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BASIC BOUNDARY VALUE PROBLEMS OF THERMOELASTICITY FOR ANISOTROPIC BODIES WITH CUTS. II

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ABSTRACT. In the first part [1] of the paper the basic boundary value problems of the mathematical theory of elasticity for three-dimensional anisotropic bodies with cuts were formulated. It is assumed that the two-dimensional surface of a cut is a smooth manifold of an arbitrary configuration with a smooth boundary. The existence and uniqueness theorems for boundary value problems were formulated in the Besov ($\mathbb{B}^s_{p,q}$) and Bessel-potential ($\mathbb{H}^s_p$) spaces. In the present part we give the proofs of the main results (Theorems 7 and 8) using the classical potential theory and the nonclassical theory of pseudodifferential equations on manifolds with a boundary.

This paper continues [1]. After recalling some auxiliary results, we prove Theorems 7 and 8 formulated in §3.

§ 4. Auxiliary Results

4.1. Convolution Operators. $\mathbb{S}(\mathbb{R}^n)$ denotes the space of C^∞-smooth fast decaying functions, while $\mathbb{S}'(\mathbb{R}^n)$ stands for the dual space of tempered distributions. The Fourier transform and its inverse

$$\mathcal{F}\varphi(x)=\int_{\mathbb{R}^n} e^{ix\xi}\varphi(\xi)d\xi,\quad \mathcal{F}^{-1}\varphi(\xi)=(2\pi)^{-n}\int_{\mathbb{R}^n} e^{-ix\xi}\psi(x)dx$$

are continuous operators in both spaces $\mathbb{S}(\mathbb{R}^n)$ and $\mathbb{S}'(\mathbb{R}^n)$. Hence the convolution operator

$$\mathbf{a}(D)\varphi=\mathcal{F}^{-1}a\mathcal{F}\varphi,\quad a\in\mathbb{S}'(\mathbb{R}^n),\quad \varphi\in\mathbb{S}(\mathbb{R}^n) \tag{4.1}$$

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is a continuous transformation

$$\mathbf{a}(D) : \mathbb{S}(\mathbb{R}^n) \to \mathbb{S}'(\mathbb{R}^n)$$

(cf. [2], [3]).

If operator (4.1) has a bounded extension

$$\mathbf{a}(D) : \mathbb{L}_p(\mathbb{R}^n) \to \mathbb{L}_p(\mathbb{R}^n), \quad 1 \le p \le \infty,$$

we write $a \in M_p(\mathbb{R}^n)$ and $a(\xi)$ is called the (Fourier) L_p-multiplier. Let

$$M_p^{(r)}(\mathbb{R}^n) = \big\{(1+|\xi|^2)^{r/2} a(\xi) : a \in M_p(\mathbb{R}^n)\big\}.$$

Recall that the Bessel potential space $\mathbb{H}_p^s(\mathbb{R}^n)$ is defined as a subset of $\mathbb{S}'(\mathbb{R}^n)$ endowed with the norm

$$\begin{gathered} \|u|\mathbb{H}_p^s(\mathbb{R}^n)\| = \|\mathcal{I}^s(D)u|\mathbb{L}_p(\mathbb{R}^n)\|, \\ \mathcal{I}^s(\xi) := (1+|\xi|^2)^{s/2}. \end{gathered} \tag{4.2}$$

Therefore due to the obvious property

$$\mathbf{a}_1(D)\mathbf{a}_2(D) = (\mathbf{a}_1\mathbf{a}_2)(D), \quad a_j \in M_p^{(r_j)}(\mathbb{R}^n) \tag{4.3}$$

we easily find that the operator

$$\mathbf{a}(D) : \mathbb{H}_p^s(\mathbb{R}^n) \to \mathbb{H}_p^{s-r}(\mathbb{R}^n), \quad s, r \in \mathbb{R}, \quad 1 \le p \le \infty, \tag{4.4}$$

is bounded if and only if $a \in M_p^{(r)}(\mathbb{R}^n)$.

The interpolation property

$$\begin{gathered} \mathbb{B}_{p,q}^s(\mathbb{R}^n) = \big[\mathbb{H}_p^{s_1}(\mathbb{R}^n),\ \mathbb{H}_p^{s_2}(\mathbb{R}^n)\big]_{\theta,q}, \\ 1 < p < \infty, \quad 1 \le p \le \infty, \quad s_1, s_2 \in \mathbb{R}, \\ s = (1-\theta)s_1 + \theta s_2, \quad 0 \le \theta \le 1 \end{gathered} \tag{4.5}$$

(see [4], [5]) for $a \in M_p^{(r)}(\mathbb{R}^n)$ ensures the boundedness of the operator

$$\mathbf{a}(D) : \mathbb{B}_{p,q}^s(\mathbb{R}^n) \to \mathbb{B}_{p,q}^{s-r}(\mathbb{R}^n), \quad 1 \le q \le \infty. \tag{4.6}$$

Equality (4.2) and boundedness (4.4) imply that the operator

$$\mathcal{I}^r : \mathbb{H}_p^s(\mathbb{R}^n) \to \mathbb{H}_p^{s-r}(\mathbb{R}^n) \tag{4.7}$$

arranges an isometric isomorphism.

Further, it is well known that the operators

$$\begin{gathered} \mathcal{I}_+^r : \widetilde{\mathbb{H}}_p^s(\mathbb{R}_+^n) \to \widetilde{\mathbb{H}}_p^{s-r}(\mathbb{R}_+^n), \\ \mathcal{I}_-^r : \mathbb{H}_p^s(\mathbb{R}_+^n) \to \mathbb{H}_p^{s-r}(\mathbb{R}_+^n), \quad \mathcal{I}_\pm^r(\xi) = (\xi_n \pm i|\xi'| \pm i)^r, \\ \mathbb{R}_+^n := \mathbb{R}^{n-1} \times \mathbb{R}^+, \quad \mathbb{R}^+ := [0, +\infty), \quad \xi = (\xi', \xi^n) \in \mathbb{R}^n, \quad \xi' \in \mathbb{R}^{n-1}, \end{gathered} \tag{4.8}$$

also arrange isomorphisms (though not isometric ones; see, for example, [3], [6]). Isomorphisms similar to (4.8) exist for any smooth manifold with a Lipschitz boundary (for details see [3], [7]).

The equality $M_2(\mathbb{R}^n) = \mathbb{L}_\infty(\mathbb{R}^n)$ is well known and trivial. A reasonable description of the class $M_p^r(\mathbb{R}^n)$ for $p \neq 2$ is less trivial and the problem still remains unsolved.

Theorem 12 (see [8], Theorem 7.9.5; [9]). *Let* $1 < p < \infty$ *and*

$$\sum_{\substack{|\beta|<[n/2]+1 \\ 0\leq\beta\leq 1}} \sup\left\{|\xi^\beta D^\beta a(\xi)|,\ \xi \in \mathbb{R}^n\right\} \leq M < \infty,$$

where for the multi-index $\beta = (\beta_1, \dots, \beta_n)$ *the inequality* $0 \leq \beta \leq 1$ *reads as* $0 \leq \beta_j \leq 1$, $j = 1, \dots, n$. *Then* $a \in \bigcap_{1<p<\infty} M_p(\mathbb{R}^n)$.

If $a \in M_p^{(r)}(\mathbb{R}^n)$, the operators

$$\begin{aligned} \mathbf{r}_+\mathbf{a}(D) : \widetilde{\mathbb{H}}_p^s(\mathbb{R}_+^n) &\to \mathbb{H}_p^{s-r}(\mathbb{R}_+^n) \\ : \widetilde{\mathbb{B}}_{p,q}^s(\mathbb{R}_+^n) &\to \mathbb{B}_{p,q}^{s-r}(\mathbb{R}_+^n) \end{aligned} \tag{4.9}$$

are bounded ($1 < p < \infty$, $s, r \in \mathbb{R}$, $1 \leq q \leq \infty$); here $\mathbf{r}_+\varphi = \varphi\big|_{\mathbb{R}_+^n}$ denotes the restriction operator.

