n \to \infty$*

$$|F_n f(x) - f(x)| \neq O(|J_n f(x) - f(x)|) \quad \textit{for a.e.} \quad x \in \mathbb{R}. \tag{1.10}$$

Comparing Theorems 1.2 and 1.3 there naturally arises the question: Is there a function $f \in C_{2\pi}$ with properties (1.9) and (1.10) when the conditions (1.4) and (1.8) are fulfilled, i.e., is there a continuous function with two-sided divergence?

2. The theorem below gives a positive answer to the above question.

Theorem 2.1. *Let $\{\chi_n\}$ be a sequence of even polynomial kernels* (1.1) *satisfying the conditions* (1.4) *and* (1.8). *Then there is a counter-example $f \in C_{2\pi}$ such that as $n \to \infty$*

$$\begin{aligned} |F_n f(x) - f(x)| &\neq O(|J_n f(x) - f(x)|), \\ |J_n f(x) - f(x)| &\neq O(|F_n f(x) - f(x)|) \end{aligned} \tag{2.1}$$

simultaneously almost for all $x \in \mathbb{R}$.

Proof. Let $\{s_k\} \subset \mathbb{N}$ be an arbitrary sequence and

$$n_1 = 4, \;\; n_{k+1} = \frac{1}{2}\big[(4s_k + 1)(2n_k + 1) - 1\big], \;\; k = 2, 3, \dots \;. \tag{2.2}$$

For natural numbers n and k we introduce the notation

$$\begin{aligned} g_n(x) &:= \cos(2n+1)x, \\ H_k &:= \bigcup_{j\in\mathbb{Z}} \frac{\pi}{2(2n_k+1)}\Big[2j + 1 - \frac{1}{k+1},\; 2j + 1 + \frac{1}{k+1}\Big], \\ D_k &:= [0, 2\pi] \cap \bigcup_{j\in\mathbb{Z}} \frac{\pi}{2(2n_k+1)}\Big[2j - \frac{1}{2},\; 2j + \frac{1}{2}\Big]. \quad \square \end{aligned} \tag{2.3}$$

The following lemma is valid.

Lemma 2.1. (*See* [2].) *Let $x \in H_k - y_k$, $y_k \in D_k$. Then for every $k \in \mathbb{N}$*

$$\begin{gathered} |J_{n_k} g_{n_k}(x + y_k) - g_{n_k}(x + y_k)| \geq C_1 - \varepsilon_k \\ (C_1 = 1/\sqrt{2},\; \varepsilon_k = \pi/2(k+1)), \\ |F_{n_k} g_{n_k}(x + y_k) - g_{n_k}(x + y_k)| \leq \varepsilon_k. \end{gathered} \tag{2.4}$$

There is also a sequence $\{y_k\}$ $(y_k \in D_k,\; k \in \mathbb{N})$ such that

$$\lim_{k\to\infty} \sup\, (H_k - y_k)$$

is a set of full measure.

Let

$$M_n := \|\chi_n\|_c \quad (n \in \mathbb{N}),$$

$$E_n := \bigcup_{j=0}^{2n} \Big(\frac{2\pi j}{2n+1} + \frac{1}{2n\sqrt{M_n}}, \frac{2\pi(j+1)}{2n+1}\Big) \quad (n \in \mathbb{N}). \tag{2.5}$$

The following assertions are valid (see [1]).

Lemma 2.2. *Let the sequence* $\{\chi_n\}$ *satisfy the conditions* (1.4) *and* (1.5). *Then* $\|\chi_n\|_c \to \infty$ $(n \to \infty)$.

