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ON A SPATIAL PROBLEM OF DARBOUX TYPE FOR A SECOND-ORDER HYPERBOLIC EQUATION

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Abstract. The theorem of unique solvability of a spatial problem of Darboux type in Sobolev space is proved for a second-order hyperbolic equation.

In the space of variables $x_1,\ x_2,\ t$ let us consider the second order hyperbolic equation

$$Lu \equiv \Box u + au_{x_1} + bu_{x_2} + cu_t + du = F, \tag{1}$$

where $\Box \equiv \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2}$ is a wave operator; the coefficients a, b, c, d and the right-hand side F of equation (1) are given real functions, and u is an unknown real function.

Denote by $D : kt < x_2 < t$, $0 < t < t_0$, $-1 < k = \text{const} < 1$, the domain lying in a half-space $t > 0$, which is bounded by a time-type plane surface $S_1 : kt - x_2 = 0$, $0 \le t \le t_0$, a characteristic surface $S_2 : t - x_2 = 0$, $0 \le t \le t_0$ of equation (1), and a plane $t = t_0$.

Let us consider the Darboux type problem formulated as follows: find in the domain D the solution $u(x_1, x_2, t)$ of equation (1) under the boundary conditions

$$u\big|_{S_i} = f_i, \quad i = 1, 2, \tag{2}$$

where f_i, $i = 1, 2$, are given real functions on S_i; moreover $(f_1 - f_2)|_{S_1 \cap S_2} = 0$.

Note that in the class of analytic functions the problem (1),(2) is considered in [1]. In the case where S_1 is a characteristic surface $t + x_2 = 0$, $0 \le t \le t_0$, the problem (1),(2) is studied in [1–3]. Some multidimensional analogues of the Darboux problems are treated in [4–6]. In the present paper the problem (1),(2) is investigated in the Sobolev space $W_2^1(D)$.

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Below we shall obtain first the solution of problem (1),(2) when equation (1) is a wave equation

$$\Box u \equiv \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x_1^2} - \frac{\partial^2 u}{\partial x_2^2} = F \tag{3}$$

and then using the estimates for that solution we shall prove the solvability of the problem (1),(2) in the Sobolev space $W_2^1(D)$.

Using the method suggested in [7], we can get an integral representation of the regular solution of the problem (3),(2). Moreover, without loss of generality we can assume that for the domain D the value $k = 0$, i.e., $D : 0 < x_2 < t$, $0 < t < t_0$, since the case $k \neq 0$ is reduced to the case $k = 0$ by a suitable Lorentz transform for which the wave operator $\Box$ is invariant. To this end we denote by $D_{\varepsilon\delta}$ a part of the domain $D : 0 < x_2 < t$, $0 < t < t_0$, bounded by the surfaces S_1 and S_2, the circular cone $K_\varepsilon : r^2 = (t - t^0)(1 - \varepsilon)$ with vertex at the point $(x^0, t^0) \in D$, and the circular cylinder $H_\delta : r^2 = \delta^2$, where $r^2 = (x_1 - x_1^0)^2 + (x_2 - x_2^0)^2$ while ε and δ are sufficiently small positive numbers.

For any two twice continuously differentiable functions u and v we have an obvious identity

$$u\Box v - v\Box u = \sum_{i=1}^{2} \frac{\partial}{\partial x_i}\Big(v\frac{\partial u}{\partial x_i} - u\frac{\partial v}{\partial x_i}\Big) - \frac{\partial}{\partial t}\Big(v\frac{\partial u}{\partial t} - u\frac{\partial v}{\partial t}\Big). \tag{4}$$

Integrating equality (4) with respect to $D_{\varepsilon\delta}$, where $u \in C^1(\bar{D}) \cap C^2(D)$ is a regular solution of the equation (3) and

$$v = E(r, t, t^0) = \frac{1}{2\pi} \log \frac{t - t^0 - \sqrt{(t - t^0)^2 - r^2}}{r},$$

we have

$$\int\limits_{\partial D_{\varepsilon\delta}} \Big[E(r,t,t^0)\frac{\partial u}{\partial N} - \frac{\partial E(r,t,t^0)}{\partial N}u\Big]ds + \int\limits_{D_{\varepsilon\delta}} F \cdot E(r,t,t^0)dxdt = 0, \tag{5}$$

where N is the unit conormal vector at the point $(x, t) = (x_1, x_2, t) \in \partial D_{\varepsilon\delta}$ with direction cosines $\cos\widehat{Nx_1} = \cos\widehat{nx_1}$, $\cos\widehat{Nx_2} = \cos\widehat{nx_2}$, $\cos\widehat{Nt} = -\cos\widehat{nt}$ and n is a unit vector of an outer normal to $\partial D_{\varepsilon\delta}$.

