

**ON SOME BOUNDARY VALUE PROBLEMS WITH
INTEGRAL CONDITIONS FOR FUNCTIONAL
DIFFERENTIAL EQUATIONS**

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ABSTRACT. For the functional differential equation $u^{(n)}(t) = f(u)(t)$ we have established the sufficient conditions for solvability and unique solvability of the boundary value problems

$$u^{(i)}(0) = c_i \quad (i = 0, \dots, m-1), \quad \int_0^{+\infty} |u^{(m)}(t)|^2 dt < +\infty$$

and

$$u^{(i)}(0) = c_i \quad (i = 0, \dots, m-1), \\ \int_0^{+\infty} t^{2j} |u^{(j)}(t)|^2 dt < +\infty \quad (j = 0, \dots, m),$$

where $n \geq 2$, m is the integer part of $\frac{n}{2}$, $c_i \in R$, and f is the continuous operator acting from the space of $(n-1)$ -times continuously differentiable functions given on an interval $[0, +\infty[$ into the space of locally Lebesgue integrable functions.

**§ 1. FORMULATION OF THE EXISTENCE AND UNIQUENESS
THEOREMS**

Let $n \geq 2$ and f be a continuous operator acting from the space of $(n-1)$ -times continuously differentiable functions given on an interval $R_+ = [0, +\infty[$ into the space of locally Lebesgue integrable functions given on the same interval. Consider the functional differential equation

$$u^{(n)}(t) = f(u)(t) \tag{1.1}$$

by whose solution we shall understand a function $u : R_+ \rightarrow R$ which is locally absolutely continuous with its derivatives up to order $n-1$ inclusive

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and satisfies (1.1) almost everywhere on R_+ . In this paper we shall be concerned with the problems of the existence and uniqueness of a solution of equation (1.1) satisfying either of the two boundary conditions

$$u^{(i)}(0) = c_i \quad (i = 0, \dots, m-1), \quad \int_0^{+\infty} |u^{(m)}(t)|^2 dt < +\infty \quad (1.2)$$

and

$$\begin{aligned} u^{(i)}(0) &= c_i \quad (i = 0, \dots, m-1), \\ \int_0^{+\infty} t^{2j} |u^{(j)}(t)|^2 dt &< +\infty \quad (j = 0, \dots, m). \end{aligned} \quad (1.3)$$

For the case $f(u)(t) = g(t, u(t), \dots, u^{(n-1)}(t))$ problems of type (1.1), (1.2) and (1.1), (1.3), as well as their closely related problems of the existence of so-called proper oscillatory and vanishing-at-infinity solutions of the equation

$$u^{(n)} = g(t, u(t), \dots, u^{(n-1)}(t)),$$

have been studied with a sufficient thoroughness (see [1,2,3] and §§4 and 14 in the monograph [4]). As to the general case, the above-mentioned problems were not previously investigated.

In this paper we establish the sufficient conditions for the solvability and unique solvability of problems (1.1), (1.2) and (1.1), (1.3). In [5] these results are specified for a differential equation with deviating arguments of the form

$$u^{(n)}(t) = g(t, u(\tau_0(t)), \dots, u^{(m-1)}(\tau_{m-1}(t))),$$

and criteria are found for the existence of a multiparameter family of vanishing-at-infinity proper oscillatory solutions of the above equation.

The following notation will be used throughout the paper.

$C^{n-1}([t_1, t_2])$ and $L([t_1, t_2])$ are respectively the spaces of $(n-1)$ -times continuously differentiable and Lebesgue integrable real functions given on the segment $[t_1, t_2]$.

\tilde{C}^{n-1} is the space of functions $u : R_+ \rightarrow R$ which are locally absolutely continuous (i.e., absolutely continuous on each finite interval from R_+) together with their derivatives up to order $n-1$ inclusive.

C^{n-1} is a topological space of $(n-1)$ -times continuously differentiable real functions given on R_+ , where by the convergence of the sequence $(u_k)_{k=1}^{+\infty}$ we understand the uniform convergence of sequences $(u_k^{(i)})_{k=1}^{+\infty}$ ($i = 0, \dots, n-1$) on each finite interval from R_+ .

$$C_0^{n-1,m} = \left\{ u \in C^{n-1} : \int_0^{+\infty} |u^{(m)}(t)|^2 dt < +\infty \right\},$$

$$C^{n-1,m} = \left\{ u \in C^{n-1} : \int_0^{+\infty} t^{2j} |u^{(i)}|^2 dt < +\infty \ (i = 0, \dots, m) \right\}.$$

If $u \in C_0^{n-1,m}$, then

$$\|u\|_{0,m} = \left[\sum_{i=0}^{m-1} |u^{(i)}(0)|^2 + \int_0^{+\infty} |u^{(m)}(s)|^2 ds \right]^{1/2};$$

if, however, $u \in C^{n-1,m}$, then

$$\|u\|_m = \left[\int_0^{+\infty} (1+s)^{2m} |u^{(m)}(s)|^2 ds \right]^{1/2}.$$

L is the space of locally Lebesgue integrable functions $v : R_+ \rightarrow R$ with the topology of convergence in the mean on each finite interval from R_+ .

μ_i^k ($i = 0, 1, \dots ; k = 2i, 2i + 1, \dots$) are the numbers given by the recurrent relations

$$\begin{aligned} \mu_0^{i+1} &= \frac{1}{2}, \quad \mu_i^{2i} = 1, \quad \mu_{i+1}^k = \mu_{i+1}^{k-1} + \mu_i^{k-2} \quad (k = 2i + 3, \dots). \\ \gamma_n &= 0 \quad \text{for } n \leq 3, \end{aligned}$$

$$\gamma_n = \sum_{j=0}^{m_0-1} \frac{n!}{(2m-2-4j)!} \mu_{m-1-2j}^n \quad \text{for } n \geq 4.$$

where m_0 is the integer part of the number $\frac{n}{4}$.

In the sequel it will always be assumed that $f : C^{n-1} \rightarrow L$ is a continuous operator.

