

**CONSTRUCTION OF ENTIRE MODULAR FORMS OF
WEIGHTS 5 AND 6 FOR THE CONGRUENCE GROUP
 $\Gamma_0(4N)$**

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ABSTRACT. Two classes of entire modular forms of weight 5 and two of weight 6 are constructed for the congruence subgroup $\Gamma_0(4N)$. The constructed modular forms as well as the modular forms from [1] will be helpful in the theory of representation of numbers by the quadratic forms in 10 and 12 variables.

The present paper is a direct continuation of [1] whose notation will be preserved here.

1.

Lemma 1. *For a given N let*

$$\begin{aligned} \Psi_3(\tau) &= \Psi_3(\tau; g_1, \dots, g_4; h_1, \dots, h_4; c_1, \dots, c_4; N_1, \dots, N_4) = \\ &= \left\{ \frac{1}{N_1} \vartheta''_{g_1 h_1}(\tau; c_1, 2N_1) \vartheta_{g_2 h_2}(\tau; c_2, 2N_2) - \right. \\ &\quad \left. - \frac{1}{N_2} \vartheta_{g_1 h_1}(\tau; c_1, 2N_1) \vartheta''_{g_2 h_2}(\tau; c_2, 2N_2) \right\} \times \\ &\quad \times \vartheta'_{g_3 h_3}(\tau; c_3, 2N_3) \vartheta_{g_4 h_4}(\tau; c_4, 2N_4) \end{aligned} \quad (1.1)$$

and

$$\begin{aligned} \Psi_4(\tau) &= \Psi_4(\tau; g_1, \dots, g_4; h_1, \dots, h_4; c_1, \dots, c_4; N_1, \dots, N_4) = \\ &= \prod_{k=1}^3 \vartheta'_{g_k h_k}(\tau; c_k, 2N_k) \vartheta_{g_4 h_4}(\tau; c_4, 2N_4), \end{aligned} \quad (1.2)$$

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where

$$2|g_k, N_k|N \quad (k = 1, 2, 3, 4), \quad 4 \left| N \sum_{k=1}^4 \frac{h_k}{N_k} \right|. \quad (1.3)$$

For all substitutions from Γ in the neighborhood of each rational point $\tau = -\frac{\delta}{\gamma}$ ($\gamma \neq 0$, $(\gamma, \delta) = 1$), we then have

$$\begin{aligned} & (\gamma\tau + \delta)^5 \Psi_j(\tau; g_1, \dots, g_4; h_1, \dots, h_4; 0, \dots, 0; N_1, \dots, N_4) = \\ & = \sum_{n=0}^{\infty} C_n^{(j)} e\left(\frac{n}{4N} \frac{\alpha\tau + \beta}{\gamma\tau + \delta}\right) \quad (j = 3, 4). \end{aligned} \quad (1.4)$$

Proof. I. Taking into account (1.19) from [1], by Lemma 4 from [1], it follows for $n = 2$ (with $g_1, h_1, g'_1, h'_1, N_1, H_1$ instead of g, h, g', h', N, H) and $n = 0$ (with $g_2, h_2, g'_2, h'_2, N_2, H_2$ instead of g, h, g', h', N, H) that

$$\begin{aligned} & \frac{1}{N_1} (\gamma\tau + \delta)^3 \vartheta''_{g'_1 h'_1}(\tau; 0, 2N_1) \vartheta_{g_2 h_2}(\tau; 0, 2N_2) = \\ & = -\frac{1}{N_1} e\left(\frac{3}{4} \operatorname{sgn} \gamma\right) (2|\gamma|(N_1 N_2)^{1/2})^{-1} \sum_{H_1 \bmod 2N_1} \varphi_{g'_1 h'_1}(0, H_1; 2N_1) \times \\ & \times \left\{ \vartheta''_{g'_1 h'_1}\left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_1, 2N_1\right) + 2A_{21} \Big|_{z=0} \cdot \vartheta_{g'_1 h'_1}\left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_1, 2N_1\right) \right\} \times \\ & \times \sum_{H_2 \bmod 2N_2} \varphi_{g'_2 h'_2}(0, H_2; 2N_2) \vartheta_{g_2 h_2}\left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_2, 2N_2\right) = \\ & = -e\left(\frac{3}{4} \operatorname{sgn} \gamma\right) (2|\gamma|(N_1 N_2)^{1/2})^{-1} \sum_{\substack{H_1 \bmod 2N_1 \\ H_2 \bmod 2N_2}} \varphi_{g'_1 h'_1}(0, H_1; 2N_1) \times \\ & \times \varphi_{g'_2 h'_2}(0, H_2; 2N_2) \left\{ \frac{1}{N_1} \vartheta''_{g'_1 h'_1}\left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_1, 2N_1\right) \times \right. \\ & \times \vartheta_{g_2 h_2}\left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_2, 2N_2\right) - 4\gamma\pi i (\gamma\tau + \delta) \vartheta_{g'_1 h'_1}\left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_1, 2N_1\right) \times \\ & \left. \times \vartheta_{g'_2 h'_2}\left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_2, 2N_2\right) \right\}. \end{aligned} \quad (1.5)$$