Passing in the equality (5) to the limit for $\varepsilon \to 0$, $\delta \to 0$, we get

$$\int\limits_{x_2^0}^{t^0} u(x_1^0, x_2^0, t)dt = \int\limits_{S_1^* \cup S_2^*} \Big[\frac{\partial E(r,t,t^0)}{\partial N}u - E(r,t,t^0)\frac{\partial u}{\partial N}\Big]ds - \\ - \int\limits_{D^*} F \cdot E(r,t,t^0)dxdt,$$

where D^* is a domain of $D_{\varepsilon\delta}$ for $\varepsilon=\delta=0$, and $S_i^*=S_i\cap\partial D^*$, $i=1,2$. Differentiation gives

$$u(x_1^0,x_2^0,t^0)=\frac{d}{dt^0}\Big[\int\limits_{S_1^*\cup S_2^*}\Big[\frac{\partial E(r,t,t^0)}{\partial N}u-E(r,t,t^0)\frac{\partial u}{\partial N}\Big]ds-\\ -\int\limits_{D^*}F\cdot E(r,t,t^0)dxdt\Big]. \tag{6}$$

Remark. Since on the characteristic surface S_2^* the direction of the conormal N coincides with that of a bicharacteristic lying on S_2^*, we can, along with $u|_{S_2^*}=f_2$, calculate also $\frac{\partial u}{\partial N}$ over S_2^*. At the same time, since the surface S_1^* is a part of the plane $x_2=0$, the direction of the conormal N coincides with that of an outer normal to ∂D^*, i.e., $\frac{\partial}{\partial N}=-\frac{\partial}{\partial x_2}$. Therefore, to obtain an integral representation of the regular solution of the problem (3),(2), we should eliminate the value $\frac{\partial u}{\partial N}|_{S_1^*}$ on the right-hand side of the representation (6).

For this let us introduce a point $P'(x_1^0,-x_2^0,t^0)$ symmetric to the point $P(x_1^0,x_2^0,t^0)$ with respect to the plane $x_2=0$. Denote by D_ε a part of the domain D bounded by the cone $K_\varepsilon^0:(x_1-x_1^0)^2+(x_2+x_2^0)^2=(t-t^0)^2(1-\varepsilon)$ with vertex at P' and a boundary ∂D. Obviously, $\partial D_\varepsilon\cap S_1\subset S_1^*$ and $\partial D_0\cap S_1=S_1^*$. Put $\partial D_0\cap S_2=\widetilde{S_2}$, $\widetilde{r}=\sqrt{(x_1-x_1^0)^2+(x_2+x_2^0)^2}$. Integrating now the equality (4) with respect to D_ε, where $u\in C^1(\overline{D})\cap C^2(D)$ is a regular solution of equation (3) and

$$v=E(\widetilde{r},t,t^0)=\frac{1}{2\pi}\log\frac{t-t^0-\sqrt{(t-t^0)^2-\widetilde{r}^2}}{\widetilde{r}},$$

and taking into account the fact that the function $E(\widetilde{r},t,t^0)$ in D_0 is nonsingular, after passing to the limit for $\varepsilon\to 0$ we get the equality

$$\frac{d}{dt^0}\Big[\int\limits_{S_1^*\cup S_2^*}\Big[\frac{\partial E(\widetilde{r},t,t^0)}{\partial N}u-E(\widetilde{r},t,t^0)\frac{\partial u}{\partial N}\Big]ds-\\ -\int\limits_{D_0}F\cdot E(\widetilde{r},t,t^0)dxdt\Big]=0. \tag{7}$$