Lemma 2.3. *Let the sequence* $\{\chi_n\}$ *satisfy the conditions* (1.4) *and* (1.8). *Then for every* $n \geq n_0$ *there is a trigonometric polynomial* $P_n(x)$ *and a set* $A_n \subset [0, 2\pi]$ *such that for* $n \in \mathbb{N}$

$$\|P_n\|_c \leq 2, \tag{2.6}$$

$$A_n \subset E_n, \tag{2.7}$$

$$\mu A_n \geq C_2, \tag{2.8}$$

where μ *is the Lebesgue measure,*

$$|F_n P_n(x) - P_n(x)| \geq C_3 \frac{\sqrt{M_n}}{n} \quad (x \in A_n), \tag{2.9}$$

$$|P_n(x)| \leq \frac{1}{nM_n} \quad (x \in E_n), \tag{2.10}$$

where n_0, C_2, C_3 *are positive constants.*

Consider the series

$$f(x) = \sum_{k=1}^{\infty} \Big[\frac{1}{\sqrt{n_k}} g_{n_k}(x + y_k) + \frac{1}{\sqrt[4]{M_{m_k}}} P_{m_k}(x + t_k)\Big]$$

(here $\sum_{k=1}^{\infty} \Big[\frac{1}{\sqrt{n_k}} + \frac{1}{\sqrt[4]{M_{m_k}}}\Big] < \infty$, whence $f \in C_{2\pi}$). The sequences $n_k \uparrow \infty$, $m_k \uparrow \infty$, $\{y_k\}$, and $\{t_k\}$ will be constructed later on.

Let for $k \geq \mathbb{N}$

$$
\begin{aligned}
\alpha_k(x) &:= \sum_{j=1}^{k-1} \frac{1}{\sqrt{n_j}} g_{n_j}(x+y_j) + \sum_{j=1}^{k-1} \frac{1}{\sqrt[4]{M_{m_j}}} P_{m_j}(x+t_j), \\
\gamma_k(x) &:= \sum_{j=k+1}^{\infty} \frac{1}{\sqrt{n_j}} g_{n_j}(x+y_j) + \sum_{j=k}^{\infty} \frac{1}{\sqrt[4]{M_{m_j}}} P_{m_j}(x+t_j), \\
\alpha'_k(x) &:= \sum_{j=1}^{k} \frac{1}{\sqrt{n_j}} g_{n_j}(x+y_j) + \sum_{j=1}^{k-1} \frac{1}{\sqrt[4]{M_{m_j}}} P_{m_j}(x+t_j), \\
\gamma'_k(x) &:= \sum_{j=k+1}^{\infty} \frac{1}{\sqrt{n_j}} g_{n_j}(x+y_j) + \sum_{j=k+1}^{\infty} \frac{1}{\sqrt[4]{M_{m_j}}} P_{m_j}(x+t_j).
\end{aligned}
\tag{2.11}
$$

We can easily conclude from (1.8) that as $n \to \infty$

$$
\begin{aligned}
&|J_n\alpha_k(x) - \alpha_k(x)| \leq S_{k-1}/n, \quad |F_n\alpha_k(x) - \alpha_k(x)| \leq S_{k-1}/n \\
&|J_n\alpha'_k(x) - \alpha'_k(x)| \leq S_{k-1}/n, \quad |F_n\alpha'_k(x) - \alpha'_k(x)| \leq S_{k-1}/n,
\end{aligned}
\tag{2.12}
$$

where S_{k-1} depends on $k-1$ only.

It is not difficult to see that for $k \in \mathbb{N}$

$$
\begin{aligned}
|J_{n_k}f(x) - f(x)| \geq \frac{1}{\sqrt{n_k}} &|J_{n_k}g_{n_k}(x+y_k) - g_{n_k}(x+y_k)| - \\
&-|J_{n_k}\alpha_k(x) - \alpha_k(x)| - |J_n\gamma_k(x) - \gamma_k(x)|, \\
|F_{n_k}f(x) - f(x)| \leq \frac{1}{\sqrt{n_k}} &|F_{n_k}g_{n_k}(x+y_k) - g_{n_k}(x+y_k)| + \\
&+|F_{n_k}\alpha_k(x) - \alpha_k(x)| - |F_{n_k}\gamma_k(x) - \gamma_k(x)|,
\end{aligned}
\tag{2.13}
$$