An equality similar to (4.3)

$$\mathbf{r}_+\mathbf{a}_1(D)\ell_0\mathbf{r}_+\mathbf{a}_2(D) = \mathbf{r}_+(\mathbf{a}_1\mathbf{a}_2)(D), \quad a_j \in M_p^{(r_j)}(\mathbb{R}^n), \tag{4.10}$$

where ℓ_0 is extension by 0 from $\mathbb{R}_+^n$ to $\mathbb{R}^n$, fails to be fulfilled in general. However, (4.10) holds if there is an analytic extension either $a_1(\xi', \xi_n - i\lambda)$ or $a_2(\xi', \xi_n + i\lambda)$, which can be estimated from above by $C(1 + |\xi| + \lambda)^N$ with $N > 0$, $\lambda > 0$, $C = const$.

4.2. Pseudodifferential operators. If the symbol $a(x, \xi)$ depends on the variable x, the corresponding convolution (cf. (4.1))

$$\mathbf{a}(x, D)\varphi(x) := \mathcal{F}_{\xi\to x}^{-1} a(x, \cdot)\mathcal{F}_{y\to\xi}\varphi(\xi) \tag{4.11}$$

is called the pseudodifferential operator ($\varphi \in \mathbb{S}(\mathbb{R}^n)$, $|a(x, \xi)| < C(1+|\xi|)^N$, $N > 0$, $C = const$).

Let $M_p^{(s,s-r)}(\mathbb{R}^n \times \mathbb{R}^n)$ denote a class of symbols $a(x, \xi)$ for which operator (4.11) can be extended to the bounded mapping

$$\mathbf{a}(x, D) : \mathbb{H}_p^s(\mathbb{R}^n) \to \mathbb{H}_p^{s-r}(\mathbb{R}^n). \tag{4.12}$$

By $S^r(\Omega\times\mathbb{R}^n)$ ($\Omega\subset\mathbb{R}^n$, $r\in\mathbb{R}$) is denoted the Hörmander class of symbols $a(x,\xi)$ if

$$\left|D_x^\alpha D_\xi^\beta a(x,\xi)\right|\le M_{\alpha,\beta}\big(1+|\xi|\big)^{r-|\beta|},\quad \forall x\in\Omega,\quad \forall\xi\in\mathbb{R}^n, \tag{4.13}$$

where $M_{\alpha,\beta}$ is independent of x and ξ.

By $S_r^{l,m}(\Omega\times\mathbb{R}^n)$ ($\Omega\subset\mathbb{R}^n$, $l,m\in\mathbb{Z}^+$, $r\in\mathbb{R}$) we denote the class of symbols $a(x,\xi)$ satisfying the estimates

$$\int\limits_\Omega \left|D_x^\alpha(\xi D_\xi)^\beta a(x,\xi)\right|dx\le M'_{\alpha,\beta}\big(1+|\xi|\big)^r$$
$$\forall\xi\in\mathbb{R}^n,\quad |\alpha|\le l,\quad |\beta|\le m,$$

where

$$(\xi D_\xi)^\beta:=(\xi_1 D_{\xi_1})^{\beta_1}\dots(\xi_n D_{\xi_n})^{\beta_n}.$$

If $\Omega\subset\mathbb{R}^n$ is compact, then $S^r(\Omega\times\mathbb{R}^n)\subset S_r^{l,m}(\Omega\times\mathbb{R}^n)$. Such an inclusion does not hold for non-compact Ω.

Theorem 13. *Let $s,r\in\mathbb{R}$, $l,m\in\mathbb{Z}^+$, $m>[n/2]+1$; then*

$$S^r(\mathbb{R}^n\times\mathbb{R}^n)\subset M_p^{(s,s-r)}(\mathbb{R}^n\times\mathbb{R}^n).$$

If, additionally, $-l+1+1/p<s-r<l+1/p$, then

$$S_r^{l+n,m}(\mathbb{R}^n\times\mathbb{R}^n)\subset M_p^{(s,s-r)}(\mathbb{R}^n\times\mathbb{R}^n).$$

Proof. When a symbol $a\in S^0(\mathbb{R}^n\times\mathbb{R}^n)$ has a compact support with respect to x, then the continuity of $\mathbf{a}(x,D)$ in $\mathbb{L}_p(\mathbb{R}^n)$ follows from Theorem 12, as shown in [10].

For an arbitrary $a\in S^0(\mathbb{R}^n\times\mathbb{R}^n)$ the above statement is proved for $\mathbb{L}_p(\mathbb{R}^n)$ using the arguments involved in the proof of Theorem 3.5 from [12]. In the general case the continuity of the mapping $\mathbb{H}_p^s(\mathbb{R}^n)\to\mathbb{H}_p^{s-r}(\mathbb{R}^n)$ is established with the aid of the order reduction operator (4.7) (see [4], [10]), while the continuity of the mapping $\mathbf{a}(x,D):\mathbb{B}_{p,q}^s(\mathbb{R}^n)\to\mathbb{B}_{p,q}^{s-r}(\mathbb{R}^n)$ is proved by interpolation (see [4]).

For a different proof of the first claim see [11].

To prove the second claim we shall introduce some notation. For a multi-index $\mu=(\mu_1,\dots,\mu_n)$, $0\le\mu\le1$ we define

$$dx^\mu:=\prod_{\substack{\mu_j=1\\ j=1,2,\dots,n}}dx_j,\quad (x,h)_\mu:=(z_1,\dots,z_n),$$
$$z_j=\begin{cases}x_j, & \text{if}\quad \mu_j=1,\\ h_j, & \text{if}\quad \mu_j=0,\end{cases}\qquad x,h\in\mathbb{R}^n.$$

Let

$$a_{(\alpha)}(x,\xi) := D_x^\alpha a(x,\xi).$$

By virtue of Theorem 12 the inclusion $a \in S_r^{l,m}(\mathbb{R}^n \times \mathbb{R}^n)$ implies

$$\int_{\mathbb{R}^n} \big\| D_x^\alpha a(x,\cdot) \big| M_p^{(r)}(\mathbb{R}^n) \big\| dx < \infty, \quad |\alpha| \le l+n.$$

From this finiteness and Fubini's theorem we get

$$\operatorname{mes}_{\mathbb{R}^n} \Delta_{\mu,\gamma} = 0 \quad \text{for any} \quad 0 \le \mu \le 1, \quad |\gamma| \le l,$$

where

$$\Delta_{\mu,\gamma} := \Big\{ h \in \mathbb{R}^n : \int_{\mathbb{R}^{|\mu|}} \big\| a_{(\mu+\gamma)}\big((y,h)_\mu,\cdot\big) \big| M_p^{(r)}(\mathbb{R}^n) \big\| dy^\mu = \infty \Big\}.$$

If now

$$\Delta = \bigcup_{\substack{0 \le \mu \le 1 \\ |\gamma| \le l}} \Delta_{\mu,\gamma}$$

then, obviously, $\operatorname{mes}_{\mathbb{R}^n} \Delta = 0$. There exists a vector $h_0 \in \mathbb{R}^n \backslash \Delta$. Then we have

$$\int_{\mathbb{R}^n} \big\| a_{(\mu+\gamma)}\big((y,h_0)_\mu,\cdot\big) \big| M_p^{(r)}(\mathbb{R}^n) \big\| dy^\mu < \infty.$$

With these conditions we can use Theorem 5.1 and Remark 5.5 from [20] where the claimed inclusion $a \in M_p^{(s,s-r)}(\mathbb{R}^n \times \mathbb{R}^n)$ is proved. □

Let

$$\mathbf{A}, \mathbf{B} : \mathbb{H}_p^s(\mathbb{R}^n) \to \mathbb{H}_p^{s-r}(\mathbb{R}^n)$$

be the bounded operators; they are called locally equivalent at $x_0 \in \mathbb{R}^n$ (see [3], [13]) if

$$\inf \big\{ \|\chi(\mathbf{A}-\mathbf{B})\| : \chi \in C_{x_0}(\mathbb{R}^n) \big\} = \inf \big\{ \|(\mathbf{A}-\mathbf{B})\chi\mathbf{I}\| : \chi \in C_{x_0}(\mathbb{R}^n) \big\} = 0,$$

where $\mathbf{I}$ is the identity operator and $C_{x_0}(\mathbb{R}^n) = \{\chi \in C_0^\infty(\mathbb{R}^n) : \chi(x) = 1$ in some neighborhood of $x_0\}$. In such a case we write $\mathbf{A} \overset{x_0}{\sim} \mathbf{B}$. In a similar manner we define the equivalence $\mathbf{A}_0 \overset{x_0}{\sim} \mathbf{B}_0$ for operators

$$\mathbf{A}_0, \mathbf{B}_0 : \widetilde{\mathbb{H}}_p^s(\mathbb{R}_+^n) \to \mathbb{H}_p^{s-r}(\mathbb{R}_+^n).$$