Theorem 1.1. *Let for any $u \in C_0^{n-1,m}$ the inequalities*

$$(-1)^{n-m-1} u(t) f(u)(t) \geq -a_1(t) \|u\|_{0,m}^2 - a_2(t), \tag{1.4}$$

$$|f(u)(t)| \leq b(t, |u(t)|, \|u\|_{0,m}) \tag{1.5}$$

hold almost everywhere on R_+ , where $a_i : R_+ \rightarrow R_+$ ($i = 1, 2$) are measurable functions such that

$$\begin{aligned} \int_0^{+\infty} (1+t)^{n-2m} a_1(t) dt &< \mu_m^n, \\ \int_0^{+\infty} (1+t)^{n-2m} a_2(t) dt &< +\infty, \end{aligned} \tag{1.6}$$

and the function $b : R_+^3 \rightarrow R_+$ is locally summable with respect to the first argument, nondecreasing with respect to the last two arguments and

$$\lim_{\substack{t \rightarrow 0 \\ y \rightarrow +\infty}} \left(y^{-2} \int_0^t b(s, x, y) ds \right) = 0 \quad \text{for } x \in R_+. \tag{1.7}$$

Then problem (1.1), (1.2) has at least one solution.

Theorem 1.2. Let for any u and $\bar{u} \in C_0^{n-1,m}$ the inequality

$$(-1)^{n-m-1}(u(t) - \bar{u}(t))(f(u)(t) - f(\bar{u})(t)) \geq -a(t)\|u - \bar{u}\|_{0,m}^2 \quad (1.8)$$

hold almost everywhere on R_+ , where $a : R_+ \rightarrow R_+$ is a measurable function such that

$$\int_0^{+\infty} (1+t)^{n-2m} a(t) dt < \mu_m^n. \quad (1.9)$$

Then problem (1.1), (1.2) has at most one solution.

Theorem 1.3. Let for any $u \in C^{n-1,m}$ the inequalities

$$\begin{aligned} & (-1)^{n-m-1} u(t) f(u)(t) \geq \\ & \geq \gamma(1+t)^{-n} |u(t)|^2 - a_1(t) \|u\|_m^2 - a_2(t), \end{aligned} \quad (1.10)$$

$$|f(u)(t)| \leq b(t, |u(t)|, \|u\|_m) \quad (1.11)$$

hold almost everywhere on R_+ , where γ and $a_i : R_+ \rightarrow R_+$ ($i = 1, 2$) are a positive number and measurable functions such that

$$\int_0^{+\infty} (1+t)^n a_i(t) dt < +\infty \quad (i = 1, 2), \quad (1.12)$$

$$\delta = \frac{n!}{(2m)!} \mu_m^n - \int_0^{+\infty} (1+t)^n a_1(t) dt > 0,$$

$$\gamma + (-1)^m \frac{n!}{2} >$$

$$> \frac{m-1}{4} \gamma_n \left[\frac{\gamma_n}{\delta} + \frac{(m-2)(4m^2 - m + 3)}{3} + 4 \right]^{m-1}, \quad (1.13)$$

and the function $b : R_+^3 \rightarrow R_+$ is locally summable with respect to the first argument, nondecreasing with respect to the last two arguments, and satisfies condition (1.7). Then problem (1.1), (1.3) has at least one solution.

Theorem 1.4. Let for any u and $\bar{u} \in C^{n-1,m}$ the inequality

$$\begin{aligned} & (-1)^{n-m-1}(u(t) - \bar{u}(t))(f(u)(t) - f(\bar{u})(t)) \geq \\ & \geq \gamma(1+t)^{-n} (u(t) - \bar{u}(t))^2 - a(t) \|u - \bar{u}\|_m^2 \end{aligned} \quad (1.14)$$

hold almost everywhere on R_+ , where γ and $a : R_+ \rightarrow R_+$ are a positive number and a measurable function such that

$$\delta = \frac{n!}{(2m)!} \mu_m^n - \int_0^{+\infty} (1+t)^n a(t) dt > 0 \quad (1.15)$$

and inequality (1.13) is fulfilled. Then problem (1.1), (1.3) has at most one solution.

§ 2. AUXILIARY STATEMENTS

2.1. Operator σ_k . For an arbitrary $i \in \{0, \dots, n-1\}$ we denote by v_i the polynomial of degree not higher than $2n-1$, satisfying the boundary conditions

$$v_i^{(j)}(0) = \delta_{ij}, \quad v_i^{(j)}(1) = 0 \quad (j = 0, \dots, n-1),$$

where δ_{ij} is the Kronecker symbol. Let v^* be the maximal value among maxima of functions $|v_i^{(j)}|$ ($i, j = 0, \dots, n-1$) on the segment $[0, 1]$. For any natural k and function $u \in C^{n-1}([0, k])$ we set

$$\varepsilon_k(u) = (k+2)^{-n} \left[1 + \sum_{j=0}^{n-1} |u^{(j)}(k)| \right]^{-2} (v^*)^{-2},$$

$$\sigma_k(u)(t) = \begin{cases} u(t) & \text{for } 0 \leq t \leq k \\ \sum_{i=0}^{n-1} \varepsilon_k^i(u) v_i \left(\frac{t-k}{\varepsilon_k(u)} \right) u^{(i)}(k) & \text{for } k < t < k + \varepsilon_k(u) \\ 0 & \text{for } t \geq k + \varepsilon_k(u) \end{cases} .$$

Lemma 2.1. For any natural k the operator $\sigma_k : C^{n-1}([0, k]) \rightarrow C^{n-1}$ is continuous and for any function $u \in C^{n-1}([0, k])$ satisfying the conditions

$$u^{(i)}(k) = 0 \quad (i = 0, \dots, m-1) \quad (2.1)$$

we have

$$0 \leq \|\sigma_k(u)\|_{0,m}^2 - \sum_{i=0}^{m-1} |u^{(i)}(0)|^2 - \int_0^k |u^{(m)}(s)|^2 ds \leq 1, \quad (2.2)$$

$$0 \leq \|\sigma_k(u)\|_m^2 - \int_0^k (1+s)^{2m} |u^{(m)}(s)|^2 ds \leq 1. \quad (2.3)$$