Since $r=\widetilde{r}$ for $x_2=0$, we have $E(\widetilde{r},t,t^0)=E(r,t,t^0)$ on S_1^*. Therefore, eliminating the value $\frac{\partial u}{\partial N}|_{S_1^*}$ from equalities (6) and (7), we finally get the integral representation of the regular solution of the problem (3),(2):

$$u(x_1^0,x_2^0,t^0)=\frac{d}{dt^0}\Big[\int\limits_{S_1^*}\Big[\frac{\partial E(r,t,t^0)}{\partial N}-\frac{\partial E(\widetilde{r},t,t^0)}{\partial N}\Big]u\,ds+$$
$$+\int\limits_{S_2^*}\Big[\frac{\partial E(r,t,t^0)}{\partial N}u-E(r,t,t^0)\frac{\partial u}{\partial N}\Big]ds-\int\limits_{\widetilde{S}_2}\Big[\frac{\partial E(\widetilde{r},t,t^0)}{\partial N}u-$$
$$-E(\widetilde{r},t,t^0)\frac{\partial u}{\partial N}\Big]ds+\int\limits_{D_0}F\cdot E(\widetilde{r},t,t^0)dx\,dt-\int\limits_{D^*}F\cdot E(r,t,t^0)dx\,dt\Big]. \quad (8)$$

Denote by $C_*^\infty(\overline{D})$ the space of functions of the class $C^\infty(\overline{D})$ having bounded supports, i.e.,

$$C_*^\infty(\overline{D})=\{u\in C^\infty(\overline{D}):\operatorname{diam}\operatorname{supp}u<\infty\}.$$

The spaces $C_*^\infty(S_i)$, $i=1,2$, are defined analogously.

According to the remark above and using the formula (8), the solution $u(x_1,x_2,t)$ of the problem (3),(2) will be defined uniquely; moreover, as is easily seen, for any $F\in C_*^\infty(\overline{D})$, $f_i\in C_*^\infty(S_i)$, $i=1,2$, this solution belongs to the class $C_*^\infty(\overline{D})$.

Denote by $W_2^1(D)$, $W_2^2(D)$ and $W_2^1(S_i)$, $i=1,2$, the well-known Sobolev spaces.

Definition. Let $f_i\in W_2^1(S_i)$, $i=1,2$, $F\in L_2(D)$. The function $u\in W_2^1(D)$ is said to be a strong solution of the problem (3),(2) of the class W_2^1 if there is a sequence $u_n\in C_*^\infty(\overline{D})$ such that $u_n\to u$, $\Box u_n\to F$ and $u_n|_{S_i}\to f_i$ in the spaces $W_2^1(D)$, $L_2(D)$ and $W_2^1(S_i)$, $i=1,2$, respectively, i.e., for $n\to\infty$

$$\|u_n-u\|_{W_2^1(D)}\to 0,\quad \|\Box u_n-F\|_{L_2(D)}\to 0,$$
$$\|u_n|_{S_i}-f_i\|_{W_2^1(S_i)}\to 0,\quad i=1,2.$$

Lemma 1. *For $-1<k<0$ the a priori estimate*

$$\|u\|_{W_2^1(D)}\le C\Big(\sum_{i=1}^2\|f_i\|_{W_2^1(S_i)}+\|F\|_{L_2(D)}\Big) \quad (9)$$

is valid for any $u\in C_*^\infty(\overline{D})$, where $f_i=u|_{S_i}$, $i=1,2$, $F=\Box u$, and the positive constant C does not depend on u.

Proof. Introduce the notations:

$$D_\tau = \{(x,t) \in D : t < \tau\}, \quad D_{0\tau} = \partial D_\tau \cap \{t = \tau\}, \quad 0 < \tau \le t_0,$$
$$S_{i\tau} = \partial D_\tau \cap S_i, \quad i = 1,2, \quad S_\tau = S_{1\tau} \cup S_{2\tau}, \quad \alpha_1 = \cos\widehat{(n,x_1)},$$
$$\alpha_2 = \cos\widehat{(n,x_2)}, \quad \alpha_3 = \cos\widehat{(n,t)}.$$

Here $n = (\alpha_1, \alpha_2, \alpha_3)$ is the unit vector of an outer normal to ∂D_τ; moreover, as is easily seen,

$$n|_{S_{1\tau}} = \Big(0, \frac{-1}{\sqrt{1+k^2}}, \frac{k}{\sqrt{1+k^2}}\Big), \quad n|_{S_{2\tau}} = \Big(0, \frac{\sqrt{2}}{2}, \frac{-\sqrt{2}}{2}\Big), \quad n|_{D_{0\tau}} = (0,0,1).$$

Hence, for $-1 < k < 0$

$$\begin{gathered} \alpha_3|_{S_{i\tau}} < 0 \;\; i = 1,2, \;\; \alpha_3^{-1}(\alpha_3^2 - \alpha_1^2 - \alpha_2^2)|_{S_1} > 0, \\ (\alpha_3^2 - \alpha_1^2 - \alpha_2^2)|_{S_2} = 0. \end{gathered} \tag{10}$$