Assume now that $\overline{S} = S \cup \partial S$ is a compact n-dimensional C^∞-smooth manifold with a C^∞-smooth boundary ∂S and

$$S = \bigcup_{j=1}^{N} V_j, \quad \varkappa_j : X_j \to V_j, \quad X_j \subset \mathbb{R}_+^n \tag{4.14}$$

are coordinate diffeomorphisms. Let $\{\chi_j\}_1^N \subset C_0^\infty(S)$ be a partition of the unity subordinated to the covering of S in (4.14); also let

$$\varkappa_{j_*}\varphi(t) = \chi_j^0\varphi\big(\chi_j(t)\big), \quad \varkappa_{j^*}^{-1}\psi(x) = \chi_j\psi\big(\varkappa_j^{-1}(x)\big),$$

where $\chi_j^0(t) := \chi_j(\varkappa_j(t))$, $t \in \mathbb{R}^n_+$, $x \in S$. The following mapping properties

$$\begin{aligned} \varkappa_{j_*} &: \mathbb{H}^r_p(S) \to \mathbb{H}^r_p(\mathbb{R}^n_+), \quad \operatorname{supp} \varkappa_j^{-1} \cap \partial S \neq \varnothing, \\ \varkappa_{j_*} &: \widetilde{\mathbb{H}}^r_p(S) \to \widetilde{\mathbb{H}}^r_p(\mathbb{R}^n_+), \quad \operatorname{supp} \varkappa_j^{-1} \cap \partial S \neq \varnothing, \\ \varkappa_{j_*} &: \mathbb{H}^r_p(S) \to \mathbb{H}^r_p(\mathbb{R}^n), \quad \operatorname{supp} \varkappa_j^{-1} \cap \partial S = \varnothing. \end{aligned} \tag{4.15}$$

are almost evident.

A bounded operator

$$\mathbf{A} : \widetilde{\mathbb{H}}^\nu_p(S) \to \mathbb{H}^{\nu-r}_p(S) \tag{4.16}$$

is called pseudodifferential (of order r) if:

(i) $\chi_1\mathbf{A}\chi_2\mathbf{I}$ is a compact operator in $\widetilde{\mathbb{H}}^r_p(S) \to \mathbb{H}^{\nu-r}_p(S)$ for any $\chi_1, \chi_2 \in C_0^\infty(S)$ with disjoint supports $\operatorname{supp}\chi_1 \cap \operatorname{supp}\chi_2 = \varnothing$;

(ii)

$$\begin{aligned} \varkappa_{j^*}\mathbf{A}\varkappa_{j^*}^{-1} &\overset{x_0}{\sim} \mathbf{a}(x_0, D), \quad x_0 \in S, \\ \varkappa_{j^*}\mathbf{A}\varkappa_{j^*}^{-1} &\overset{x_0}{\sim} \mathbf{r}_+\mathbf{a}(x_0, D), \quad x_0 \in \partial S, \end{aligned} \tag{4.17}$$

where $a(x_0, \cdot) \in M_p^{(r)}(\mathbb{R}^n)$ for any $x_0 \in \overline{S}$.

Example 14 (see [3], Example 3.19]). . Let $\overline{\Omega} \subset \mathbb{R}^n$ be a compact domain with a smooth boundary $\partial\Omega \neq \varnothing$.

The operator $\mathbf{r}_\Omega\mathbf{a}(x, D)$, where $a(x, \xi) \in S^r(\Omega \times \mathbb{R}^n)$ and $\mathbf{r}_\Omega\varphi = \varphi\big|_\Omega$ denotes the restriction, is a pseudodifferential one of order r and

$$\begin{aligned} \mathbf{r}_\Omega\mathbf{a}(x, D) &\overset{x_0}{\sim} \mathbf{a}(x_0, D), \quad x_0 \notin \partial\Omega, \\ \mathbf{r}_\Omega\mathbf{a}(x, D) &\overset{x_0}{\sim} \mathbf{r}_+\mathbf{a}(x_0, D), \quad x_0 \in \partial\Omega. \end{aligned} \tag{4.18}$$

If $a(x_0, \xi)$ has the radial limits

$$a^\infty(x_0, \xi) = \lim_{\lambda\to\infty} \lambda^{-r} a(x_0, \lambda\xi) \tag{4.19}$$

which are nontrivial bounded functions of ξ, then $a^\infty(x_0, \xi)$ is a homogeneous function of order r with respect to ξ:

$$a^\infty(x_0, \lambda\xi) = \lambda^r a^\infty(x_0, \xi), \quad \lambda > 0.$$

Let

$$a^0(x_0, \xi) = a^\infty\big(x_0, (1 + |\xi'|)|\xi'|^{-1}\xi', \xi_n\big) \tag{4.20}$$

represent the modified symbol (see [6], Section 3). Assume that $a^0 \in M_p^{(r)}(\mathbb{R}^n)$; then using (4.17) and the relation

$$\lim_{R\to\infty}\sup_{|\xi|\geq R}|\xi|^{-r}\big|a(x_0,\xi)-a^0(x_0,\xi)\big|=0$$

we obtain

$$\begin{aligned}\varkappa_{j_*}\mathbf{A}\varkappa_{j^*}^{-1} &\overset{x_0}{\sim} \mathbf{a}^0(x_0,D), \quad x_0\notin\Omega,\\ \varkappa_{j_*}\mathbf{A}\varkappa_{j^*}^{-1} &\overset{x_0}{\sim} \mathbf{r}_+\mathbf{a}^0(x_0,D), \quad x_0\in\Omega.\end{aligned} \tag{4.21}$$

Thus the operators $\chi[\mathbf{a}(x_0,D)-\mathbf{a}^0(x_0,D)]$, $[\mathbf{a}(x_0,D)-\mathbf{a}^0(x_0,D)]\chi\mathbf{I}$ with $\chi\in C_0^\infty(\mathbb{R}^n)$ are compact in $\mathbb{H}_p^\nu(\mathbb{R}^n)\to\mathbb{H}_p^{\nu-r}(\mathbb{R}^n)$ (see [3]). As for the compact operator $\mathbf{T}:\mathbb{H}_p^\nu(\mathbb{R}^n)\to\mathbb{H}_p^{\nu-r}(\mathbb{R}^n)$, the equivalence $\mathbf{T}\overset{x_0}{\sim}\mathbf{0}$ holds automatically.

The functions $a^\infty(x_0,\xi)$ (see (4.19)) and $a^0(x_0,\xi)$ (see (4.20)) are respectively called the homogeneous principal symbol and the modified principal symbol of the operator $\mathbf{A}$.

Theorem 15 (see [3])). *Let* (4.16) *be a pseudodifferential operator* $(r,\nu\in\mathbb{R}, 1<p<\infty)$. $\mathbf{A}$ *is a Fredholm operator if and only if the following conditions are fulfilled:*

(i) $\inf\{|\det a^\infty(x_0,\xi)| : x_0\in\overline{S},\ \xi\in\mathbb{R}^n\}>0$;

(ii) $\mathbf{r}_+\mathbf{a}_{\nu,r}(x_0,D)$ *is a Fredholm operator in the space* $\mathbb{L}_p(\mathbb{R}^n_+)$ *for any* $x_0\in\partial S$, *where*

$$a_{\nu,r}(x_0,\xi)=\big(\xi_n-i|\xi'|-i\big)^{\nu-r}a^0(x_0,\xi)\big(\xi_n+i|\xi'|+i\big)^{-\nu},\\ \xi=(\xi',\xi_n),\quad \xi'\in\mathbb{R}^{n-1}.$$