Proof. The continuity of the operator σ_k is obvious. We shall prove the validity of inequality (2.2). In view of (2.1)

$$\begin{aligned} 0 &\leq \|\sigma_k(u)\|_{0,m}^2 - \left(\sum_{i=0}^{m-1} |u^{(i)}(0)|^2 \right) - \int_0^k |u^{(m)}(s)|^2 ds = \\ &= \int_k^{k+\varepsilon_k(u)} \left[\sum_{i=m}^{n-1} \varepsilon_k^{i-m}(u) v_i^{(m)} \left(\frac{s-k}{\varepsilon_k(u)} \right) u^{(i)}(k) \right]^2 ds \leq \end{aligned}$$

$$\leq \varepsilon_k(u)(v^*)^2 \left[\sum_{i=m}^{n-1} |u^{(i)}(k)| \right]^2 < 1.$$

Inequality (2.3) is proved likewise. \square

2.2. Lemmas on A Priori Estimates. In this subsection we shall derive a priori estimates of the function $u \in \tilde{C}^{n-1}$ which for some natural k satisfy either of the two systems of differential inequalities

$$\begin{aligned} |u^{(n)}(t)| &\leq b_0(t, |u(t)|, \|\sigma_k(u)\|_{0,m}), \quad (-1)^{n-m-1}u^{(n)}(t)u(t) \geq \\ &\geq -a_1(t)\|\sigma_k(u)\|_{0,m}^2 - a_2(t) \quad \text{for } 0 \leq t \leq k \end{aligned} \quad (2.4)$$

and

$$\begin{aligned} |u^{(n)}(t)| &\leq b_0(t, |u(t)|, \|\sigma_k(u)\|_m), \quad (-1)^{n-m-1}u^{(n)}(t)u(t) \geq \\ &\geq \gamma(1+t)^{-n}|u(t)|^2 - a_1(t)\|\sigma_k(u)\|_m^2 - a_2(t) \quad \text{for } 0 \leq t \leq k \end{aligned} \quad (2.5)$$

and the boundary conditions

$$\begin{aligned} |u^{(i)}(0)| &\leq \rho_0 \quad (i = 0, \dots, m-1), \\ u^{(j)}(k) &= 0 \quad (j = 0, \dots, n-m-1), \end{aligned} \quad (2.6)$$

where $\rho_0 > 0$ and m is the integer part of the number $\frac{n}{2}$.

Along with the above-mentioned numbers μ_i^k we shall introduce the numbers μ_{ij}^k ($i = 0, 1, \dots$; $j = i, i+1, \dots$; $k = i+j+1, i+j+2, \dots$) using the following recurrent relations:

$$\begin{aligned} \mu_{00}^k &= \frac{1}{2}, \quad \mu_{0j}^k = 1 \quad (k = 1, 2, \dots; j = 1, \dots, k-1), \\ \mu_{ii}^{2i+1} &= \frac{1}{2}, \quad \mu_{i, k-i-1}^k = 1 \quad (i = 1, 2, \dots; k = 2i+2, 2i+3, \dots), \\ \mu_{ij}^k &= \mu_{ij}^{k-1} + \mu_{i-1, j-1}^{k-2} \\ &(i = 1, 2, \dots; j = i, i+1, \dots; k = i+j+2, i+j+3, \dots). \end{aligned}$$

The following three lemmas are proved in [4] (see Lemmas 4.1-4.3 and 5.1).

Lemma 2.2. *If the functions u and $w : [0, t_0] \rightarrow R$ are absolutely continuous together with their derivatives up to order $n-1$ inclusive, then*

$$\begin{aligned} &\int_0^{t_0} w(t)u(t)u^{(n)}(t)dt = \\ &= \sum_{i=0}^{n-m-1} \sum_{j=i}^{n-1-i} (-1)^{n-1-j} \mu_{ij}^n (w^{(n-1-i-j)}(t_0)u^{(i)}(t_0)u^{(j)}(t_0) - \\ &\quad - w^{(n-1-i-j)}(0)u^{(i)}(0)u^{(j)}(0)) + \end{aligned}$$

$$+ \sum_{i=0}^m (-1)^{n-i} \mu_i^n \int_0^{t_0} w^{(n-2i)}(t) |u^{(i)}(t)|^2 dt.$$

Lemma 2.3. *Let $m \geq 2$, $t_0 > 0$, $r_0 \geq 0$ and the function $u : [0, t_0] \rightarrow R$ be m -times continuously differentiable and*

$$(1 + t_0)^{2i} u^{(i)}(t_0) u^{(i-1)}(t_0) - i(1 + t_0)^{2i-1} |u^{(i-1)}(t_0)|^2 - u^{(i)}(0) u^{(i-1)}(0) + i |u^{(i-1)}(0)|^2 \leq r_0 \quad (i = 1, \dots, m-1).$$

Then for any $\eta > \frac{1}{3}(m-2)(4m^2 - m + 3)$ the estimates

$$\int_0^{t_0} (1+t)^{2i} |u^{(i)}(t)|^2 dt \leq 2\eta^{m-2} r_0 + \alpha_i(\eta) \int_0^{t_0} |u(t)|^2 dt + \beta_i(\eta) \int_0^{t_0} (1+t)^{2m} |u^{(m)}(t)|^2 dt \quad (i = 1, \dots, m-1),$$

hold, where

$$\begin{aligned} \alpha_1(\eta) &= (m-1) \left(1 + \frac{\eta}{4}\right), \\ \alpha_i(\eta) &= (m-i) \left(1 + \frac{\eta}{4}\right) \prod_{j=1}^{i-1} \left(\eta - \frac{(j-1)(4j^2 + 7j + 6)}{3}\right) \\ &\quad (i = 2, \dots, m-1), \\ \beta_i(\eta) &= \prod_{j=i}^{m-1} \left(\eta - \frac{(j-1)(4j^2 + 7j + 6)}{3}\right)^{-1} \quad (i = 1, \dots, m-1). \end{aligned} \tag{2.7}$$

Lemma 2.4. *Let $\delta > 0$ and the function $u : [0, \delta] \rightarrow R$ be i -times continuously differentiable. Then there exists a point $t^* \in [0, \delta]$ such that*

$$|u^{(i)}(t^*)| \leq (i+1)!(2i+1)^{i+\frac{1}{2}} \delta^{-\frac{1}{2}-i} \left(\int_0^\delta |u(t)|^2 dt\right)^{\frac{1}{2}}.$$