If in (1.5) $N_1, g_1, h_1, H_1, g'_1, h'_1$ are replaced by $N_2, g_2, h_2, H_2, g'_2, h'_2$, and vice versa, then we have

$$\begin{aligned} & \frac{1}{N_2} (\gamma\tau + \delta)^3 \vartheta''_{g_2 h_2}(\tau; 0, 2N_2) \vartheta_{g_1 h_1}(\tau; 0, 2N_1) = \\ & = -e\left(\frac{3}{4} \operatorname{sgn} \gamma\right) (2|\gamma|(N_1 N_2)^{1/2})^{-1} \sum_{\substack{H_2 \bmod 2N_2 \\ H_1 \bmod 2N_1}} \varphi_{g'_2 h'_2}(0, H_2; 2N_2) \times \end{aligned}$$

$$\begin{aligned}
& \times \varphi_{g'_1 g_1 h_1}(\tau; 0, H_1; 2N_1) \times \\
& \times \left\{ \frac{1}{N_2} \vartheta''_{g'_2 h'_2} \left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_2, 2N_2 \right) \vartheta_{g'_1 h'_1} \left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_1, 2N_1 \right) - \right. \\
& \quad - 4\gamma\pi i(\gamma\tau + \delta) \vartheta_{g'_2 h'_2} \left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_2, 2N_2 \right) \times \\
& \quad \left. \times \vartheta_{g'_1 h'_1} \left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_1, 2N_1 \right) \right\}. \tag{1.6}
\end{aligned}$$

Subtracting (1.6) from (1.5), we obtain

$$\begin{aligned}
& (\gamma\tau + \delta)^3 \left\{ \frac{1}{N_1} \vartheta''_{g_1 h_1}(\tau; 0, 2N_1) \vartheta_{g_2 h_2}(\tau; 0, 2N_2) - \right. \\
& \quad \left. - \frac{1}{N_2} \vartheta_{g_1 h_1}(\tau; 0, 2N_1) \vartheta''_{g_2 h_2}(\tau; 0, 2N_2) \right\} = -e\left(\frac{3}{4} \operatorname{sgn} \gamma\right) \times \\
& \times (2|\gamma|(N_1 N_2)^{1/2})^{-1} \sum_{\substack{H_1 \bmod 2N_1 \\ H_2 \bmod 2N_2}} \varphi_{g'_1 g_1 h_1}(0, H_1; 2N_1) \varphi_{g'_2 g_2 h_2}(0, H_2; 2N_2) \times \\
& \times \left\{ \frac{1}{N_1} \vartheta''_{g'_1 h'_1} \left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_1, 2N_1 \right) \vartheta_{g'_2 h'_2} \left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_2, 2N_2 \right) - \right. \\
& \quad \left. - \frac{1}{N_2} \vartheta_{g'_1 h'_1} \left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_1, 2N_1 \right) \vartheta''_{g'_2 h'_2} \left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_2, 2N_2 \right) \right\}. \tag{1.7}
\end{aligned}$$

Analogously, by Lemma 4 from [1], for $n - 1$ and $n = 0$ we obtain

$$\begin{aligned}
& (\gamma\tau + \delta)^2 \vartheta'_{g_3 h_3}(\tau; 0, 2N_3) \vartheta_{g_4 h_4}(\tau; 0, 2N_4) = \\
& = -e\left(\frac{1}{2} \operatorname{sgn} \gamma\right) (2|\gamma|(N_3 N_4)^{1/2})^{-1} i \operatorname{sgn} \gamma \times \\
& \times \sum_{\substack{H_3 \bmod 2N_3 \\ H_4 \bmod 2N_4}} \varphi_{g'_3 g_3 h_3}(0, H_3; 2N_3) \varphi_{g'_4 g_4 h_4}(0, H_4; 2N_4) \times \\
& \times \vartheta'_{g'_3 h'_3} \left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_3, 2N_3 \right) \vartheta_{g'_4 h'_4} \left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_4, 2N_4 \right). \tag{1.8}
\end{aligned}$$