Multiplying both parts of equation (3) by $2u_t$, where $u \in C_*^\infty(\overline{D})$, $F = \Box u$, integrating the obtained expression over the region to D_τ, and taking into account (10), we get

$$\begin{aligned} 2\int\limits_{D_\tau} F u_\tau \,dx dt &= \int\limits_{D_\tau} \Big(\frac{\partial u_t^2}{\partial t} + 2u_{x_1}u_{tx_1} + 2u_{x_2}u_{tx_2}\Big) dx dt - \\ -2\int\limits_{S_\tau} (u_{x_1}u_t\alpha_1 + u_{x_2}u_t\alpha_2)\,ds &= \int\limits_{D_{0\tau}} (u_t^2 + u_{x_1}^2 + u_{x_2}^2)\,dx + \\ &+ \int\limits_{S_\tau} [(u_t^2 + u_{x_1}^2 + u_{x_2}^2)\alpha_3 - 2(u_{x_1}u_t\alpha_1 + u_{x_2}u_t\alpha_2)]\,ds = \\ = \int\limits_{D_{0\tau}} (u_t^2 + u_{x_1}^2 + u_{x_2}^2)\,dx &+ \int\limits_{S_\tau} \alpha_3^{-1}\big[(\alpha_3 u_{x_1} - \alpha_1 u_t)^2 + (\alpha_3 u_{x_2} - \alpha_2 u_t)^2 + \\ &+ (\alpha_3^2 - \alpha_1^2 - \alpha_2^2)u_t^2\big]\,ds \ge \int\limits_{D_{0\tau}} (u_t^2 + u_{x_1}^2 + u_{x_2}^2)\,dx + \\ &+ \int\limits_{S_\tau} \alpha_3^{-1}\big[(\alpha_3 u_{x_1} - \alpha_1 u_t)^2 + (\alpha_3 u_{x_2} - \alpha_2 u_t)^2\big]\,ds. \end{aligned} \tag{11}$$

Putting

$$W(\tau) = \int\limits_{D_{0\tau}} (u_t^2 + u_{x_1}^2 + u_{x_2}^2)\,dx, \quad \widetilde{u}_i = \alpha_3 u_{x_i} - \alpha_i u_t, \quad i = 1,2,$$

from (11) we have

$$W(\tau) \leq \frac{\sqrt{1+k^2}}{|k|} \int\limits_{S_{1\tau}} (\widetilde{u}_1^2 + \widetilde{u}_2^2) ds + \sqrt{2} \int\limits_{S_{2\tau}} (\widetilde{u}_1^2 + \widetilde{u}_2^2) ds +$$

$$+ \int\limits_{D_\tau} (F^2 + u_t^2) dx dt \leq \frac{\sqrt{1+k^2}}{|k|} \int\limits_{S_{1\tau}} (\widetilde{u}_1^2 + \widetilde{u}_2^2) ds + \sqrt{2} \int\limits_{S_{2\tau}} (\widetilde{u}_1^2 + \widetilde{u}_2^2) ds +$$

$$+ \int\limits_0^\tau d\xi \int\limits_{D_{0\xi}} u_t^2 dx + \int\limits_{D_\tau} F^2 dx dt \leq \frac{\sqrt{1+k^2}}{|k|} \int\limits_{S_\tau} (\widetilde{u}_1^2 + \widetilde{u}_2^2) ds +$$

$$+ \int\limits_0^\tau W(\xi) d\xi + \int\limits_{D_\tau} F^2 dx dt. \tag{12}$$

Let (x, τ_x) be a point of intersection of the surface $S_1 \cup S_2$ with a straight line parallel to the axis t and passing through the point $(x, 0)$. We have

$$u(x,\tau) = u(x,\tau_x) + \int\limits_{\tau_x}^{\tau} u_t(x,t) dt,$$

whence it follows that

$$\int\limits_{D_{0\tau}} u^2(x,\tau) dx \leq 2 \int\limits_{D_{0\tau}} u^2(x,\tau_x) dx +$$

$$+2|\tau - \tau_x| \cdot \int\limits_{D_{0\tau}} dx \int\limits_{\tau_x}^{\tau} u_t^2(x,t) dt = 2 \int\limits_{S_\tau} \alpha_3^{-1} u^2 ds +$$

$$+2|\tau - \tau_x| \int\limits_{D_\tau} u_t^2 dx dt \leq C_k \Big(\int\limits_{S_\tau} u^2 ds + \int\limits_{D_\tau} u_t^2 dx dt \Big), \tag{13}$$

where $C_k = 2 \max \Big(\frac{\sqrt{1+k^2}}{|k|}, t_0 \Big)$.