$$
\begin{aligned}
|F_{m_k}f(x) - f(x)| \geq \frac{1}{\sqrt[4]{M_{m_k}}} &|F_{m_k}P_{m_k}(x+t_k) - P_{m_k}(x+t_k)| - \\
&-|F_{m_k}\alpha'_k(x) - \alpha'_k(x)| - |F_{m_k}\gamma'_k(x) - \gamma'_k(x)|, \\
|J_{m_k}f(x) - f(x)| \leq \frac{1}{\sqrt[4]{M_{m_k}}} &|J_{m_k}P_{m_k}(x+t_k) - \\
-P_{m_k}(x+t_k)| + |J_{m_k}\alpha'_k(x) &- \alpha'_k(x)| + |J_{m_k}\gamma'_k(x) - \gamma'_k(x)|
\end{aligned}
\tag{2.14}
$$

and for $C_4 = const$ and $n \in \mathbb{N}$

$$
|F_nf(x) - f(x)| \leq C_4\|f\|_c, \quad |J_nf(x) - f(x)| \leq C_4\|f\|_c. \tag{2.15}
$$

Let us construct the sequences $\{n_k\}$ and $\{m_k\}$ such that for $k \in \mathbb{N}$

$$
\begin{gathered}
n_0 < n_1 < m_1 < \cdots < n_k < m_k < n_{k+1} < m_{k+1} < \dots \quad , \\
\sqrt{n_k} > kS_{k-1}, \\
\sqrt[4]{M_{m_k}} > kS_{k-1}, \\
\frac{1}{\sqrt[4]{M_{m_k}}} < \frac{1}{2n_k}, \\
\frac{1}{\sqrt{n_{k+1}}} < \frac{1}{2m_k}.
\end{gathered} \tag{2.16}
$$

Put

$$
T_k := \bigcup_{j=0}^{2m_k} \Big[\frac{2\pi j}{2m_k+1} + \frac{1}{2m_k\sqrt{M_{m_k}}}, \frac{2\pi(j+1)}{2m_k+1} - \frac{1}{2m_k\sqrt{M_{m_k}}}\Big].
$$

Let $A^*_{m_k}$ and $E^*_{m_k}$ be 2π-periodic extensions of the sets A_{m_k} and E_{m_k}, respectively.

We can easily show that

$$
\frac{2\pi j}{2m_k+1} + t_k \in E^*_{m_k} \quad (k \in \mathbb{N},\ t_k \in T_k),
$$

whence by Lemma 2.3 we have

$$
\Big|P_{m_k}\Big(\frac{2\pi j}{2m_k+1} + t_k\Big)\Big| \le \frac{1}{m_k M_{m_k}} \quad (k \in \mathbb{N},\ t_k \in T_k,\ \ j = 0, 1, \dots, 2n). \tag{2.17}
$$

From (2.17) we obtain that if $x \in A^*_{m_k} - t_k$, $t_k \in T_k$, then for every $k \in \mathbb{N}$

$$
\begin{gathered}
|J_{m_k} P_{m_k}(x+t_k) - P_{m_k}(x+t_k)| \le \\
\le |J_{m_k} P_{m_k}(x+t_k)| + |P_{m_k}(x+t_k)| \le \frac{1}{m_k} + \frac{1}{m_k M_{m_k}}.
\end{gathered} \tag{2.18}
$$

We can prove that (see [1]) there is a sequence $\{t_k\}$ $(t_k \in T_k,\ k \in \mathbb{N})$ such that

$$
\limsup_{k\to\infty} (A^*_{m_k} - t_k) \tag{2.19}
$$

is the set of full measure.