Theorem 16 (see [3]). *Let* $\mathbf{a}(x,D)$ *be a pseudodifferential operator of the order* $r\in\mathbb{R}$ *with the* $N\times N$ *matrix symbol* $a(x,\cdot)\in S^r(\mathbb{R}^n)$ *for any* $x\in\overline{S}$. *If* $a(x,\xi)$ *is positive definite, i.e.,*

$$\begin{aligned}&\big(a(x,\xi)\eta,\eta\big)\geq\delta_0|\xi|^r|\eta|^2 \quad \textit{for some}\quad \delta_0>0\\ &\textit{and any}\quad \xi\in\mathbb{R}^n,\quad x\in\overline{S},\quad \eta\in\mathbb{C}^N,\end{aligned} \tag{4.22}$$

then

$$\mathbf{a}(x,D):\widetilde{\mathbb{H}}_2^{\frac{r}{2}+\nu}(S)\to\mathbb{H}_2^{-\frac{r}{2}+\nu}(S) \tag{4.23}$$

is a Fredholm operator for any $|\nu|<\frac12$ *and*

$$\operatorname{Ind}\mathbf{a}(x,D)=0. \tag{4.24}$$

4.3. Further Auxiliary Results. Let $\mathcal{H}^r(\mathbb{R}^n)$ denote the class of functions with the properties

(i) $a(\lambda\xi) = \lambda^r a(\xi)$, $\lambda > 0$, $\xi \in \mathbb{R}^n$;

(ii) $a \in C^\infty(S^{n-1})$, $S^{n-1} := \{\omega \in \mathbb{R}^n : |\omega| = 1\}$;

(iii) if $a(\xi) = a_0(\omega', t, \xi_n)$, where $\omega' = |\xi'|^{-1}\xi'$, $t = |\xi'|$, $\xi = (\xi', \xi_n) \in \mathbb{R}^n$, then

$$\begin{gathered}\lim_{t\to 0} D_t^k a_0(\omega', t, -1) = (-1)^k \lim_{t\to 0} D_t^k a_0(\omega', t, 1),\\ \omega' \in S^{n-2}, \quad k = 0, 1, 2, \dots .\end{gathered} \tag{4.25}$$

For $r = 0$ condition (4.25) coincides with the well-known transmission property (see [6,14]).

Lemma 17. *Let $a \in \mathcal{H}^r(\mathbb{R}^n)$ be a positive definite $N\times N$ matrix-function (cf.* (4.22))

$$\begin{gathered}\big(a(\xi)\eta, \eta\big) \geq \delta_0 |\xi|^r |\eta|^2 \quad \textit{for some} \quad \delta_0 > 0\\ \textit{and any} \quad \xi \in \mathbb{R}^n, \quad \eta \in \mathbb{C}^N.\end{gathered} \tag{4.26}$$

Then $a(\xi)$ admits the factorization

$$a(\xi) = a_-(\xi) a_+(\xi), \quad a_\pm(\xi) = \big(\xi_n \pm i|\xi'|\big)^{-\frac{r}{2}} b_\pm(\xi), \tag{4.27}$$

where $b_+^{\pm 1}(\xi', \xi_n + i\lambda)$, $b_-^{\pm 1}(\xi', \xi_n - i\lambda)$ have uniformly bounded analytic extensions for $\lambda > 0$, $\xi' \in \mathbb{R}^{n-1}$, $\xi_n \in \mathbb{R}$ and

$$\sum_{|\alpha|\leq m} \sup\big\{|\xi^\alpha D^\alpha b_\pm^{\pm 1}(\xi)| : \xi \in \mathbb{R}^n\big\} \leq M_m < \infty, \quad m = 0, 1, 2, \dots . \tag{4.28}$$

Proof. For the proof of this lemma see [2,9,15]. □

Remark 18. A lemma similar to the above one but for a general elliptic symbol was proved in [2,9] (see [6] for the scalar case $N = 1$). In [15, §2] a similar but more general assertion is proved when $a(x, \xi)$ depends smoothly on a parameter $x \in S$.

A pair of Banach spaces $\{\mathbb{X}_0, \mathbb{X}_1\}$ embedded in some topological space $\mathbb{E}$ is called an interpolation pair. For such a pair we can introduce the following two spaces: $\mathbb{X}_{\min} = \mathbb{X}_0 \cap \mathbb{X}_1$ and $\mathbb{X}_{\max} = \mathbb{X}_0 + \mathbb{X}_1 := \big\{x \in \mathbb{E} : x = x_0 + x_1,\ x_j \in \mathbb{X}_j,\ j = 0, 1\big\}$; $\mathbb{X}_{\min}$ and $\mathbb{X}_{\max}$ become Banach spaces if they are endowed with the norms

$$\begin{gathered}\|x|\mathbb{X}_{\min}\| = \max\big\{\|x|\mathbb{X}_0\|,\ \|x|\mathbb{X}_1\|\big\},\\ \|x|\mathbb{X}_{\max}\| = \inf\big\{\|x_0|\mathbb{X}_0\| + \|x_1|\mathbb{X}_1\| : x = x_0 + x_1,\ x_j \in \mathbb{X}_j,\ j = 0, 1\big\},\end{gathered}$$

respectively.

Moreover, we have the continuous embeddings

$$\mathbb{X}_{\min} \subset \mathbb{X}_0,\ \mathbb{X}_1 \subset \mathbb{X}_{\max}. \tag{4.29}$$

For any interpolation pairs $\{\mathbb{X}_0, \mathbb{X}_1\}$ and $\{\mathbb{Y}_0, \mathbb{Y}_1\}$ the space $\mathcal{L}(\{\mathbb{X}_0\mathbb{X}_1\}, \{\mathbb{Y}_0\mathbb{Y}_1\})$ consists of all linear operators from $\mathbb{X}_{\max}$ into $\mathbb{Y}_{\max}$ whose restrictions to $\mathbb{X}_j$ belong to $\mathcal{L}(\mathbb{X}_j, \mathbb{Y}_j)$ $(j = 0, 1)$. The notation $\mathcal{L}(\mathbb{X}, \mathbb{Y})$ is used for the space of all linear bounded operators $\mathbf{A} : \mathbb{X} \to \mathbb{Y}$.

Lemma 19. *Assume $\{\mathbb{X}_0, \mathbb{X}_1\}$ and $\{\mathbb{Y}_0, \mathbb{Y}_1\}$ to be interpolation pairs and the embeddings $\mathbb{X}_{\min} \subset \mathbb{X}_{\max}$, $\mathbb{Y}_{\min} \subset \mathbb{Y}_{\max}$ to be dense. Let an operator $\mathbf{A} \in \mathcal{L}(\mathbb{X}_0, \mathbb{Y}_0) \cap \mathcal{L}(\mathbb{X}_1, \mathbb{Y}_1)$ have a common regularizer: let $\mathbf{R} \in \mathcal{L}(\mathbb{Y}_0, \mathbb{X}_0) \cap \mathcal{L}(\mathbb{Y}_1, \mathbb{X}_1)$ and $\mathbf{RA} - \mathbf{I} \in \mathcal{L}(\mathbb{X}_0\mathbb{X}_0) \cap \mathcal{L}(\mathbb{X}_1, \mathbb{X}_1)$ be compact. Then*

$$\mathbf{A} : \mathbb{X}_{\min} \to \mathbb{Y}_{\min}, \quad \mathbf{A} : \mathbb{X}_{max} \to \mathbb{Y}_{\max}$$

are Fredholm operators and

$$\mathrm{Ind}_{\mathbb{X}_{\min}\to\mathbb{Y}_{\min}} \mathbf{A} = \mathrm{Ind}_{\mathbb{X}_{\max}\to\mathbb{Y}_{\max}} \mathbf{A} = \mathrm{Ind}_{\mathbb{X}_j\to\mathbb{Y}_j} \mathbf{A}, \quad j = 0, 1. \tag{4.30}$$

If $y \in \mathbb{Y}_j$, then any solution $x \in \mathbb{X}_{\max}$ of the equation $\mathbf{A}x = y$ belongs to $\mathbb{X}_j$. In particular,

$$\ker_{\mathbb{X}_{\min}} \mathbf{A} = \ker_{\mathbb{X}_j} \mathbf{A} = \ker_{\mathbb{X}_{\max}} \mathbf{A}, \quad j = 0, 1. \tag{4.31}$$

Proof. We begin by noting that the definition of a norm in $\mathbb{X}_{\min}, \dots, \mathbb{Y}_{\max}$ implies

$$\big\|\mathbf{A}|\mathcal{L}(\mathbb{X}_{\min}, \mathbb{Y}_{\min})\big\| \le \max\big\{\|\mathbf{A}|\mathcal{L}(\mathbb{X}_j, \mathbb{Y}_j)\| : j = 0, 1\big\},$$
$$\big\|\mathbf{A}|\mathcal{L}(\mathbb{X}_{\max}, \mathbb{Y}_{\max})\big\| \le \max\big\{\|\mathbf{A}|\mathcal{L}(\mathbb{X}_j, \mathbb{Y}_j)\| : j = 0, 1\big\}.$$