Introducing the notation

$$W_0(\tau) = \int\limits_{D_{0\tau}} (u^2 + u_t^2 + u_{x_1}^2 + u_{x_2}^2) dx$$

and adding the inequalities (12) and (13) we obtain

$$W_0(\tau) \leq C_k \Big[\int\limits_{S_\tau} (u^2 + \widetilde{u}_1^2 + \widetilde{u}_2^2) ds + \int\limits_0^\tau W_0(\xi) d\xi + \int\limits_{D_\tau} F^2 dx dt \Big]$$

from which by Gronwall's lemma we find that

$$W_0(\tau) \le C_{1k}\Big[\int\limits_{S_\tau}(u^2+\widetilde{u}_1^2+\widetilde{u}_2^2)ds+\int\limits_{D_\tau}F^2dxdt\Big]. \tag{14}$$

We can easily see that $\alpha_3\frac{\partial}{\partial x_i}-\alpha_i\frac{\partial}{\partial t}$ is the interior differential operator on the surface S_τ. Therefore, by virtue of (2), the inequality

$$\int\limits_{S_\tau}(u^2+\widetilde{u}_1^2+\widetilde{u}_2^2)ds \le \widetilde{C}_3\sum_{i=1}^{2}\|f\|^2_{W_2^1(S_{i\tau})} \tag{15}$$

is valid.

It follows from (14) and (15) that

$$W_0(\tau) \le C_{2k}\Big(\sum_{i=1}^{2}\|f_i\|^2_{W_2^1(S_{i\tau})}+\|F\|^2_{L_2(D_\tau)}\Big). \tag{16}$$

Integrating both parts of the inequality (16) with respect to τ, we obtain the estimate (9). □

Remark. It is easy see that the a priori estimate (9) is also valid for a function u of the class $W_2^2(D)$, since the space $C_*^\infty(\overline{D})$ is everywhere a dense subset of the space $W_2^2(D)$. It should be noted that the constant C in (9) tends to infinity for $k\to 0$ and it becomes, generally speaking, invalid in the limit for $k=0$, i.e. for $S_1 : x_2 = 0,\ 0\le t\le t_0$. At the same time, following the proof of Lemma 1, we can see that the estimate (9) is also valid for $k=0$ if $f_1 = u|_{S_1} = 0$.

The following theorem holds.

Theorem 1. *Let $-1<k<0$. Then for every $f_i\in W_2^1(S_i)$, $i=1,2$, $F\in L_2(D)$ there exists a unique strong solution of the problem* (3), (2) *of the class W_2^1 for which the estimate* (9) *is valid.*

Proof. It is known that the spaces $C_*^\infty(\overline{D})$ and $C_*^\infty(S_i)$, $i=1,2$, are dense everywhere in the spaces $L_2(D)$ and $W_2^1(S_i)$, $i=1,2$, respectively. Therefore there exist sequences $F_n\in C_*^\infty(D)$ and $f_{in}\in C_*^\infty(S_i)$, $i=1,2$, such that

$$\lim_{n\to\infty}\|F-F_n\|_{L_2(D)}=\lim_{n\to\infty}\|f_i-f_{in}\|_{W_2^1(S_i)}=0, \quad i=1,2. \tag{17}$$

Moreover, because of the condition $(f_1-f_2)|_{S_1\cap S_2}=0$, the sequences f_{1n} and f_{2n} can be chosen so that

$$(f_{1n}-f_{2n})|_{S_1\cap S_2}=0, \quad n=1,2,\dots .$$

According to the integral representation (8) of the regular solutions of the problem (3),(2), there exists a sequence $u_n \in C_*^\infty(\overline{D})$ of solutions of that problem for $F = F_n$, $f_i = f_{in}$, $i = 1,2$.