From Lemma 2.3 and relations (2.13)–(2.16) and (2.18) we get that if $x \in A^*_{m_k} - t_k$, $t_k \in T_k$, then for every $k \in \mathbb{N}$

$$
\begin{aligned}
|F_{m_k} f(x) - f(x)| &\geq \frac{1}{\sqrt[4]{M_{m_k}}} C_3 \frac{\sqrt{M_{m_k}}}{m_k} - \frac{S_{k-1}}{m_k} - \\
&\quad - C_4 \|\gamma'_k\|_c = \frac{\sqrt[4]{M_{m_k}}}{m_k}(C_3 - o(1)), \\
|J_{m_k} f(x) - f(x)| &\leq \frac{1}{\sqrt[4]{M_{m_k}}}\Big(\frac{1}{m_k} + \frac{1}{m_k M_{m_k}}\Big) + \\
&\quad + \frac{S_{k-1}}{m_k} + C_4 \|\gamma'_k\|_c = \frac{\sqrt[4]{M_{m_k}}}{m_k} o(1).
\end{aligned}
\tag{2.20}
$$

From (2.19) and (2.20) we obtain that as $n \to \infty$

$$
|F_n f(x) - f(x)| \neq O(|J_n f(x) - f(x)|) \quad \text{for a.e. } x \in \mathbb{R}. \tag{2.21}
$$

On the other hand, from Lemma 2.1 and conditions (2.13)–(2.15) we easily see that if $x \in H_k - y_k$, $y_k \in D_k$, then for $k \in \mathbb{N}$

$$
\begin{aligned}
|J_{n_k} f(x) - f(x)| &\geq \frac{1}{\sqrt{n_k}} C_1 - \frac{S_{k-1}}{n_k} - C_4 \|\gamma_k\|_c = \frac{1}{\sqrt{n_k}}(C_1 - o(1)), \\
|F_{n_k} f(x) - f(x)| &\leq \frac{1}{\sqrt{n_k}} \varepsilon_k + \frac{S_{k-1}}{n_k} + C_4 \|\gamma_k\|_c = \frac{1}{\sqrt{n_k}} o(1)).
\end{aligned}
\tag{2.22}
$$

It follows from (2.22) that as $n \to \infty$

$$
|J_n f(x) - f(x)| \neq O(|F_n f(x) - f(x)|) \quad \text{for a.e. } x \in \mathbb{R}. \tag{2.23}
$$

With the help of (2.21) and (2.23) we can conclude that f is the sought for counter-example.

Here without proof we shall give the result concerning the smoothness of the counter-example constructed in Theorem 2.1.

Theorem 2.2. *Let the sequence* $\{\chi_n\}$ *satisfy the conditions* (1.4) *and*

$$
1 - \rho_{k,n} = O_k\Big(\frac{M_n}{n^+}\Big) \quad (k \in \mathbb{N}, \quad n \to \infty).
$$

Then for every $\omega \in \Omega$ *there is a counter-example* $f_\omega \in \mathrm{Lip}_1\, \omega$ *such that*

$$
\begin{aligned}
|F_n f_\omega(x) - f_\omega(x)| &\neq O(|J_n f_\omega(x) - f_\omega(x)|) \quad (n \to \infty), \\
|J_n f_\omega(x) - f_\omega(x)| &\neq O(|F_n f_\omega(x) - f_\omega(x)|) \quad (n \to \infty)
\end{aligned}
$$

simultaneously for a.e. $x \in \mathbb{R}$.

3. As we have mentioned, if the sequence (1.1) is of Fejér type (i.e., $\|\chi_n\|_1 = O(1)$ and $1-\rho_{k,n} = o_k(1)$ $(k \in \mathbb{N},\ n \to \infty)$), then $\|\chi_n\|_c \to \infty$ $(n \to \infty)$ (see Lemma 2.2).

In this section we shall investigate the rate of the above-mentioned convergence.