Multiplying (1.7) by (1.8), on account of (1.1), we obtain

$$\begin{aligned}
& (\gamma\tau + \delta)^5 \Psi_3(\tau; g_1, \dots, g_4; h_1, \dots, h_4; 0, \dots, 0; N_1, \dots, N_4) = \\
& = e\left(\frac{5}{4} \operatorname{sgn} \gamma\right) \left(4\gamma^2 \left(\prod_{k=1}^4 N_k \right)^{1/2} \right)^{-1} i \operatorname{sgn} \gamma \times \\
& \times \sum_{\substack{H_k \bmod 2N_k \\ (k=1,2,3,4)}} \prod_{k=1}^4 \varphi_{g'_k g_k h_k}(0, H_k; 2N_k) \times
\end{aligned}$$

$$\times \Psi_3\left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; g'_1, \dots, g'_4; h'_1, \dots, h'_4; H_1, \dots, H_4; N_1, \dots, N_4\right). \quad (1.9)$$

Further, applying the same reasoning as in [1, Lemma 5, pp. 62-63], we obtain (1.4) if $j = 3$.

II. As in Subsection I, by Lemma 4 from [1], for $n = 1$ (with $g_k, h_k, N_k, g'_k, h'_k, H_k$ for all $1 \leq k \leq 4$ instead of g, h, N, g', h', H) and $n = 0$ (with $g_4, h_4, N_4, g'_4, h'_4, H_4$ instead of g, h, N, g', h', H), we obtain

$$\begin{aligned} & (\gamma\tau + \delta)^5 \prod_{k=1}^3 \vartheta'_{g_k h_k}(\tau; 0, 2N_k) \vartheta_{g_4 h_4}(\tau; 0, 2N_4) = \\ & = e\left(\frac{5}{4} \operatorname{sgn} \gamma\right) \left(4\gamma^2 \left(\prod_{k=1}^4 N_k\right)^{1/2}\right)^{-1} i \operatorname{sgn} \gamma \times \\ & \quad \times \sum_{\substack{H_k \bmod 2N_k \\ (k=1,2,3,4)}} \prod_{k=1}^4 \varphi_{g'_k h_k} (0, H_k; 2N_k) \times \\ & \quad \times \prod_{k=1}^3 \vartheta'_{g'_k h'_k} \left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_k, 2N_k\right) \vartheta_{g'_4 h'_4} \left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_4, 2N_4\right). \end{aligned}$$

Hence, according to (1.2), it follows that

$$\begin{aligned} & (\gamma\tau + \delta)^5 \Psi_4(\tau; g_1, \dots, g_4; h_1, \dots, h_4; 0, \dots, 0; N_1, \dots, N_4) = \\ & = e\left(\frac{5}{4} \operatorname{sgn} \gamma\right) \left(4\gamma^2 \left(\prod_{k=1}^4 N_k\right)^{1/2}\right)^{-1} i \operatorname{sgn} \gamma \times \\ & \quad \times \sum_{\substack{H_k \bmod 2N_k \\ (k=1,2,3,4)}} \prod_{k=1}^4 \varphi_{g'_k h_k} (0, H_k; 2N_k) \times \\ & \quad \times \Psi_4\left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; g'_1, \dots, g'_4; h'_1, \dots, h'_4; H_1, \dots, H_4; N_1, \dots, N_4\right). \quad (1.10) \end{aligned}$$

Further, applying the same reasoning as in [1, Lemma 5], we obtain (1.4) if $j = 4$. \square

Theorem 1. *For a given N the functions $\Psi_3(\tau)$ and $\Psi_4(\tau)$ with $c_1 = c_2 = c_3 = c_4 = 0$ are entire modular forms of weight 5 and character $\chi(\delta) = \operatorname{sgn} \gamma \left(\frac{-\Delta}{|\delta|}\right)$ (Δ is the determinant of an arbitrary positive quadratic form in 10 variables) for the group $\Gamma_0(4N)$ if the following conditions hold:*

$$1) \quad 2|g_k, N_k|N \quad (k = 1, 2, 3, 4), \quad (1.11)$$

$$2) \quad 4|N \sum_{k=1}^4 \frac{h_k^2}{N_k}, \quad 4 \left| \sum_{k=1}^4 \frac{g_k^2}{4N_k} \right., \quad (1.12)$$