By virtue of the inequality (9) we have

$$\|u_n - u_m\|_{W_2^1(D)} \leq$$
$$\leq C\Big(\sum_{i=1}^{2} \|f_{in} - f_{im}\|_{W_2^1(S_i)} + \|F_n - F_m\|_{L_2(D)}\Big). \tag{18}$$

It follows from (17) and (18) that the sequence u_n of the functions is fundamental in the space $W_2^1(D)$. Therefore, since the space $W_2^1(D)$ is complete, there exists a function $u \in W_2^1(D)$ such that $u_n \to u$, $\Box u_n \to F$, and $u_n|_{S_i} \to f_i$ in $W_2^1(D)$, $L_2(D)$, and $W_2^1(S_i)$, $i = 1,2$, respectively, for $n \to \infty$. Hence the function u is the strong solution of the problem (3),(2) of the class W_2^1. The uniqueness of the strong solution of the problem (3),(2) of the class W_2^1 follows from the inequality (9). $\square$

Remark. Theorem 1 remains also valid for $k = 0$, i.e., for $S_1 : x_2 = 0$, $0 \leq t \leq t_0$ if $f_1 = u|_{S_1} = 0$.

Now for the problem (3),(2) let us introduce the notion of a weak solution of the class W_2^1. Put $S_3 = \partial D \cap \{t = t_0\}$, $V = \{v \in W_2^1(D) : v|_{S_1 \cup S_3} = 0\}$.

Definition. Let $f_i \in W_2^1(S_i)$, $i = 1,2$, $F \in L_2(D)$. The function $u \in W_2^1(D)$ is said to be a weak solution of the problem (3),(2) of the class W_2^1 if it satisfies both the boundary conditions (2) and the identity

$$\int_D (u_t v_t - u_{x_1} v_{x_1} - u_{x_2} v_{x_2}) dx dt + \int_{S_2} \frac{\partial f_2}{\partial N} v ds + \int_D F v dx dt = 0 \tag{19}$$

for any $v \in V$, where $\frac{\partial}{\partial N}$ is a derivative with respect to a conormal to S_2.

Obviously, every strong solution of the problem (3),(2) of the class W_2^1 is a weak solution of the same class.

Lemma 2. *For $k = 0$, i.e., for $S_1 : x_2 = 0$, $0 \leq t \leq t_0$ the problem* (3), (2) *cannot have more than one weak solution of the class W_2^1.*

Proof. Let the function $u \in W_2^1(D)$ satisfy the identity (19) for $u|_{S_i} = f_i = 0$, $i = 1,2$, $F = 0$. In this identity we take as v the function

$$v(x_1, x_2, t) = \begin{cases} 0 \quad \text{for} \quad t \geq \tau, \\ \int\limits_{\tau}^{t} u(x_1, x_2, \sigma) d\sigma \quad \text{for} \quad |x_2| \leq t \leq \tau, \end{cases} \tag{20}$$

where $0 < \tau \leq t_0$.

Obviously, $v \in V$ and

$$v_t = u, \quad v_{x_i} = \int_{\tau}^{t} u_{x_i}(x_1, x_2, \sigma) d\sigma, \quad i = 1, 2, \tag{21}$$
$$v_{tx_i} = u_{x_i}, \quad v_{tt} = u_t.$$

By virtue of (20) and (21), the identity (19) for $f_2 = 0$, $F = 0$ takes the form

$$\int_{D_\tau} (v_{tt} v_t - v_{tx_1} v_{x_1} - v_{tx_2} v_{x_2}) dx dt = 0$$

or

$$\int_{D_\tau} \frac{\partial}{\partial t} (v_t^2 - v_{x_1}^2 - v_{x_2}^2) dx dt = 0, \tag{22}$$

where $D_\tau = D \cap \{t < \tau\}$.

Using the Gauss–Ostrogradsky formula on the left-hand side of (22), we obtain

$$\int_{\partial D_\tau} (v_t^2 - v_{x_1}^2 - v_{x_2}^2) \cos \widehat{nt} ds = 0. \tag{23}$$

Since $\partial D_\tau = S_{1\tau} \cup S_{2\tau} \cup S_{3\tau}$, where $S_{i\tau} = \partial D_\tau \cap S_i$, $i = 1, 2$, $S_{3\tau} = \partial D_\tau \cap \{t = \tau\}$ and

$$\cos \widehat{nt}|_{S_{1\tau}} = 0, \quad \cos \widehat{nt}|_{S_{2\tau}} = -\frac{1}{\sqrt{2}}, \quad \cos \widehat{nt}|_{S_{3\tau}} = 1,$$
$$u|_{S_{i\tau}} = f_i = 0, \quad i = 1, 2, \quad v_{x_i}|_{S_{3\tau}} = 0, \quad i = 1, 2, \quad v_t = u,$$

it follows from (23) that

$$\int_{S_{3\tau}} u^2 dx_1 dx_2 + \frac{1}{\sqrt{2}} \int_{S_{2\tau}} (v_{x_1}^2 + v_{x_2}^2) ds = 0.$$