Whence we find

$$\mathcal{L}(\mathbb{X}_0, \mathbb{Y}_0) \cap \mathcal{L}(\mathbb{X}_1, \mathbb{Y}_1) \subset \mathcal{L}(\mathbb{X}_{\min}, \mathbb{Y}_{\min}) \cap \mathcal{L}(\mathbb{X}_{\max}, \mathbb{Y}_{\max}).$$

Next we shall prove that $\mathbf{A}$ is a Fredholm operator in the spaces $\mathbb{X}_{\min} \to \mathbb{Y}_{\min}$ and $\mathbb{X}_{\max} \to \mathbb{Y}_{\max}$. For this it suffices to show that $\mathbf{AR}-\mathbf{I}$, $\mathbf{RA}-\mathbf{I}$ are compact in the spaces $\mathbb{X}_{\min}$ and $\mathbb{X}_{\max}$, since by the conditions of the lemma they are compact in $\mathbb{X}_0$ and $\mathbb{X}_1$. Let us prove a more general inclusion

$$\mathrm{Com}(\mathbb{X}_0, \mathbb{Y}_0) \cap \mathrm{Com}(\mathbb{X}_1, \mathbb{Y}_1) \subset \mathrm{Com}(\mathbb{X}_{\min}, \mathbb{Y}_{\min}) \cap \mathrm{Com}(\mathbb{X}_{\max}, \mathbb{Y}_{\max}),$$

that implies the claimed assertion.

Assume $\mathbf{T} : \mathbb{X}_j \to \mathbb{Y}_j$ $(j = 0, 1)$ to be compact and $\{x_k\}_{k\in\mathbb{N}}$ to be an arbitrary bounded sequence in $\mathbb{X}_{\min}$. Then $\{x_k\}_{k\in\mathbb{N}}$ is bounded in both spaces $\mathbb{X}_0$ and $\mathbb{X}_1$. It can be assumed without loss of generality that the sequences $\{\mathbf{T}x_k\}_{k\in\mathbb{N}}$ are convergent in both $\mathbb{Y}_0$ and $\mathbb{Y}_1$ (otherwise we can select subsequences). Then $\{\mathbf{T}x_k\}_{k\in\mathbb{N}}$ is convergent in $\mathbb{Y}_{\min}$ and therefore $\mathbf{T} \in \mathrm{Com}(\mathbb{X}_{\min}, \mathbb{Y}_{\min})$.

If S_0, S_1, and $S_{\max}$ denote the unit balls in $\mathbb{X}_0$, $\mathbb{X}_1$, and $\mathbb{X}_{\max}$, respectively, then $S_{\max} \subset S_0 + S_1$. Due to the compactness of $\mathbf{T} : \mathbb{X}_j \to \mathbb{Y}_j$ ($j = 0, 1$), there exist $\varepsilon/2$-grids $\{y_k^{(j)}\}_{k=1}^{m_j} \subset \mathbf{T}(S_j)$ ($j = 0, 1$), $\varepsilon > 0$. Then $\{y_k^{(0)} + y_n^{(1)}\}_{k,n} \subset \mathbf{T}(S_0) + \mathbf{T}(S_1)$ defines an ε-grid in $\mathbf{T}(S_{\max})(\subset \mathbf{T}(S_0) + \mathbf{T}(S_1))$. Since $\varepsilon > 0$ is arbitrary, $\mathbf{T} : \mathbb{X}_{\max} \to \mathbb{Y}_{\max}$ is compact.

Now we shall show that the density of the embedding $\mathbb{Y}_{\min} \subset \mathbb{Y}_{\max}$ implies the density of $\mathbb{Y}_{\min} \subset \mathbb{Y}_j$ ($j = 0, 1$). For the sake of definiteness assume that $j = 0$. By the condition of the lemma for any $\varepsilon > 0$ and $a \in \mathbb{Y}_0$ there exists $b \in \mathbb{Y}_{\min}$ with the property

$$\|(a - b)|\mathbb{Y}_{\max}\| < \varepsilon;$$

i.e., there exist $a_0 \in \mathbb{Y}_0$, $a_1 \in \mathbb{Y}_1$ such that $a - b = a_0 + a_1$,

$$\|a_0|\mathbb{Y}_0\| + \|a_1|\mathbb{Y}_1\| < \varepsilon.$$

Since $a \in \mathbb{Y}_0$ and $b \in \mathbb{Y}_{\min} \subset \mathbb{Y}_0$, we obtain $a - b \in \mathbb{Y}_0$ and $a_1 = (a-b) - a_0 \in \mathbb{Y}_0$, so that $a_1 \in \mathbb{Y}_0 \cap \mathbb{Y}_1 = \mathbb{Y}_{\min}$ and $a_1 + b \in \mathbb{Y}_{\min}$. Therefore

$$\Big\|[a - (a_1 + b)]|\mathbb{Y}_0\Big\| = \|a_0|\mathbb{Y}_0\| < \varepsilon,$$

which proves that the embedding $\mathbb{Y}_{\min} \subset \mathbb{Y}_0$ is dense.

The density of the embeddings $\mathbb{Y}_{\min} \subset \mathbb{Y}_j \subset \mathbb{Y}_{\max}$, $j = 0, 1$, yields

$$\mathbb{Y}^*_{\max} \subset \mathbb{Y}^*_j \subset \mathbb{Y}^*_{\min}, \quad j = 0, 1.$$

Since $\mathbb{X}_{\min} \subset \mathbb{X}_j \subset \mathbb{X}_{\max}$ and $\mathbf{A}^* : \mathbb{Y}^*_j \to \mathbb{X}^*_j$ ($j = 0, 1$), $\mathbf{A}^* : \mathbb{Y}^*_{\min} \to \mathbb{X}^*_{\min}$, $\mathbf{A}^* : \mathbb{Y}^*_{\max} \to \mathbb{X}^*_{\max}$ are Fredholm, we have

$$\ker_{\mathbb{X}_{\min}} \mathbf{A} \subset \ker_{\mathbb{X}_j} \mathbf{A} \subset \ker_{\mathbb{X}_{\max}} \mathbf{A}, \tag{4.32}$$

$$\ker_{\mathbb{Y}^*_{\max}} \mathbf{A}^* \subset \ker_{\mathbb{Y}^*_j} \mathbf{A}^* \subset \ker_{\mathbb{Y}^*_{\min}} \mathbf{A}^*. \tag{4.33}$$

The dimensions of the kernels ($\dim \ker \mathbf{A}$) in appropriate spaces will be denoted by $\alpha_{\min}$, α_j, $\alpha_{\max}$, while the notation $\beta_{\min}$, β_j, $\beta_{\max}$ will be used for the dimensions of cokernels ($\dim \operatorname{Coker} \mathbf{A}$). Note that for a Fredholm operator we have

$$\dim \operatorname{Coker} \mathbf{A} = \dim \ker \mathbf{A}^*.$$

Embeddings (4.32) and (4.33) imply

$$\alpha_{\min} \le \alpha_j \le \alpha_{\max}, \quad j = 0, 1, \tag{4.34}$$

$$\beta_{\max} \le \beta_j \le \beta_{\min}, \quad j = 0, 1. \tag{4.35}$$

By the definition of $\operatorname{Ind} \mathbf{A}$ we obtain

$$\operatorname{Ind}_{\mathbb{X}_{\min} \to \mathbb{Y}_{\min}} \mathbf{A} \le \operatorname{Ind}_{\mathbb{X}_j \to \mathbb{Y}_j} \mathbf{A} \le \operatorname{Ind}_{\mathbb{X}_{\max} \to \mathbb{Y}_{\max}} \mathbf{A}. \tag{4.36}$$

A similar inequality for indices of the regularizer $\mathbf{R}$ is proved just in the same manner. Since $\operatorname{Ind} \mathbf{R} = -\operatorname{Ind} \mathbf{A}$, the inequalities inverse to (4.36) are valid and therefore (4.30) holds. Now from (4.34) and (4.35) we obtain $\alpha_{\min} = \alpha_j = \alpha_{\max}$. The latter equality and (4.32) give (4.31). $\square$

Remark 20. Similar statements under different conditions on spaces and operators are well known (see, for example, [16], [17], [18]).