3) for all α and δ with $\alpha\delta \equiv 1 \pmod{4N}$

$$\begin{aligned} & \left(\frac{\prod_{k=1}^4 N_k}{|\delta|} \right) \Psi_j(\tau; \alpha g_1, \dots, \alpha g_4; h_1, \dots, h_4; 0, \dots, 0; N_1, \dots, N_4) = \\ & = \operatorname{sgn} \delta \left(\frac{-\Delta}{|\delta|} \right) \Psi_j(\tau; g_1, \dots, g_4; h_1, \dots, h_4; 0, \dots, 0; N_1, \dots, N_4) \quad (1.13) \\ & \quad (j = 3, 4). \end{aligned}$$

Proof. I. As in the case of Theorem 1 from [1], the functions $\Psi_3(\tau)$ and $\Psi_4(\tau)$ with $c_1 = c_2 = c_3 = c_4 = 0$ satisfy the condition 1) and, by Lemma 1, also the condition 4) of the definition from [1, p. 53–54].

II. From (1.12), since $2 \nmid \delta$, it follows that

$$4 \left| N \delta^2 \sum_{k=1}^4 \frac{h_k^2}{N_k} \right., \quad 4 \left| \sum_{k=1}^4 \frac{g_k^2}{4N_k} \delta^{2\varphi(2N_k)-2} \right. \quad (1.14)$$

Taking into account (1.11), by Lemma 3 from [1], for $n = 2$ and $n = 0$ (with g_r, h_r, N_r and g_s, h_s, N_s instead of g, h, N), we obtain for each substitution from $\Gamma_0(4N)$

$$\begin{aligned} & \vartheta''_{g_r h_r} \left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; 0, 2N_r \right) \vartheta_{g_s h_s} \left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; 0, 2N_s \right) = i^{3\eta(\gamma)(\operatorname{sgn}\delta-1)} i^{1-|\delta|} \times \\ & \times \left(\frac{N_r N_s}{|\delta|} \right) (\gamma\tau + \delta)^3 e \left(\frac{\beta\delta}{4} \left(\frac{g_r^2}{4N_r} \delta^{2\varphi(2N_r)-2} + \frac{g_s^2}{4N_s} \delta^{2\varphi(2N_s)-2} \right) \right) \times \\ & \times e \left(-\frac{\alpha\gamma\delta^2}{4} \left(\frac{h_r^2}{4N_r} + \frac{h_s^2}{4N_s} \right) \right) \times \\ & \times \vartheta''_{\alpha g_r, h_r}(\tau; 0, 2N_r) \vartheta'_{\alpha g_s, h_s}(\tau; 0, 2N_s) \quad (1.15) \end{aligned}$$

for $r = 1, s = 2$ and $r = 2, s = 1$.

Analogously, by Lemma 3 from [1], for $n = 1$ and $n = 0$, we have

$$\begin{aligned} & \vartheta'_{g_3 h_3} \left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; 0, 2N_3 \right) \vartheta_{g_4 h_4} \left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; 0, 2N_4 \right) = \operatorname{sgn} \delta i^{2\eta(\gamma)(\operatorname{sgn}\delta-1)} \times \\ & \times i^{1-|\delta|} \left(\frac{N_3 N_4}{|\delta|} \right) (\gamma\tau + \delta)^2 e \left(\frac{\beta\delta}{4} \left(\frac{g_3^2}{4N_3} \delta^{2\varphi(2N_3)-2} + \frac{g_4^2}{4N_4} \delta^{2\varphi(2N_4)-2} \right) \right) \times \\ & \times e \left(-\frac{\alpha\gamma\delta^2}{4} \left(\frac{h_3^2}{4N_3} + \frac{h_4^2}{4N_4} \right) \right) \times \\ & \times \vartheta'_{\alpha g_3, h_3}(\tau; 0, 2N_3) \vartheta_{\alpha g_4, h_4}(\tau; 0, 2N_4). \quad (1.16) \end{aligned}$$

Hence, by (1.1), (1.15), (1.16), and (1.14), we obtain

$$\begin{aligned} & \Psi_3\left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; g_1, \dots, g_4; h_1, \dots, h_4; 0, \dots, 0; N_1, \dots, N_4\right) = \\ & = \operatorname{sgn} \delta i^{\eta(\gamma)(\operatorname{sgn} \delta - 1)} (-1)^{1 - |\delta|} \left(\frac{\prod_{k=1}^4 N_k}{|\delta|}\right) (\gamma\tau + \delta)^5 \times \\ & \times \Psi_3(\tau; \alpha g_1, \dots, \alpha g_4; h_1, \dots, h_4; 0, \dots, 0; N_1, \dots, N_4), \end{aligned}$$

from which according to (1.13) it follows for all α and δ with $\alpha\delta \equiv 1 \pmod{4N}$ that

$$\begin{aligned} & \Psi_3\left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; g_1, \dots, g_4; h_1, \dots, h_4; 0, \dots, 0; N_1, \dots, N_4\right) = \\ & = \operatorname{sgn} \delta \left(\frac{-\Delta}{|\delta|}\right) (\gamma\tau + \delta)^5 \Psi_3(\tau; g_1, \dots, g_4; h_1, \dots, h_4; 0, \dots, 0; N_1, \dots, N_4). \end{aligned}$$