Hence, $u|_{S_{3\tau}} = 0$ for any τ from the interval $(0, t_0]$. Therefore, $u \equiv 0$ in the domain D. □

Due to the fact that the strong solution of the problem (3),(2) of the class W_2^1 is at the same time a weak solution of the class W_2^1, from Lemma 2 and the remark following after Theorem 1 we have

Theorem 2. *Let $k = 0$, i.e., $S_1 : x_2 = 0$, $0 \le t \le t_0$ and $u|_{S_1} = f_1 = 0$. Then for any $f_2 \in W_2^1(S_2)$ and $F_2 \in L_2(D)$ there exists a unique weak solution u of the problem* (3), (2) *of the class W_2^1 for which the estimate* (9) *is valid.*

To prove the solvability of the problem (1),(2) we shall use the solvability of the problem (3),(2) and the fact that in the specifically chosen equivalent norms of the spaces $L_2(D)$, $W_2^1(D)$, $W_2^1(S_i)$, $i = 1, 2$, the lowest terms in equation (1) give arbitrarily small perturbations.

Introduce in the space $W_2^1(D)$ an equivalent norm depending on the parameter γ,

$$\|u\|^2_{D,1,\gamma} = \int\limits_D e^{-\gamma t}(u^2 + u_t^2 + u_{x_1}^2 + u_{x_2}^2)dxdt, \quad \gamma > 0.$$

In the same manner we introduce the norms $\|F\|_{D,0,\gamma}$, $\|f_i\|_{S_i,1,\gamma}$ in the spaces $L_2(D)$, $W_2^1(S_i)$, $i = 1, 2$.

Making use of the inequality (16), we obtain the a priori estimate for $u \in C_*^\infty(D)$ with respect to the norms $\|\cdot\|_{D,1,\gamma}$, $\|\cdot\|_{S_{i,1,\gamma}}$, $i = 1, 2$. Multiplying both parts of the inequality (16) by $e^{-\gamma t}$ and integrating the obtained inequality with respect to τ from 0 to t_0 we get

$$\|u\|^2_{D,1,\gamma} = \int\limits_0^{t_0} e^{-\gamma\tau} W_0(\tau)d\tau \le C_{2k}\Big(\sum_{i=1}^2 \int\limits_0^{t_0} e^{-\gamma t}\|f_i\|^2_{W_2^1(S_{i\tau})}d\tau + \\ + \int\limits_0^{t_0} e^{-\gamma\tau}\|F\|^2_{L_2(D_\tau)}d\tau\Big). \tag{24}$$

We have

$$\int\limits_0^{t_0} e^{-\gamma t}\|F\|^2_{L_2(D_\tau)}d\tau = \int\limits_0^{t_0} e^{-\gamma t}\Big[\int\limits_0^{\tau}\Big(\int\limits_{D_{0\sigma}} F^2dx\Big)d\sigma\Big]d\tau = \\ = \int\limits_0^{t_0}\Big[\int\limits_{D_{0\sigma}} F^2dx\int\limits_\sigma^{t_0} e^{-\gamma\tau}d\tau\Big]d\sigma = \frac{1}{\gamma}\int\limits_0^{t_0}(e^{-\gamma\sigma} - e^{-\gamma t_0})\Big[\int\limits_{D_{0\sigma}} F^2dx\Big]d\sigma \le \\ \le \frac{1}{\gamma}\int\limits_0^{t_0} e^{-\gamma\sigma}\Big[\int\limits_{D_{0\sigma}} F^2dx\Big]d\sigma = \frac{1}{\gamma}\|F\|^2_{D,0,\gamma}, \tag{25}$$

where $D_{0\tau} = \partial D_\tau \cap \{t = \tau\}$, $0 < \tau \le t_0$.

Analogously we obtain

$$\int_0^{t_0} e^{-\gamma\tau}\|f_i\|^2_{W_2^1(S_{i\tau})}d\tau \leq \frac{C_3}{\gamma}\|f_i\|^2_{S_i,1,\gamma}, \quad i=1,2, \tag{26}$$

where C_3 is a positive constant independent of f_i and the parameter γ.