§ 5. Proofs of Theorems

5.1. *Proof of Theorem* 7. In the first place we shall prove that $\mathbf{P}^1_S$ (see (3.2), (3.6), (3.7)) is a pseudodifferential operator according to the definition given in Subsection 4.2.

Let $U_1, \dots, U_N$ be a covering of $S \subset \mathbb{R}^3$ (see (4.14), where $n = 2$), $\varkappa_1, \dots, \varkappa_N$ be coordinate diffeomorphisms, and

$$\begin{gathered}\widetilde{\varkappa}_j : \widetilde{X}_j \to \widetilde{U}_j, \quad \widetilde{X}_j, \widetilde{U}_j \subset \mathbb{R}^3, \quad \widetilde{U}_j \cap S = V_j, \\ \widetilde{X}_j = (-\varepsilon, \varepsilon) \times X_j, \quad \widetilde{\varkappa}_j|_{X_j} = \varkappa_j, \quad j = 1, \dots, N,\end{gathered} \tag{5.1}$$

be extensions of diffeomorphisms (4.14). By $d\varkappa_j(t) = \varkappa_j'(t)$ and $d\widetilde{\varkappa}_j(\widetilde{t}) = \widetilde{\varkappa}_j'(\widetilde{t})$ ($t = (t_1, t_2) \in \mathbb{R}^2_+$, $\widetilde{t} = (t_0, t_1, t_2) \in \mathbb{R}^3_+$) we denote the corresponding Jacobian matrices of orders 3×2 and 3×3. $\varkappa_j'(t)$ will coincide with $\widetilde{\varkappa}_j'(0, t) (t \in X_j \subset \mathbb{R}^2_+)$ if the first column in these matrices is deleted.

Let further

$$\Gamma_{\chi_j}(t) = \big(\det \|(\partial_k \varkappa_j, \partial_l \varkappa_j)\|_{2\times 2}\big)^{1/2}, \quad \partial_k \varkappa_j = (\partial_k \varkappa_{j1}, \partial_k \varkappa_{j2}, \partial_k \varkappa_{j3})$$

denote the square root of the Gramm determinant of the vector-function $\varkappa_j = (\varkappa_{j1}, \varkappa_{j2}, \varkappa_{j3})$.

If the operator $\mathbf{P}^1_S$ is lifted locally from the manifold S onto the half-space $\mathbb{R}^2_+$ by means of operators (4.15), then we obtain the operator (cf. (4.17))

$$\begin{gathered}\mathbf{P}^1_{s,\varkappa_j} v(t) = \varkappa_{j*} \mathbf{P}^1_s \varkappa_{j*}^{-1} v(t) = \chi^0_j(t) \int\limits_{\mathbb{R}^2_+} \Phi\big((\varkappa_j(t) - \\ -\varkappa_j(\theta), \tau\big) \chi^0_j(\theta) \Gamma_{\varkappa_j}(\theta) v(\theta) d\theta, \quad t \in \mathbb{R}^2_+, \quad \chi^0_j \in C^\infty_0(\mathbb{R}^2_+).\end{gathered}$$

From the last equality it follows that operator (3.7) is bounded. Moreover,

$$\begin{gathered}\mathbf{K}_j v(t) := \chi^0_j(t) \int\limits_{\mathbb{R}^2_+} \big[\Phi(\varkappa_j(t) - \varkappa_j(\theta), \tau) \Gamma_{\varkappa_j}(\theta) - \\ -\Phi(\varkappa_j'(t)(t - \theta), \tau) \Gamma_{\varkappa_j}(t)\big] \chi^0_j(\theta) v(\theta) d\theta\end{gathered}$$

has the order -2, i.e., the operator

$$\mathbf{K}_j : \widetilde{\mathbb{H}}^\nu_p(\mathbb{R}^2_+) \to \mathbb{H}^{\nu+2}_p(\mathbb{R}^2_+) \tag{5.2}$$

is bounded for any $\nu \in \mathbb{R}$ (see [19, Section 33.2 and Theorem 13]). Due to (5.2) the operator

$$\mathbf{K}_j : \widetilde{\mathbb{H}}^{\nu}_p(\mathbb{R}^2_+) \to \mathbb{H}^{\nu+1}_p(\mathbb{R}^2_+) \tag{5.3}$$

is compact, since $\chi^0_j \in C^\infty_0(\mathbb{R}^2_+)$ [see (4.19)]. From (5.3), Example 14, and (2.1), it follows that the symbol of the pseudodifferential operator $\mathbf{P}^1_S$ reads $(x \in \overline{S}, \xi \in \mathbb{R}^2)$

$$\mathcal{P}^1_S(x,\xi) = \Gamma_{\varkappa_j}(t) \int\limits_{\mathbb{R}^2} e^{i\xi\eta} \Phi\big(\varkappa'_j(t)\eta, \tau\big) d\eta =$$

$$= \Gamma_{\varkappa_j}(t) \int\limits_{\mathbb{R}^2} e^{i\xi\eta} \Phi\big(\widetilde{\varkappa}'_j(0,t)(0,\eta), \tau\big) d\eta =$$

$$= \frac{\Gamma_{\varkappa_j}(t)}{(2\pi)^3} \int\limits_{\mathbb{R}^2} e^{i\xi\eta} \int\limits_{\mathbb{R}^3} e^{-i(\widetilde{\varkappa}'_j(0,t)(0,\eta),\widetilde{y})} \mathcal{A}^{-1}(\widetilde{y},\tau) d\widetilde{y} d\eta =$$

$$= \frac{\Gamma_{\varkappa_j}(t)}{(2\pi)^3 \det \widetilde{\varkappa}'_j(0,t)} \int\limits_{\mathbb{R}^2} e^{i\xi\eta} \int\limits_{\mathbb{R}^2} e^{-i\eta y} \int\limits_{-\infty}^{\infty} \mathcal{A}^{-1}\Big(\big[(\widetilde{\varkappa}'_j(0,t))^T \big]^{-1} \widetilde{y}, \tau \Big) dy_0 dy d\eta =$$

$$= \frac{\Gamma_{\varkappa_j}(t)}{2\pi \det \widetilde{\varkappa}'_j(0,t)} \int\limits_{-\infty}^{\infty} \mathcal{A}^{-1}\Big(\big[(\widetilde{\varkappa}'_j(0,t))^T \big]^{-1} \zeta, \tau \Big) dy_0. \tag{5.4}$$

for $t = \varkappa_j^{-1}(x)$, $x \in S$, $t \in \mathbb{R}^2_+$, $\xi \in \mathbb{R}^2$, $\widetilde{y} = (y_0, y) \in \mathbb{R}^3$, $\zeta = (y_0, \xi)$. By (2.3) the principal homogeneous symbol of $\mathbf{P}^1_S$ (see (2.18)) is written in the form

$$(\mathcal{P}^1_S)^\infty(x,\xi) = \frac{\Gamma_{\varkappa_j}(t)}{2\pi \det \widetilde{\varkappa}'_j(0,t)} \int\limits_{-\infty}^{\infty} \mathcal{A}_0^{-1}\Big(\big[(\widetilde{\varkappa}'_j(0,t))^T \big]^{-1} \zeta \Big) dy_0, \tag{5.5}$$

$$x \in \overline{S}, \quad \xi \in \mathbb{R}^2, \quad t = \varkappa_j^{-1}(x) \in \mathbb{R}^2_+, \quad \zeta = (y_0, \xi),$$

$$\mathcal{A}_0^{-1}(\widetilde{\xi}) = \begin{Vmatrix} \mathcal{C}^{-1}(\widetilde{\xi}) & 0 \\ 0 & \mathbf{\Lambda}^{-1}(-i\widetilde{\xi}) \end{Vmatrix}, \quad \widetilde{\xi} \in \mathbb{R}^3, \tag{5.6}$$

where $\mathcal{C}(\widetilde{\xi})$ and $\mathbf{\Lambda}(\widetilde{\xi})$ are defined by (2.4). Since $-\mathcal{C}(\widetilde{\xi})$ and $-\mathbf{\Lambda}(-i\widetilde{\xi})$ are positive-definite (see (1.12) and (1.14)), the same is true for $-\mathcal{A}_0^{-1}(\widetilde{\xi})$:

$$\big(- \mathcal{A}_0^{-1}(\widetilde{\xi})\eta, \eta \big) \geq \delta_2 |\eta|^2 |\widetilde{\xi}|^{-2}, \quad \delta_2 > 0, \quad \eta \in \mathbb{C}^4, \quad \widetilde{\xi} \in \mathbb{R}^3.$$