Analogously, by the just mentioned Lemma 3, for $n = 1$ and $n = 0$ (with g_k, h_k, N_k ($k = 1, 2, 3, 4$) instead of g, h, N), according to (1.11) and (1.14), we find for all substitutions from $\Gamma_0(4N)$ that

$$\begin{aligned} & \prod_{k=1}^3 \vartheta'_{g_k h_k} \left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; 0, 2N_k\right) \vartheta_{g_4 h_4} \left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; 0, 2N_4\right) = \\ & = \operatorname{sgn} \delta i^{\eta(\gamma)(\operatorname{sgn} \delta - 1)} (-1)^{1 - |\delta|} \left(\frac{\prod_{k=1}^4 N_k}{|\delta|}\right) (\gamma\tau + \delta)^5 \times \\ & \times \prod_{k=1}^3 \vartheta'_{\alpha g_k, h_k}(\tau; 0, 2N_k) \vartheta'_{\alpha g_4, h_4}(\tau; 0, 2N_4). \quad (1.17) \end{aligned}$$

Hence, by (1.2), (1.17), and (1.13), for all α and δ with $\alpha\delta \equiv 1 \pmod{4N}$ we have

$$\begin{aligned} & \Psi_4\left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; g_1, \dots, g_4; h_1, \dots, h_4; 0, \dots, 0; N_1, \dots, N_4\right) = \\ & = \operatorname{sgn} \delta \left(\frac{-\Delta}{|\delta|}\right) (\gamma\tau + \delta)^5 \Psi_4(\tau; g_1, \dots, g_4; h_1, \dots, h_4; 0, \dots, 0; N_1, \dots, N_4). \end{aligned}$$

Thus the functions $\Psi_3(\tau)$ and $\Psi_4(\tau)$ with $c_1 = c_2 = c_3 = c_4 = 0$ satisfy the condition 2) of the definition from [1].

III. According to (13) from [1] we have

$$\begin{aligned} & 1) \vartheta''_{g_r h_r}(\tau; 0, 2N_r) \vartheta_{g_s h_s}(\tau; 0, 2N_s) = \\ & = -\pi^2 \sum_{m_r, m_s = -\infty}^{\infty} (-1)^{h_r m_r + h_s m_s} (4N_r m_r + g_r)^2 e(\Lambda_1 \tau) \quad (1.18) \end{aligned}$$

for $r = 1, s = 2$ and $r = 2, s = 1$, where

$$\begin{aligned}\Lambda_1 &= \sum_{k=1}^2 \frac{1}{4N_k} (2N_k m_k + g_k/2)^2 = \\ &= \sum_{k=1}^2 \left(N_k m_k^2 + \frac{1}{2} m_k g_k \right) + \frac{1}{4} \sum_{k=1}^2 g_k^2 / 4N_k;\end{aligned}\quad (1.19)$$

$$\begin{aligned}2) \quad \vartheta'_{g_3 h_3}(\tau; 0, 2N_3) \vartheta_{g_4 h_4}(\tau; 0, 2N_4) &= \\ = \pi i \sum_{m_3, m_4 = -\infty}^{\infty} (-1)^{h_3 m_3 + h_4 m_4} (4N_3 m_3 + g_3) e(\Lambda_2 \tau),\end{aligned}\quad (1.20)$$

where

$$\begin{aligned}\Lambda_2 &= \sum_{k=3}^4 \frac{1}{4N_k} (2N_k m_k + g_k/2)^2 = \\ &= \sum_{k=3}^4 \left(N_k m_k^2 + \frac{1}{2} m_k g_k \right) + \frac{1}{4} \sum_{k=3}^4 g_k^2 / 4N_k;\end{aligned}\quad (1.21)$$