Lemma 2.5. *Let $r_1 > 0$ and the function $b_0 : [0, 1] \times R_+^2 \rightarrow R_+$ be summable with respect to the first argument, nondecreasing with respect to the last two arguments and*

$$\lim_{\substack{t \rightarrow 0 \\ y \rightarrow +\infty}} \left(y^{-2} \int_0^t b_0(s, x, y) ds\right) = 0 \quad \text{for } x \in R_+. \tag{2.8}$$

Then there exists a positive number r_2 such that any function $u : [0, 1] \rightarrow R$, absolutely continuous together with its derivatives up to order $n-1$ inclusive and satisfying the inequalities

$$\int_0^1 |u^{(n)}(s)u(s)|ds + \int_0^1 |u^{(m)}(s)|^2 ds \leq r_1 \left(1 + \sum_{i=m}^{n-1} |u^{(i)}(0)|\right), \quad (2.9)$$

$$|u^{(n)}(t)| \leq b_0(t, |u(t)|, r_1^{\frac{1}{2}} \left(1 + \sum_{i=m}^{n-1} |u^{(i)}(0)|\right)^{\frac{1}{2}}) \quad (2.10)$$

for $0 \leq t \leq 1$,

admits the estimate

$$|u^{(i)}(0)| \leq r_2 \quad (i = m, \dots, n-1). \quad (2.11)$$

Proof. By virtue of condition (2.8) there exist numbers $\delta \in]0, 1]$ and $y_0 \in R_+$ such that

$$\int_0^\delta b_0(s, 8nr_1, y)ds \leq \frac{1}{8nr_1} y^2 \quad \text{for } y \geq y_0,$$

and therefore

$$\begin{aligned} & \int_0^\delta b_0(s, 8nr_1, y)ds \leq \\ & \leq \int_0^\delta b_0(s, 8nr_1, y_0)ds + \frac{1}{8nr_1} y^2 \quad \text{for } y \geq 0. \end{aligned} \quad (2.12)$$

According to Lemma 2.4 there exist point $t_i \in [0, \delta]$ ($i = m, \dots, n-1$) such that

$$|u^{(i)}(t_i)| \leq (i-m+1)!(2i-2m+1)^{i-m+\frac{1}{2}} \delta^{-\frac{1}{2}-i+m} \left(\int_0^\delta |u^{(m)}(s)|^2 ds \right)^{\frac{1}{2}} \quad (i = m, \dots, n-1).$$

Therefore from the equalities

$$u^{(i)}(t) = u^{(i)}(t_i) + \int_{t_i}^t u^{(i+1)}(s)ds \quad (i = m, \dots, n-1)$$

we find

$$\begin{aligned} |u^{(i)}(t)| & \leq \sum_{j=i}^{n-1} \delta^{j-i} |u^{(j)}(t_j)| + \delta^{n-1-i} \int_0^\delta |u^{(n)}(s)|ds \leq \\ & \leq n^n \delta^{-n} \left(\int_0^1 |u^{(m)}(s)|^2 ds \right)^{\frac{1}{2}} + \int_0^\delta |u^{(n)}(s)|ds \\ & \quad \text{for } 0 \leq t \leq \delta \quad (i = m, \dots, n-1). \end{aligned}$$

Hence by (2.9) it follows that

$$\begin{aligned} \sum_{i=m}^{n-1} |u^{(i)}(0)| &\leq n^{1+n} \delta^{-n} \left(\int_0^1 |u^{(m)}(s)|^2 ds \right)^{\frac{1}{2}} + n \int_0^\delta |u^{(n)}(s)| ds \leq \\ &\leq n^{2+2n} \delta^{-2n} r_1 + \frac{1}{4r_1} \int_0^1 |u^{(m)}(s)|^2 ds + n \int_0^\delta |u^{(n)}(s)| ds \leq \\ &\leq n^{2+2n} \delta^{-2n} r_1 + \frac{1}{4} \left(1 + \sum_{i=m}^{n-1} |u^{(i)}(0)| \right) + n \int_0^\delta |u^{(n)}(s)| ds. \end{aligned} \quad (2.13)$$

Let I_1 and I_2 be respectively the set of points of the interval $[0, \delta]$ at which the inequalities $|u(t)| \leq 8nr_1$ and $|u(t)| > 8nr_1$ hold. Then on account of inequalities (2.9), (2.10), and (2.12) we have

$$\begin{aligned} \int_0^\delta |u^{(n)}(s)| ds &= \int_{I_1} |u^{(n)}(s)| ds + \int_{I_2} |u^{(n)}(s)| ds \leq \\ &\leq \int_{I_1} b_0 \left(s, 8nr_1, r_1^{\frac{1}{2}} \left(1 + \sum_{i=m}^{n-1} |u^{(i)}(0)| \right)^{\frac{1}{2}} \right) ds + \frac{1}{8nr_1} \int_{I_2} |u^{(n)}(s)u(s)| ds \leq \\ &\leq \int_0^\delta b_0(s, 8nr_1, y_0) ds + \frac{1}{4n} \left(1 + \sum_{i=m}^{n-1} |u^{(i)}(0)| \right). \end{aligned}$$

Taking the latter inequality into account, from (2.13) we obtain

$$\sum_{i=m}^{n-1} |u^{(i)}(0)| \leq n^{2+2n} \delta^{-2n} r_1 + n \int_0^\delta b_0(s, 8nr_1, y_0) ds + \frac{1}{2} \left(1 + \sum_{i=m}^{n-1} |u^{(i)}(0)| \right).$$