Applying this fact, we proceed as follows:

$$\big((-\mathcal{P}^1_S)^\infty(x,\xi)\eta, \eta \big) =$$

$$= \frac{\Gamma_{\varkappa_j}(t)}{2\pi \det \varkappa_j'(t)} \int\limits_{-\infty}^{+\infty} \Big(-\mathcal{A}_0^{-1}\big(\big[(\widetilde{\varkappa}_j'(0,t))^T\big]^{-1}\zeta\big)\eta, \eta\Big) dy_0 \geq$$

$$\geq \delta_2 |\eta|^2 \int\limits_{-\infty}^{+\infty} \big|\widetilde{\varkappa}_j'(0,t)\zeta\big|^{-2} dy_0 \geq$$

$$\geq \delta_3 |\eta|^2 \int\limits_{-\infty}^{+\infty} \frac{dy_0}{y_0^2 + |\xi|^2} = \delta_4 |\eta|^2 |\xi|^{-1}, \tag{5.7}$$

$$\eta \in \mathbb{C}^4, \quad \xi \in \mathbb{R}^2, \quad \zeta = (y_0, \xi), \quad \delta_k = const > 0, \quad k = 2, 3, 4.$$

Formulas (1.6), (5.5) and (5.6) also imply

$$\begin{gathered} D_x^\alpha D_{\xi_1}^m (\mathcal{P}_S^1)^\infty (x, \lambda\xi) = |\lambda|^{-1} \lambda^{-m} D_x^\alpha D_{\xi_1}^m (\mathcal{P}_S^1)^\infty (x, \xi), \\ |\alpha| < \infty, \quad m = 0, 1, \dots, \quad \xi \in \mathbb{R}^2, \quad \lambda \in \mathbb{R}. \end{gathered} \tag{5.8}$$

Hence we have the equivalences (see (4.18), (4.21), (5.1), (5.2))

$$\begin{gathered} \varkappa_{j*} \mathbf{P}_S^1 \varkappa_{j*}^{-1} \overset{x_0}{\approx} (\mathbf{P}_S^1)^0 (x_0, D), \quad x_0 \in U_j \subset S, \quad x_0 \notin \partial S, \\ \varkappa_{j*} \mathbf{P}_S^1 \varkappa_{j*}^{-1} \overset{x_0}{\approx} \mathbf{r}_+ (\mathbf{P}_S^1)^0 (x_0, D), \quad x_0 \in U_j \cap \partial S, \end{gathered}$$

where (see (4.20))

$$(\mathcal{P}_S^1)^0 (x, \xi) := (\mathcal{P}_S^1)^\infty \big(x, (1 + |\xi_1|)|\xi_1|^{-1}\xi_1, \xi_2\big).$$

Due to (5.7) the symbol $(\mathcal{P}_S^1)^0(x,\xi)$ is an elliptic one,

$$\inf\{|\det(\mathcal{P}_S^1)^\infty(x,\xi)| : x \in \overline{S}, \quad |\xi| = 1\} > 0.$$

Since condition (5.8) implies the continuity property (4.25) for the symbol $(\mathcal{P}_S^1)^\infty(x,\xi)$, by virtue of Lemma 17 it admits the factorization

$$\begin{gathered} (\mathcal{P}_S^1)^0 (x, \xi) = \big[(\xi_2 - i|\xi_1| - i)^{-1/2} \mathcal{P}_-(x, \xi)\big] \big[(\xi_2 + i|\xi'| + i)^{-1/2} \mathcal{P}_+(x, \xi)\big], \\ \mathcal{P}_-^{\pm 1}(x, \cdot),\ \mathcal{P}_+^{\pm 1}(x, \cdot) \in M_p(\mathbb{R}^2), \quad x \in \partial S, \end{gathered}$$

where $\mathcal{P}_-^{\pm 1}(x, \xi_1 - i\lambda)$, $\mathcal{P}_+^{\pm 1}(x, \xi_1 + i\lambda)$ have bounded analytic extensions for $\lambda > 0$. According to Theorem 15 operator (3.7) is a Fredholm one if and only if the operators $\mathbf{r}_+(\mathbf{P}_S^1)_{\nu,-1}(x_0, D)$ are Fredholm ones in $\mathbb{L}_p(\mathbb{R}_+^2)$ for all $x_0 \in \partial S$, where

$$(\mathcal{P}_S^1)_{\nu,-1}(x_0, \xi) = \frac{(\xi_2 - i|\xi_1| - i)^{\nu+1}}{(\xi_2 + i|\xi_1| + i)^{\nu}} (\mathcal{P}_S^1)^0 (x_0, \xi) =$$

$$= \Big(\frac{\xi_2 - i|\xi_1| - i}{\xi_2 + i|\xi_1| + i}\Big)^{\nu + 1/2} \mathcal{P}_-(x_0, \xi) \mathcal{P}_+(x_0, \xi), \quad x_0 \in \partial S. \tag{5.9}$$

Therefore (see (4.10), (5.9))

$$\mathbf{r}_+(\mathbf{P}^1_S)_{\nu,-1}(x_0,D)=\mathbf{r}_+\mathbf{P}_-(x_0,D)\ell_0\mathbf{r}_+\mathbf{G}_\nu(D)\mathbf{P}_+(x_0,D),\tag{5.10}$$

with

$$\mathcal{G}_\nu(\xi)=\Big(\frac{\xi_2-i|\xi_1|-i}{\xi_2+i|\xi_1|+i}\Big)^{\nu+1/2}\tag{5.11}$$

and since $\mathbf{r}_+\mathbf{P}_\pm(x_0,D)$ are invertible (according to (4.10) the inverses read $\mathbf{r}_+\mathbf{P}^{-1}_\pm(x,D)$). The proof will be completed if we find invertibility conditions for $\mathbf{r}_+\mathbf{G}_\nu(D)$ in $\mathbb{L}_p(\mathbb{R}^2_+)$; the latter is invertible if and only if

$$1/p-1<\nu+1/2<1/p\tag{5.12}$$

and the inverse reads $(\mathbf{r}_+\mathbf{G}_\nu(D))^{-1}=\mathcal{I}^{\nu+1/2}_+(D)\ell_0\mathbf{r}_+\mathcal{I}^{-\nu-1/2}_-(D)$ (see [2], §2). Conditions (5.12) coincide with (3.8).

The local inverses to $\mathbf{P}^1_S:\widetilde{\mathbb{H}}^\nu_p(S)\to\mathbb{H}^{\nu+1}_p(S)$ are, therefore, independent of the parameters p and ν if conditions (3.8) are fulfilled.

In fact, the operator

$$(\mathbf{r}_+\mathbf{P}^1_S)^{-1}_{\nu,-1}(x_0,D):=\mathbf{P}^{-1}_+(x_0,D)\mathcal{I}^{\nu+1/2}_+(D)\ell_0\mathbf{r}_+\mathcal{I}^{-\nu-1/2}_-(D)\mathbf{P}^{-1}_+(x_0,D)$$

is inverse to $(\mathbf{r}_+\mathbf{P}^1_S)_{\nu,-1}(x_0,D)$ in $L_p(\mathbb{R}^2_+)$; if we “lift” these operators from the space $L_p(\mathbb{R}^2_+)$ to the Bessel potential spaces by means of the Bessel potentials $\mathcal{I}^\mu_\pm(D)$ defined by (4.8), we shall come to the following conclusion: if (3.8) holds, the operator

$$\begin{gathered}\mathcal{I}^{-\nu}_+(D)\ell_0(\mathbf{r}_+\mathbf{P}^1_S)^{-1}_{\nu,-1}(x_0,D)\mathcal{I}^{\nu+1}_-(D)=\\=(D)\mathbf{P}^{-1}_+(x_0,D)\mathcal{I}^{1/2}_+(D)\ell_0\mathbf{r}_+\mathcal{I}^{1/2}_-\mathbf{P}^{-1}_-(x_0,D)\end{gathered}$$

inverts the operator

$$\begin{gathered}\mathcal{I}^{-\nu}_+(D)\ell_0(\mathbf{r}_+\mathbf{P}^1_S)_{\nu,-1}(x_0,D)\mathcal{I}^{\nu+1}_-(D)=\\=\mathbf{P}^1_S(x_0,D):\widetilde{\mathbb{H}}^\nu_p(\mathbb{R}^2_+)\to\mathbb{H}^{\nu+1}_p(\mathbb{R}^2_+),\quad x_0\in\partial S\end{gathered}$$

which is a local representation of $\mathbf{P}^1_S=\mathbf{P}^1_S(x,D)$ $(x\in S,\ x_0\in\partial S)$.