Does any condition of the type $1-\rho_{k,n} = O_k(\varepsilon_n)$ $(k \in \mathbb{N},\ \varepsilon_n \downarrow 0,\ n \to \infty)$ improve the rate of convergence of $\|\chi_n\|_c$ to ∞? Theorem below gives a negative answer to this question.

Theorem 3.1. *For every sequence $\alpha_n \to \infty$ there is a sequence of kernels $\{\chi_n\}$ such that as $n \to \infty$*

$$\|\chi_n\|_1 = O(1),$$
$$1-\rho_{k,n} = 0 \quad (k \in \mathbb{N},\ n > n_k), \tag{3.1}$$
$$\|\chi_n\|_c = O(\alpha_n) \tag{3.2}$$

(i.e., convergence to infinity can be arbitrarily slow).

Proof. Let

$$K_n(x) := \frac{1}{2} + \frac{n}{n+1}\cos x + \dots + \frac{n+1-k}{n+1}\cos kx + + \dots + \frac{1}{n+1}\cos nx \quad \text{(Fejér type kernel)}, \tag{3.3}$$
$$R_{k,n}(x) := \frac{1}{n+1}\cos x + \frac{2}{n+1}\cos 2x + + \dots + \frac{k}{n+1}\cos kx \quad (k \le n). \tag{3.4}$$

Clearly,

$$|R_{k,n}(x)| \le \frac{k^2}{n+1} \quad (k, n \in \mathbb{N}). \tag{3.5}$$

Consider the sequence $n_k \uparrow \infty$ with the following properties:

$$n_k \ge 2k^2 \quad (k \in \mathbb{N}), \tag{3.6}$$
$$\alpha_n \ge 2k^2 \quad (n \ge n_k). \tag{3.7}$$

Let $\chi_n(x) = 1$ $(n \in [1, n_1])$, and

$$\frac{\chi_n(x)}{2} := K_{k^2}(x) + R_{k,k^2}(x) \quad (n \in [n_k+1, n_{k+1}]), \tag{3.8}$$

i.e.,

$$\chi_n(x)/2 = \frac{1}{2} + \frac{k^2}{k^2+1}\cos x + \dots + \frac{k^2+1-k}{k^2+1}\cos kx +$$

$$+\cdots+\frac{1}{k^2+1}\cos k^2x+\Big(\frac{1}{k^2+1}\cos x+\cdots+\frac{k}{k^2+1}\cos kx\Big)=$$
$$\frac{1}{2}+\cos x+\cdots+\cos kx+\Big(\frac{k^2-k}{k^2+1}\cos(k+1)x+\cdots+$$
$$+\frac{1}{k^2+1}\cos k^2x\Big)\quad (n\in[n_k+1,n_{k+1}]). \tag{3.9}$$

It is easily seen from (3.5) that

$$\|\chi_n\|_1\le 2\|K_{k^2}\|_1+2\|R_{k,k^2}\|_1<2\pi+4\pi=$$
$$=6\pi\quad (n\in[n_k+1,n_{k+1}]). \tag{3.10}$$

From (3.9) we obtain that

$$1-\rho_{k,n}=0\quad (k\in\mathbb{N},\ n>n_k). \tag{3.11}$$

The relations (3.5)–(3.8) show that

$$\|\chi_n\|_c\le 2\|K_{k^2}\|_c+2\|R_{k,k^2}\|_c\le 2\frac{k^2+1}{2}+2<$$
$$<2k^2<\alpha_n\quad (n\in[n_k+1,n_{k+1}]). \tag{3.12}$$

From (3.9), (3.10), and (3.11) we can conclude that $\{\chi_n\}$ is the desired sequence. □

It is not difficult to see that

$$\|\chi_n\|_c=O(n)\quad (n\to\infty).$$

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Author's address:
Faculty of Mechanics and Mathematics
I. Javakhishvili Tbilisi State University
2, University St., Tbilisi 380043
Republic of Georgia