This immediately implies estimates (2.11), where

$$r_2 = 2n^{2+2n} \delta^{-2n} r_1 + 2n \int_0^\delta b_0(s, 8nr_1, y_0) ds + 1$$

is the positive number independent of u . \square

Lemma 2.6. *Let $\rho_0 > 0$, the functions $a_i : R_+ \rightarrow R_+$ ($i = 1, 2$) be measurable and satisfy conditions (1.6), and the function $b_0 : R_+^3 \rightarrow R_+$ be locally summable with respect to the first argument, nondecreasing with respect to the last two arguments, and satisfies condition (2.8). Then there exists a positive number r such that any function $u \in \tilde{C}^{n-1}$ satisfying for some natural k conditions (2.4) and (2.6) admits the estimates*

$$\int_0^k |u^{(m)}(s)|^2 ds + \int_0^k (1+s)^{n-2m} |u^{(n)}(s)u(s)| ds \leq r, \quad (2.14)$$

$$|u^{(i)}(t)| \leq r(1+t)^{m-i-\frac{1}{2}} \quad \text{for } 0 \leq t \leq k \quad (i = 0, \dots, m-1), \quad (2.15)$$

$$|u^{(i)}(t)| \leq r(1+t)^{n-i-1} + \int_0^t (t-s)^{n-i-1} b_0(s, r(1+s)^{m-\frac{1}{2}}, r) ds$$

for $0 \leq t \leq k$ ($i = m, \dots, n-1$). (2.16)

Proof. By virtue of (1.6)

$$\varepsilon = \mu_m^n - \int_0^{+\infty} (1+s)^{n-2m} a_1(s) ds > 0, \quad (2.17)$$

$$\rho_1 = \left(1 + m\rho_0^2\right) \mu_m^n + \int_0^{+\infty} (1+s)^{n-2m} a_2(s) ds < +\infty. \quad (2.18)$$

For

$$r_1 = (\varepsilon^{-1} + \varepsilon^{-1} \mu_m^n + 2)\rho_1 + (\varepsilon^{-1} + \varepsilon^{-1} \mu_m^n + 1)m\rho_0 + 1 + m^2 \rho_0^2$$

we shall choose $r_2 > 0$ such that the conclusion of Lemma 2.5 is valid and put $r = r_1(1 + r_2 + mr_2) + m\rho_0$.

Let $u \in \tilde{C}^{n-1}$ be an arbitrary function satisfying, for some natural k , inequalities (2.4) and (2.6). Then

$$\begin{aligned} & (-1)^{n-m}(1+t)^{n-2m} u^{(n)}(t)u(t) + w_k(t) = \\ & = (1+t)^{n-2m} a_1(t) \|\sigma_k(u)\|_{0,m}^2 + (1+t)^{n-2m} a_2(t), \end{aligned}$$

where

$$\begin{aligned} w_k(t) & = (1+t)^{n-2m} |(-1)^{n-m} u^{(n)}(t)u(t) - \\ & \quad - a_1(t) \|\sigma_k(u)\|_{0,m}^2 - a_2(t)|. \end{aligned} \quad (2.19)$$

Hence due to Lemma 2.2 we find

$$\begin{aligned} & \mu_m^n \int_0^k |u^{(m)}(s)|^2 ds + \int_0^k w_k(s) ds = l(u) + \\ & + \|\sigma_k(u)\|_{0,m}^2 \int_0^k (1+s)^{n-2m} a_1(s) ds + \int_0^k (1+s)^{n-2m} a_2(s) ds, \end{aligned} \quad (2.20)$$

where

$$l(u) = \sum_{i=0}^{m-1} (-1)^{m-i} u^{(i)}(0) u^{(n-1-i)}(0) \quad \text{for } n = 2m$$

and

$$\begin{aligned} l(u) & = -\frac{1}{2} |u^{(m)}(0)|^2 + \sum_{i=0}^{m-1} (-1)^{m-i} \left[(i+1) u^{(n-2-i)}(0) - \right. \\ & \quad \left. - u^{(n-1-i)}(0) \right] u^{(i)}(0) \quad \text{for } n = 2m + 1. \end{aligned}$$

By conditions (2.2) and (2.6)

$$\|\sigma_k(u)\|_{0,m}^2 \leq 1 + m\rho_0^2 + \int_0^k |u^{(m)}(s)|^2 ds \tag{2.21}$$

and

$$l(u) \leq m\rho_0 \sum_{i=m}^{n-1} |u^{(i)}(0)|.$$

Taking into account these inequalities and conditions (1.6), (2.17), and (2.18), from (2.19) and (2.20) we obtain

$$\begin{aligned} & \int_0^k w_k(s) ds \geq \int_0^k (1+s)^{n-2m} |u^{(n)}(s)u(s)| ds - \\ & - \|\sigma_k(u)\|_{0,m}^2 \int_0^k (1+s)^{n-2m} a_1(s) ds - \int_0^k (1+s)^{n-2m} a_2(s) ds \geq \\ & \geq \int_0^k (1+s)^{n-2m} |u^{(n)}(s)u(s)| ds - \mu_m^n \int_0^k |u^{(m)}(s)|^2 ds - \rho_1, \tag{2.22} \\ & \mu_m^n \int_0^k |u^{(m)}(s)|^2 ds + \int_0^k w_k(s) ds \leq (\mu_m^n - \varepsilon) \int_0^k |u^{(m)}(s)|^2 ds + \rho_1 + \\ & + m\rho_0 \sum_{i=m}^{n-1} |u^{(i)}(0)| \end{aligned}$$

and

$$\varepsilon \int_0^k |u^{(m)}(s)|^2 ds + \int_0^k w_k(s) ds \leq \rho_1 + m\rho_0 \sum_{i=m}^{n-1} |u^{(i)}(0)|. \tag{2.23}$$

Since w_k is nonnegative, from (2.22) and (2.23) we obtain

$$\begin{aligned} & \int_0^k |u^{(m)}(s)|^2 ds \leq \varepsilon^{-1} \rho_1 + \varepsilon^{-1} m\rho_0 \sum_{i=m}^{n-1} |u^{(i)}(0)|, \tag{2.24} \\ & \int_0^k (1+s)^{n-2m} |u^{(n)}(s)u(s)| ds \leq (\varepsilon^{-1} \mu_m^n + 2) \rho_1 + \\ & + (1 + \varepsilon^{-1} \mu_m^n) m\rho_0 \sum_{i=m}^{n-1} |u^{(i)}(0)|, \\ & \int_0^k |u^{(m)}(s)|^2 ds + \int_0^k (1+s)^{n-2m} |u^{(n)}(s)u(s)| ds \leq \\ & \leq r_1 \left(1 + \sum_{i=m}^{n-1} |u^{(i)}(0)| \right). \tag{2.25} \end{aligned}$$

On the other hand, using (2.21) and (2.24), from (2.4) we have

$$|u^{(n)}(t)| \leq b_0 \left(t, |u(t)|, r_1^{\frac{1}{2}} \left(1 + \sum_{i=m}^{n-1} |u^{(i)}(0)| \right)^{\frac{1}{2}} \right) \text{ for } 0 \leq t \leq k. \quad (2.26)$$

Therefore inequalities (2.9) and (2.10) are fulfilled. Thus estimates (2.11) hold by virtue of the choice of r_2 .