$$\begin{aligned}3) \quad \prod_{k=1}^3 \vartheta'_{g_k h_k}(\tau; 0, 2N_k) \vartheta_{g_4 h_4}(\tau; 0, 2N_4) &= \\ = -\pi^3 i \sum_{m_1, \dots, m_4 = -\infty}^{\infty} (-1)^{\sum_{k=1}^4 h_k m_k} \prod_{k=1}^3 (4N_k m_k + g_k) e(\Lambda \tau),\end{aligned}$$

where

$$\begin{aligned}\Lambda &= \sum_{k=1}^4 \frac{1}{4N_k} (2N_k m_k + g_k/2)^2 = \\ &= \sum_{k=1}^4 \left(N_k m_k^2 + \frac{1}{2} m_k g_k \right) + \frac{1}{4} \sum_{k=1}^4 g_k^2 / 4N_k.\end{aligned}\quad (1.22)$$

$\Lambda_1 + \Lambda_2$ and Λ are integers by virtue of (1.11) and (1.12). Therefore the functions $\Psi_3(\tau)$ and $\Psi_4(\tau)$ satisfy the condition 3) of the definition from [1]. \square

2.

Lemma 2. For a given N let

$$\begin{aligned} \Phi_3(\tau) &= \Phi_3(\tau; g_1, \dots, g_4; h_1, \dots, h_4; c_1, \dots, c_4; N_1, \dots, N_4) = \\ &= \left\{ \frac{1}{N_1} \vartheta''_{g_1 h_1}(\tau; c_1, 2N_1) \vartheta_{g_2 h_2}(\tau; c_2, 2N_2) - \right. \\ &\left. - \frac{1}{N_2} \vartheta_{g_1 h_1}(\tau; c_1, 2N_1) \vartheta''_{g_2 h_2}(\tau; c_2, 2N_2) \right\} \prod_{k=3}^4 \vartheta'_{g_k h_k}(\tau; c_k, 2N_k) \quad (2.1) \end{aligned}$$

and

$$\begin{aligned} \Phi_4(\tau) &= \Phi_4(\tau; g_1, \dots, g_4; h_1, \dots, h_4; c_1, \dots, c_4; N_1, \dots, N_4) = \\ &= \prod_{k=1}^4 \vartheta'_{g_k h_k}(\tau; c_k, 2N_k), \quad (2.2) \end{aligned}$$

where

$$2|g_k, N_k|N \ (k = 1, 2, 3, 4), \quad 4|N \sum_{k=1}^4 \frac{h_k}{N_k}. \quad (2.3)$$

For all substitutions from Γ in the neighborhood of each rational point $\tau = -\frac{\delta}{\gamma}$ ($\gamma \neq 0$, $(\gamma, \delta) = 1$), we then have

$$(\gamma\tau + \delta)^6 \Phi_j(\tau) = \sum_{n=0}^{\infty} D_n^{(j)} e\left(\frac{n}{4N} \frac{\alpha\tau + \beta}{\gamma\tau + \delta}\right) \quad (j = 3, 4). \quad (2.4)$$

Proof. I. As in the proof of Lemma 1, by Lemma 4 from [1], it follows for $n = 1$ that

$$\begin{aligned} &(\gamma\tau + \delta)^3 \vartheta'_{g_3 h_3}(\tau; 0, 2N_3) \vartheta'_{g_4 h_4}(\tau; 0, 2N_4) = \\ &= -e\left(\frac{3}{4} \operatorname{sgn} \gamma\right) (2|\gamma|(N_3 N_4)^{1/2})^{-1} \times \\ &\times \sum_{\substack{H_3 \bmod 2N_3 \\ H_4 \bmod 2N_4}} \prod_{k=1}^4 \varphi_{g'_k h'_k}(0, H_k; 2N_k) \varphi_{g_4 h_4}(\tau; 0, H_4; 2N_4) \times \\ &\times \vartheta'_{g'_3 h'_3}\left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_3, 2N_3\right) \vartheta'_{g'_4 h'_4}\left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_4, 2N_4\right). \quad (2.5) \end{aligned}$$

Multiplying (1.7) by (2.5), by virtue of (2.1), we obtain

$$(\gamma\tau + \delta)^6 \Phi_3(\tau; g_1, \dots, g_4; h_1, \dots, h_4; 0, \dots, 0; N_1, \dots, N_4) =$$

$$\begin{aligned}
&= e\left(\frac{3}{2} \operatorname{sgn} \gamma\right) \left(4\gamma^2 \left(\prod_{k=1}^4 N_k\right)^{1/2}\right)^{-1} \sum_{\substack{H_k \bmod 2N_k \\ (k=1,2,3,4)}} \varphi_{g'_k g_k h_k}(0, H_k; 2N_k) \times \\
&\quad \times \Phi_3\left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; g'_1, \dots, g'_4; h'_1, \dots, h'_4; H_1, \dots, H_4; N_1, \dots, N_4\right). \quad (2.6)
\end{aligned}$$

Further, applying the same reasoning as in [1, Lemma 5], we obtain (2.4) if $j = 3$.