From the inequalities (24)–(26) we have the a priori estimate for $u \in C_*^\infty(\overline{D})$

$$\|u\|_{D,1,\gamma} \leq \frac{C_4}{\sqrt{\gamma}}\Big(\sum_{i=1}^{2}\|f_i\|_{S_i,1,\gamma}+\|F\|_{D,0,\gamma}\Big) \tag{27}$$

for $-1<k<0$, where $C_4=\text{const}>0$ does not depend on u and the parameter γ.

Below, the coefficients a,b,c, and d in equation (1) are assumed to be bounded measurable functions in the domain D.

Consider the space

$$V=L_2(D)\times W_2^1(S_1)\times W_2^1(S_2).$$

To the problem (1),(2) there corresponds an unbounded operator

$$T: W_2^1(D)\to V$$

with the domain of definition $\Omega_T = C_*^\infty(D)\subset W_2^1(D)$, acting by the formula

$$Tu=(Lu, u|_{S_1}, u|_{S_2}), \quad u\in\Omega_T.$$

We can easily prove that the operator T admits a closure $\overline{T}$. In fact, let $u_n\in\Omega_T$, $u_n\to 0$ in $W_2^1(D)$ and $Tu_n\to(F,f_1,f_2)$ in the space V. First we shall show that $F=0$. For $\varphi\in C_0^\infty(D)$ we have

$$(Lu_n,\varphi)=(u_n,\Box\varphi)+(Ku,\varphi), \tag{28}$$

where $Ku=au_{x_1}+bu_{x_2}+cu_t+du$. Since $u_n\to 0$ in $W_2^1(D)$, it follows from (28) that $(Lu_n,\varphi)\to 0$. On the other hand, by the definition of a strong solution, we have the convergence $Lu_n\to F$ in $L_2(D)$. Therefore $(f,\varphi)=0$ for any $\varphi\in C_0^\infty(D)$, and hence, $F=0$. That $f_1=f_2=0$ follows from the fact that $u_n\to 0$ in $W_2^1(D)$ and the contraction operator $u\to(u|_{S_1},u|_{S_2})$ acts boundedly from $W_2^1(D)$ to $L_2(S_1)\times L_2(S_2)$.

To the problem (3),(2) there corresponds an unbounded operator T_0 : $W_2^1(D)\to V$ obtained from the operator T for $a=b=c=d=0$. As was shown above, the operator T_0 also admits a closure $\overline{T}_0$. Obviously, the operator $K_0: W_2^1(D)\to V$ acting by the formula $K_0u=(Ku,0,0)$ is bounded and

$$T=T_0+K_0. \tag{29}$$

Note that the domains of definition $\Omega_{\overline{T}}$ and $\Omega_{\overline{T}_0}$ of the closed operators $\overline{T}$ and $\overline{T}_0$ coincide by virtue of (29) and the fact that the operator K_0 is bounded.

We can easily see that the existence and uniqueness of the strong solution of the problem (1),(2) of the class W_2^1 as well as the estimate (9) for this solution follow from the existence of the bounded right operator $\overline{T}^{-1}$ inverse to $\overline{T}$ and defined in a whole space V.

The fact that the operator $\overline{T}_0$ has a bounded right inverse operator $\overline{T}_0^{-1}$: $V \to W_2^1(D)$ for $-1 < k < 0$ follows from Theorem 1 and the estimate (9) which, as we have shown above, can be written in equivalent norms in the form of (27). It is easy to see that the operator

$$K_0\overline{T}_0^{-1} : V \to V$$

is bounded and by virtue of (27) its norm admits the following estimate

$$\|K_0\overline{T}_0^{-1}\| \leq \frac{C_4C_5}{\sqrt{\gamma}}, \tag{30}$$

where C_5 is a positive constant depending only on the coefficients a, b, c, and d of equation (1).

Taking into account (30), we note that the operator $(I+K_0\overline{T}_0^{-1}) : V \to V$ has a bounded inverse operator $(I+K_0\overline{T}_0^{-1})^{-1}$ for sufficiently large γ, where I is the unit operator. Now it remains only to note that the operator

$$\overline{T}_0^{-1}(I + K_0\overline{T}_0^{-1})^{-1}$$

is a bounded operator right inverse to $\overline{T}$ and defined in a whole space V.

Thus the following theorem is proved.

Theorem 3. *Let $-1 < k < 0$. Then for any $f_i \in W_2^1(S_i)$, $i = 1, 2$, $F \in L_2(D)$ there exists a unique strong solution u of the problem* (1), (2) *of the class W_2^1 for which the estimate* (9) *is valid.*

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