With (2.11) taken into account, inequalities (2.25) and (2.26) imply estimates (2.14) and (2.16). On the other hand, using (2.6) and (2.11), from (2.25) we find

$$\begin{aligned} |u^{(i)}(t)| &= \left| \sum_{j=i}^{m-1} \frac{t^{j-i}}{(j-i)!} u^{(j)}(0) + \frac{1}{(m-1-i)!} \int_0^t (t-s)^{m-1-i} u^{(m)}(s) ds \right| \leq \\ &\leq m\rho_0(1+t)^{m-1-i} + \left[\int_0^t (t-s)^{2m-2-2i} ds \right]^{\frac{1}{2}} \left[\int_0^t |u^{(m)}(s)|^2 ds \right]^{\frac{1}{2}} \leq \\ &\leq r(1+t)^{m-i-\frac{1}{2}} \text{ for } 0 \leq t \leq k \quad (i = 0, \dots, m-1). \end{aligned}$$

Therefore estimates (2.15) hold as well. \square

Lemma 2.7. *Let γ and $a_i : R_+ \rightarrow R_+$ ($i = 1, 2$) be a positive constant and measurable functions, satisfying conditions (1.12) and (1.13), $\rho_0 > 0$, and $b_0 : R_+^3 \rightarrow R_+$ be locally summable with respect to the first argument, nondecreasing with respect to the last two arguments, and satisfies condition (2.8). Then there is a positive number r such that any function $u \in \tilde{C}^{m-1}$, satisfying for some natural k conditions (2.5) and (2.6), admits estimates*

$$\sum_{i=0}^m \int_0^k (1+s)^{2i} |u^{(i)}(s)|^2 ds + \int_0^k (1+s)^n |u^{(n)}(s)u(s)| ds \leq r, \quad (2.27)$$

$$|u^{(i)}(t)| \leq r(1+t)^{-i-\frac{1}{2}} \text{ for } 0 \leq t \leq k \quad (i = 0, \dots, m-1), \quad (2.28)$$

$$\begin{aligned} |u^{(i)}(t)| &\leq r(1+t)^{n-i-1} + \int_0^t (t-s)^{n-i-1} b_0(s, r(1+s)^{-\frac{1}{2}}, r) ds \\ &\text{for } 0 \leq t \leq k \quad (i = m, \dots, n-1). \end{aligned} \quad (2.29)$$

Proof. According to (1.13) there exists $\varepsilon \in]0, \delta[$ such that

$$\gamma + (-1)^m \frac{n!}{2} > \frac{m-1}{4} \gamma_n (\eta + 4)^{m-1} + \varepsilon, \quad (2.30)$$

where

$$\eta = \frac{\gamma_n}{\delta - \varepsilon} + \frac{(m-2)(4m^2 - m + 3)}{3}.$$

Let

$$\rho_1 = \sum_{i=1}^2 \int_0^{+\infty} (1+s)^n a_i(s) ds, \quad \rho_2 = (1+\rho_0^2) \sum_{i=0}^{m-1} \sum_{j=i}^{n-1-i} \frac{n!}{(1+i+j)!} \mu_{ij}^n,$$

$$\rho_3 = 2m\rho_0^2 \eta^{m-2} \gamma_n, \quad r_1 = \left[\varepsilon^{-1} \left(1 + \gamma + \frac{n!}{(2m)!} \mu_m^n \right) + 2 \right] (\rho_1 + \rho_2 + \rho_3) + 1.$$

Choose $r_2 > 0$ such that the conclusion of Lemma 2.5 is valid and put

$$r = 2(m-1)m\rho_0^2 \eta^{m-2} + r_1 [(m-1)^2(\eta+4)^{m-1} + m](1+nr_2).$$

Let $u \in \tilde{C}^{n-1}$ be an arbitrary function satisfying for some natural k conditions (2.5) and (2.6). Then

$$\begin{aligned} & (-1)^{n-m} (1+s)^n u^{(n)}(s) u(s) + \gamma |u(s)|^2 + w_k(s) = \\ & (1+s)^n a_1(s) \|\sigma_k(u)\|_m^2 + (1+s)^n a_2(s) \quad \text{for } 0 \leq s \leq k, \end{aligned}$$

where

$$\begin{aligned} w_k(s) = & \left| (-1)^{n-m} (1+s)^n u^{(n)}(s) u(s) + \gamma |u(s)|^2 - \right. \\ & \left. - (1+s)^n a_1(s) \|\sigma_k(u)\|_m^2 - (1+s)^n a_2(s) \right|. \end{aligned}$$

Hence by virtue of Lemma 2.2 we find

$$\begin{aligned} & \sum_{i=0}^m (-1)^{m-i} \frac{n!}{(2i)!} \mu_i^n \int_0^k (1+s)^{2i} |u^{(i)}(s)|^2 ds + \gamma \int_0^k |u(s)|^2 ds + \\ & + \int_0^k w_k(s) ds = l(u) + \|\sigma_k(u)\|_m^2 \int_0^k (1+s)^n a_1(s) ds + \int_0^k (1+s)^n a_2(s) ds, \end{aligned}$$

where

$$l(u) = \sum_{i=0}^{n-m-1} \sum_{j=i}^{n-1-i} (-1)^{m-1-j} \frac{n!}{(i+j+1)!} \mu_{ij}^n u^{(i)}(0) u^{(j)}(0).$$