II. By Lemma 4 from [1], for $n = 1$ we have

$$\begin{aligned}
(\gamma\tau + \delta)^6 \prod_{k=1}^4 \vartheta'_{g_k h_k}(\tau; 0, 2N_k) &= e\left(\frac{3}{2} \operatorname{sgn} \gamma\right) \left(4\gamma^2 \left(\prod_{k=1}^4 N_k\right)^{1/2}\right)^{-1} \times \\
&\quad \times \sum_{\substack{H_k \bmod 2N_k \\ (k=1,2,3,4)}} \prod_{k=1}^4 \varphi_{g'_k g_k h_k}(0, H_k; 2N_k) \prod_{k=1}^4 \vartheta'_{g'_k h'_k}\left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; H_k, 2N_k\right).
\end{aligned}$$

Hence, according to (2.2), it follows that

$$\begin{aligned}
&(\gamma\tau + \delta)^6 \Phi_4(\tau; g_1, \dots, g_4; h_1, \dots, h_4; 0, \dots, 0; N_1, \dots, N_4) = \\
&= e\left(\frac{3}{2} \operatorname{sgn} \gamma\right) \left(4\gamma^2 \left(\prod_{k=1}^4 N_k\right)^{1/2}\right)^{-1} \sum_{\substack{H_k \bmod 2N_k \\ (k=1,2,3,4)}} \prod_{k=1}^4 \varphi_{g'_k g_k h_k}(0, H_k; 2N_k) \times \\
&\quad \times \Phi_4\left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; g'_1, \dots, g'_4; h'_1, \dots, h'_4; H_1, \dots, H_4; N_1, \dots, N_4\right).
\end{aligned}$$

Further, applying the same reasoning as in [1, Lemma 5], we obtain (2.4) if $j = 4$. \square

Theorem 2. For a given N the functions $\Phi_3(\tau)$ and $\Phi_4(\tau)$ with $c_1 = c_2 = c_3 = c_4 = 0$ are entire modular forms of weight 6 and character $\chi(\delta) = \left(\frac{\Delta}{|\delta|}\right)$ (Δ is the determinant of an arbitrary positive quadratic form in 12 variables) for the group $\Gamma_0(4N)$ provided that the following conditions hold:

$$1) \quad 2|g_k, N_k|N \quad (k = 1, 2, 3, 4), \quad (2.7)$$

$$2) \quad 4|N \sum_{k=1}^4 \frac{h_k^2}{N_k}, 4| \sum_{k=1}^4 \frac{g_k^2}{4N_k}, \quad (2.8)$$

$$3) \quad \text{for all } \alpha \text{ and } \delta \text{ with } \alpha\delta \equiv 1 \pmod{4N}$$

$$\left(\frac{\prod_{k=1}^4 N_k}{|\delta|}\right) \Phi_j(\tau; \alpha g_1, \dots, \alpha g_4; h_1, \dots, h_4; 0, \dots, 0; N_1, \dots, N_4) =$$

$$= \left(\frac{\Delta}{|\delta|} \right) \Phi_j(\tau; g_1, \dots, g_4; h_1, \dots, h_4; 0, \dots, 0; N_1, \dots, N_4) \quad (2.9)$$

$$(j = 3, 4).$$

Proof. I. As in the case of Theorem 1, the functions $\Phi_3(\tau)$ and $\Phi_4(\tau)$ with $c_1 = c_2 = c_3 = c_4 = 0$ satisfy the condition 1) and, on account of Lemma 2, also the condition 4) of the definition from [1].