Thus the regularizer constructed by means of the local inverses (see, for example, [2], [3], [13]) can be chosen independent of p and ν if (3.8) holds. Now we can take $p=2$ and by Theorem 16 and Lemma 19 we get $\operatorname{Ind}\mathbf{P}^1_S=0$.

To complete the proof for the space $H^\nu_p(S)$ it remains to check that $\ker\mathbf{P}^1_S=0$. We need to do this only for $\nu=-1/2$ and $p=2$, since $\ker\mathbf{P}^1_S$ is also independent of the parameters p and ν (see Lemma 19).

The equality $\ker \mathbf{P}_S^1 = 0$, in turn, follows from the triviality of a solution of the homogeneous Problem D. Actually, formula (1.17) implies that for any solution $U = (u_1, \dots, u_4)$ of the homogeneous Problem D we have

$$\int\limits_{\mathbb{R}_S^3} \Big\{ c_{ijkl} D_l u_k D_j \overline{u}_i + \rho\tau^2 u_k \overline{u}_k + \frac{1}{\overline{\tau} T_0} \lambda_{ij} D_j u_4 D_i \overline{u}_4 + \frac{c_0}{T_0} u_4 \overline{u}_4 \Big\} dx = 0;$$

recalling that $\tau = \sigma + i\omega$ and separating the real and the imaginary part, we obtain

$$\begin{aligned} &\int\limits_{\mathbb{R}_S^3} \Big\{ c_{ijkl} D_l u_k D_j \overline{u}_i + \rho(\sigma^2 - \omega^2) u_k \overline{u}_k + \\ &+ \frac{\sigma}{|\tau|^2 T_0} \lambda_{ij} D_j u_4 D_i \overline{u}_4 + \frac{c_0}{T_0} u_4 \overline{u}_4 \Big\} dx = 0, \\ &\frac{\omega}{T_0} \int\limits_{\mathbb{R}_S^3} \Big\{ 2\sigma T_0 u_k \overline{u}_k + \lambda_{ij} D_j u_4 D_i \overline{u}_4 \Big\} dx = 0. \end{aligned} \tag{5.13}$$

Whence by (1.12) and (1.14) we find $U = 0$ for an arbitrary τ with $\operatorname{Re}\tau > 0$. For $\tau = 0$ we obtain

$$D_j u_k(x) + D_k u_j(x) = 0, \quad u_4 = 0, \quad k, j = 1, 3, \quad x \in \mathbb{R}_S^3. \tag{5.14}$$

The general solution of this system is (see [1])

$$U = [a \times x] + b,$$

where a and b are the constant three-dimensional vectors with complex entries and $[\cdot \times \cdot]$ denotes the vector product of two vectors. From conditions (1.10) and (5.14) it follows that $U = 0$.

Thus the homogeneous Problem D has only a trivial solution and $\ker \mathbf{P}_S^1 = \{0\}$.

To prove the theorem for the Besov space $\mathbb{B}_{p,p}^{\nu}(S)$ recall the following interpolation property from (4.5):

If $\mathbf{A} : \widetilde{\mathbb{H}}_p^{\nu}(S) \to \mathbb{H}_p^{\nu+r}(S)$ is bounded for any $\nu_0 < \nu < \nu_1$ and some $1 < p < \infty$, then the operator $\mathbf{A} : \widetilde{\mathbb{B}}_{p,q}^{\nu}(S) \to \mathbb{B}_{p,q}^{\nu+r}(S)$ is also bounded for any $\nu_0 < \nu < \nu_1$, $1 \le q \le \infty$.

Let conditions (3.8) be fulfilled. Then the operator $\mathbf{P}_S^1 : \widetilde{\mathbb{H}}_p^{\nu}(S) \to \mathbb{H}_p^{\nu+1}(S)$ has the bounded inverse $(\mathbf{P}_S^1)^{-1} : \mathbb{H}_p^{\nu+1}(S) \to \widetilde{\mathbb{H}}_p^{\nu}(S)$; due to the above-mentioned interpolation property the operator $(\mathbf{P}_S^1)^{-1} : \mathbb{B}_{p,q}^{\nu+1}(S) \to \widetilde{\mathbb{B}}_{p,q}^{\nu}(S)$ will also be bounded and therefore the operator $\mathbf{P}_S^1$ in (3.6) has the bounded inverse.

5.2. *Proof of Theorem* 8. After the localization and local transformation of variables (see (5.1)–(5.9)) we obtain the equivalences

$$\varkappa_{j*}\mathbf{P}_S^4\varkappa_{j*}^{-1} \overset{x_0}{\approx} (\mathbf{P}_S^4)^0(x_0,D), \quad x_0\in U_j\subset S, \quad x_0\notin\partial S,$$
$$\varkappa_{j*}\mathbf{P}_S^4\varkappa_{j*}^{-1} \overset{x_0}{\approx} \mathbf{r}_+(\mathbf{P}_S^4)^0(x_0,D), \quad x_0\in U_j\cap\partial S,$$

where

$$(\mathcal{P}_S^4)^0(x_0,\xi) = \mathcal{B}^0(x_0,\xi)(\mathcal{P}_S^1)^0(x_0,\xi)(\mathcal{B}^0)^T(x_0,\xi) \tag{5.15}$$

and $\mathcal{B}^0(x_0,\xi)$ represents the modified principal symbol of the operators $\mathbf{B}(D_x,n(x))$ and $\mathbf{Q}(D_x,n(x))$ (whose principal symbols coincide). The order of $\mathcal{B}^0(x_0,\xi)$ is 1 and therefore (5.15), (5.7) yield

$$((\mathcal{P}_S^4)^\infty(x_0,\xi)\eta,\eta) \geq \delta_5|\xi||\eta|^2, \quad \xi\in\mathbb{R}^2, \quad \eta\in\mathbb{C}^4, \quad \delta_5>0.$$

The homogeneity property

$$D_{\xi_1}^m D_x^\alpha(\mathcal{P}_S^4)^\infty(x,\lambda\xi) = |\lambda|\lambda^{-m} D_{\xi_1}^m D_x^\alpha(\mathcal{P}_S^4)^\infty(x,\xi),$$
$$|\alpha|<\infty, \quad m=0,1,\dots, \quad \xi\in\mathbb{R}^2, \quad \lambda\in\mathbb{R}$$

holds as well (see (5.8)).

Thus the symbol $(\mathcal{P}_S^4)^\infty(x,\xi)$ is elliptic

$$\inf\{|\det(\mathcal{P}_S^4)^\infty(x,\xi)| : x\in\overline{S}, \quad |\xi|=1\}>0$$

and operator (3.10) is a Fredholm one if and only if the operators $\mathbf{r}_+(\mathbf{P}_S^4)^0_{\nu+1,1}(x_0,D)$ are Fredholm in $\mathbb{L}_p(\mathbb{R}^2_+)$ for all $x_0\in\partial S$; here

$$(\mathcal{P}_S^4)^0_{\nu+1,1}(x_0,\xi) = \frac{(\xi_2-i|\xi_1|-i)^\nu}{(\xi_2+i|\xi_1|+i)^{\nu+1}}(\mathcal{P}_S^4)^0(x_0,\xi) =$$
$$= \Big(\frac{\xi_2-i|\xi_1|-i}{\xi_2+i|\xi_1|+i}\Big)^{\nu+1/2}\mathcal{P}_-^4(x_0,\xi)\mathcal{P}_+^4(x_0,\xi),$$
$$(\mathcal{P}_+^4)^{\pm1}(x,\cdot),\ (\mathcal{P}_-^4)^{\pm1}(x,\cdot)\in M_p(\mathbb{R}^2), \quad x_0\in\partial S,$$

and $(\mathcal{P}_+^4)^{\pm1}(x_0,\xi_1,\xi_2+i\lambda)$, $(\mathcal{P}_-^4)^{\pm1}(x_0,\xi_1,\xi_2-i\lambda)$ have bounded analytic extensions for $\lambda>0$. The proof is completed similarly to that of Theorem 7. □

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