If we set $c_m(u) = 0$ for $m = 1$,

$$c_m(u) = \sum_{i=1}^{m-1} (-1)^{m-i-1} \frac{n!}{(2i)!} \mu_i^n \int_0^k (1+s)^{2i} |u^{(i)}(s)|^2 ds \quad \text{for } m > 1,$$

then the latter equality can be rewritten as

$$\begin{aligned} & \frac{n!}{(2m)!} \mu_m^n \int_0^k (1+s)^{2m} |u^{(m)}(s)|^2 ds + \left[\gamma + (-1)^m \frac{n!}{2} \right] \int_0^k |u(s)|^2 ds + \\ & + \int_0^k w_k(s) ds = c_m(u) + l(u) + \|\sigma_k(u)\|_m^2 \int_0^k (1+s)^n a_1(s) ds + \end{aligned}$$

$$+ \int_0^k (1+s)^n a_2(s) ds. \quad (2.31)$$

By conditions (1.12), (2.3), and (2.6)

$$\begin{aligned} & \|\sigma_k(u)\|_m^2 \int_0^k (1+s)^n a_1(s) ds + \int_0^k (1+s)^n a_2(s) ds \leq \\ & \leq \left(\frac{n!}{(2m)!} \mu_m^n - \delta \right) \int_0^k (1+s)^{2m} |u^{(m)}(s)|^2 ds + \rho_1, \quad (2.32) \\ l(u) & \leq \sum_{i=0}^{m-1} \sum_{j=i}^{n-1-i} (-1)^{m-1-j} \frac{n!}{(1+i+j)!} \mu_{ij}^n u^{(i)}(0) u^{(j)}(0) \leq \\ & \leq \sum_{i=0}^{m-1} \left[\rho_0^2 \sum_{j=i}^{m-1} \frac{n!}{(1+i+j)!} \mu_{ij}^n + \rho_0 \sum_{j=m}^{n-1-i} \frac{n!}{(1+i+j)!} \mu_{ij}^n |u^{(j)}(0)| \right] \leq \\ & \leq \rho_2 \left[1 + \sum_{i=m}^{n-1} |u^{(i)}(0)| \right]. \quad (2.33) \end{aligned}$$

If $m > 1$, then on account of Lemma 2.3 we have

$$\begin{aligned} c_m(u) & \leq \sum_{j=0}^{m_0-1} \frac{n!}{(2m-2-4j)!} \mu_{m-1-2j}^n \int_0^k (1+s)^{2m-2-4j} |u^{(m-1-2j)}(s)|^2 ds \leq \\ & \leq \rho_3 + \sum_{j=0}^{m_0-1} \frac{n!}{(2m-2-4j)!} \mu_{m-1-2j}^n \alpha_{m-1-2j}(\eta) \int_0^k |u(s)|^2 ds + \\ & + \sum_{j=0}^{m_0-1} \frac{n!}{(2m-2-4j)!} \mu_{m-1-2j}^n \beta_{m-1-2j}(\eta) \int_0^k (1+s)^{2m} |u^{(m)}(s)|^2 ds. \end{aligned}$$

On the other hand, by virtue of the inequality

$$\eta > 1 + \frac{(m-2)(4m^2 - m + 3)}{3}$$

(2.7) implies clearly that

$$\begin{aligned} \alpha_i(\eta) & \leq \frac{m-1}{4} (\eta+4)^{m-1}, \\ \beta_i(\eta) & \leq \left[\eta - \frac{(m-2)(4m^2 - m + 3)}{3} \right]^{-1} = \frac{\delta - \varepsilon}{\gamma_n} \quad (2.34) \\ & (i = 1, \dots, m-1). \end{aligned}$$

Therefore

$$c_m(u) \leq \rho_3 + \frac{m-1}{4}(\eta+4)^{m-1}\gamma_n \int_0^k |u(s)|^2 ds + (\delta-\varepsilon) \int_0^k (1+s)^{2m} |u^{(m)}(s)|^2 ds.$$

If along with this inequality we take into consideration inequalities (2.30), (2.32), and (2.33), then from (2.31) we have

$$\begin{aligned} \varepsilon \left[\int_0^k (1+s)^{2m} |u^{(m)}(s)|^2 ds + \int_0^k |u(s)|^2 ds \right] + \int_0^k w_k(s) ds &\leq \\ &\leq (\rho_1 + \rho_2 + \rho_3) \left[1 + \sum_{i=m}^{n-1} |u^{(i)}(0)| \right]. \end{aligned}$$

Therefore

$$\begin{aligned} \int_0^k (1+s)^{2m} |u^{(m)}(s)|^2 ds + \int_0^k |u(s)|^2 ds &\leq \\ &\leq \varepsilon^{-1} (\rho_1 + \rho_2 + \rho_3) \left[1 + \sum_{i=m}^{n-1} |u^{(i)}(0)| \right], \quad (2.35) \\ \int_0^k (1+s)^n |u^{(n)}(s)u(s)| ds &\leq \int_0^k w_k(s) ds + \gamma \int_0^k |u(s)|^2 ds + \\ + \left[1 + \int_0^k (1+s)^{2m} |u^{(m)}(s)|^2 ds \right] &\int_0^k (1+s)^n a_1(s) ds + \int_0^k (1+s)^n a_2(s) ds \leq \\ \leq \rho_1 + \left[\varepsilon^{-1} \left(\gamma + \frac{n!}{(2m)!} \mu_m^n \right) + 1 \right] &(\rho_1 + \rho_2 + \rho_3) \left[1 + \sum_{i=m}^{n-1} |u^{(i)}(0)| \right], \\ \int_0^k (1+s)^{2m} |u^{(m)}(s)|^2 ds + \int_0^k |u(s)|^2 ds &+ \\ + \int_0^k (1+s)^n |u^{(n)}(s)u(s)| ds &\leq r_1 \left[1 + \sum_{i=m}^{n-1} |u^{(i)}(0)| \right]. \quad (2.36) \end{aligned}$$

On the other hand, using (2.3) and (2.35), from (2.5) we obtain inequality (2.26). Therefore inequalities (2.9) and (2.10) are fulfilled. Thus estimates (2.11) hold by virtue of the choice of r_2 .