II. Since $2 \nmid \delta$, from (2.8) it follows that

$$4 \left| N \delta^2 \sum_{k=1}^4 \frac{h_k^2}{N_k} \right|, 4 \left| \sum_{k=1}^4 \frac{g_k^2}{4N_k} \delta^{2\varphi(2N_k)-2} \right|. \quad (2.10)$$

Taking into account (2.7), by Lemma 3 from [1], for $n = 1$ we find that for all substitutions from $\Gamma_0(4N)$ we have

$$\begin{aligned} \prod_{k=3}^4 \vartheta'_{g_k h_k} \left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; 0, 2N_k \right) &= i^{3\eta(\gamma)(\text{sgn } \delta - 1)} i^{1-|\delta|} \left(\frac{N_3 N_4}{|\delta|} \right) \times \\ &\times (\gamma\tau + \delta)^3 e \left(\frac{\beta\delta}{4} \left(\frac{g_3^2}{4N_3} \delta^{2\varphi(2N_3)-2} + \frac{g_4^2}{4N_4} \delta^{2\varphi(2N_4)-2} \right) \right) \times \\ &\times e \left(-\frac{\alpha\gamma\delta^2}{4} \left(\frac{h_3^2}{4N_3} + \frac{h_4^2}{4N_4} \right) \right) \prod_{k=3}^4 \vartheta'_{\alpha g_k, h_k}(\tau; 0, 2N_k). \end{aligned} \quad (2.11)$$

Taking into account (2.10) and (2.9), by (2.1), (1.15), and (2.11), for all α and δ with $\alpha\delta \equiv 1 \pmod{4N}$ we obtain

$$\begin{aligned} \Phi_3 \left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; g_1, \dots, g_4; h_1, \dots, h_4; 0, \dots, 0; N_1, \dots, N_4 \right) &= \\ &= i^{2\eta(\gamma)(\text{sgn } \delta - 1)} (-1)^{1-|\delta|} \left(\frac{\prod_{k=1}^4 N_k}{|\delta|} \right) (\gamma\tau + \delta)^6 \times \\ &\times \Phi_3(\tau; \alpha g_1, \dots, \alpha g_4; h_1, \dots, h_4; 0, \dots, 0; N_1, \dots, N_4) = \\ &= \left(\frac{\Delta}{|\delta|} \right) (\gamma\tau + \delta)^6 \Phi_3(\tau; g_1, \dots, g_4; h_1, \dots, h_4; 0, \dots, 0; N_1, \dots, N_4). \end{aligned}$$

Analogously, taking into account (2.7) and (2.10), by Lemma 3 from [1], for $n = 1$, we have

$$\begin{aligned} \prod_{k=1}^4 \vartheta'_{g_k h_k} \left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; 0, 2N_k \right) &= \\ &= \left(\frac{\prod_{k=1}^4 N_k}{|\delta|} \right) (\gamma\tau + \delta)^6 \prod_{k=1}^4 \vartheta'_{\alpha g_k, h_k}(\tau; 0, 2N_k) \end{aligned} \quad (2.12)$$

for all substitutions from $\Gamma_0(4N)$.

Hence, by (2.2), (2.12), and (2.9), for all α and δ with $\alpha\delta \equiv 1 \pmod{4N}$, we have

$$\begin{aligned} & \Phi_4\left(\frac{\alpha\tau + \beta}{\gamma\tau + \delta}; g_1, \dots, g_4; h_1, \dots, h_4; 0, \dots, 0; N_1, \dots, N_4\right) = \\ & = \left(\frac{\Delta}{|\delta|}\right)(\gamma\tau + \delta)^6 \Phi_4(\tau; g_1, \dots, g_4; h_1, \dots, h_4; 0, \dots, 0; N_1, \dots, N_4). \end{aligned}$$

Thus the functions $\Phi_3(\tau)$ and $\Phi_4(\tau)$ with $c_1 = c_2 = c_3 = c_4 = 0$ satisfy the condition 2) of the definition from [1].

III. According to (13) from [1], we have

1) (1.18) with Λ_1 defined by (1.19);

2)

$$\begin{aligned} & \prod_{k=3}^4 \vartheta'_{g_k h_k}(\tau; 0, 2N_k) = \\ & = -\pi^2 \sum_{m_3, m_4 = -\infty}^{\infty} (-1)^{h_3 m_3 + h_4 m_4} \prod_{k=3}^4 (4N_k m_k + g_k) e(\Lambda_2 \tau), \end{aligned}$$

where Λ_2 is defined by (1.20);

3)

$$\begin{aligned} & \prod_{k=1}^4 \vartheta'_{g_k h_k}(\tau; 0, 2N_k) = \\ & = \pi^4 \sum_{m_1, \dots, m_4 = -\infty}^{\infty} (-1)^{\sum_{k=1}^4 h_k m_k} \prod_{k=3}^4 (4N_k m_k + g_k) e(\Lambda \tau), \end{aligned}$$

where Λ_2 is defined by (1.21).

Thus the functions $\Phi_3(\tau)$ and $\Phi_4(\tau)$ with $c_1 = c_2 = c_3 = c_4 = 0$ satisfy the condition 3) of the definition from [1]. \square

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