By inequalities (2.11), (2.34), (2.36) and Lemma 2.3 we have

$$\begin{aligned} \int_0^k (1+s)^{2m} |u^{(m)}(s)|^2 ds + \int_0^k |u(s)|^2 ds &+ \\ + \int_0^k (1+s)^n |u^{(n)}(s)u(s)| ds &\leq r_1(1 + nr_2), \end{aligned}$$

$$\begin{aligned}
& \sum_{i=1}^{m-1} \int_0^k (1+s)^{2i} |u^{(i)}(s)|^2 ds \leq 2m(m-1)\rho_0^2 \eta^{m-2} + \\
& + \sum_{i=1}^{m-1} \alpha_i(\eta) \int_0^k |u(s)|^2 ds + \sum_{i=1}^{m-1} \beta_i(\eta) \int_0^k (1+s)^{2m} |u^{(m)}(s)|^2 ds \leq \\
& \leq 2m(m-1)\rho_0^2 \eta^{m-2} + (m-1) \left[(m-1)(\eta+4)^{m-1} + 1 \right] \times \\
& \quad \times \left[\int_0^k (1+s)^{2m} |u^{(m)}(s)|^2 ds + \int_0^k |u(s)|^2 ds \right] \leq \\
& \leq 2m(m-1)\rho_0^2 \eta^{m-2} + r_1 \left[(m-1)^2 (\eta+4)^{m-1} + m-1 \right] (1+nr_2).
\end{aligned}$$

Therefore estimate (2.27) is valid.

In view of (2.6)

$$\begin{aligned}
|u^{(i)}(t)| &= \left| \int_t^k u^{(i+1)}(s) ds \right| \leq \\
&\leq \left| \int_t^k (1+s)^{-2i-2} ds \right|^{\frac{1}{2}} \left[\int_t^k (1+s)^{2i+2} |u^{(i+1)}(s)|^2 ds \right]^{\frac{1}{2}} \leq \\
&\leq (1+t)^{-i-\frac{1}{2}} \left[\int_t^k (1+s)^{2i+2} |u^{(i+1)}(s)|^2 ds \right]^{\frac{1}{2}} \\
&\text{for } 0 \leq t \leq k \quad (i = 0, \dots, m-1).
\end{aligned}$$

Hence, with (2.27) taken into account, we obtain estimates (2.28). As to estimates (2.29), they follow from inequalities (2.10), (2.11), and (2.28). \square

2.3. Some Properties of Functions from the Classes $C_0^{n-1,m}$ and $C^{n-1,m}$.

Lemma 2.8. *If $u \in C_0^{n-1,m}$, then*

$$\lim_{t \rightarrow +\infty} t^{i+\frac{1}{2}-m} u^{(i)}(t) = 0 \quad (i = 0, \dots, m-1) \quad (2.37)$$

and for any constants c_{ij} ($i = 0, \dots, n-m-1; j = i, \dots, n-1-i$) the function

$$w(t) = \sum_{i=0}^{n-m-1} \sum_{j=i}^{n-1-i} c_{ij} t^{i+j+1-2m} u^{(i)}(t) u^{(j)}(t)$$

satisfies the condition

$$\liminf_{t \rightarrow +\infty} |w(t)| = 0. \quad (2.38)$$

Lemma 2.9. *If $u \in C^{n-1,m}$, then*

$$\lim_{t \rightarrow +\infty} t^{i+\frac{1}{2}} u^{(i)}(t) = 0 \quad (i = 0, \dots, m-1) \quad (2.39)$$

and for any constants c_{ij} ($i = 0, \dots, n-m-1; j = i, \dots, n-1-i$) the function

$$w(t) = \sum_{i=0}^{n-m-1} \sum_{j=i}^{n-1-i} c_{ij} t^{i+j+1} u^{(i)}(t) u^{(j)}(t)$$

satisfies condition (2.38).

Lemma 2.10. *If $m \geq 2$, $r_0 \geq 0$, $u \in C^{n-1,m}$,*

$$i|u^{(i-1)}(0)|^2 - u^{(i)}(0)u^{(i-1)}(0) \leq r_0 \quad (i = 1, \dots, m-1),$$

then for any $\eta > \frac{1}{3}(m-2)(4m^2 - m + 3)$ we have the estimates

$$\begin{aligned} \int_0^{+\infty} (1+t)^{2i} |u^{(i)}(t)|^2 dt &\leq 2\eta^{m-2} r_0 + \alpha_i(\eta) \int_0^{+\infty} |u(t)|^2 dt + \\ &+ \beta_i(\eta) \int_0^{+\infty} (1+t)^{2m} |u^{(m)}(t)|^2 dt \quad (i = 1, \dots, m-1). \end{aligned}$$

These lemmas follow immediately from Lemmas 4.3-4.5 in the monograph [4].

2.4. Lemma on the Solvability of an Auxiliary Two-Point Boundary Value Problem. Let $t_0 \in]0, +\infty[$, $c_i \in R$ ($i = 0, \dots, m-1$), $\bar{c}_j \in R$ ($j = 0, \dots, n-m-1$), $p : [0, t_0] \rightarrow R$ be a summable function and $q : C^{n-1}([0, t_0]) \rightarrow L([0, t_0])$ be a continuous operator. Consider the boundary value problem

$$u^{(n)}(t) = p(t)u(t) + q(u)(t), \quad (2.40)$$

$$u^{(i)}(0) = c_i \quad (i = 0, \dots, m-1), \quad (2.41)$$

$$u^{(j)}(t_0) = \bar{c}_j \quad (j = 0, \dots, n-m-1).$$

Lemma 2.11. *Let $(-1)^{n-m-1}p(t) \geq 0$ for $0 \leq t \leq t_0$, and let there exist a summable function $q^* : [0, t_0] \rightarrow R_+$ such that the inequality*

$$|q(u)(t)| \leq q^*(t) \quad \text{for } 0 \leq t \leq t_0 \quad (2.42)$$

holds for any function $u \in C^{n-1}([0, t_0])$ satisfying the boundary conditions (2.41). Then problem (2.40), (2.41) is solvable.

Proof. We shall show in the first place that the homogeneous problem

$$u^{(n)}(t) = p(t)u(t), \quad (2.40_0)$$

$$u^{(i)}(0) = 0 \quad (i = 0, \dots, m-1), \quad (2.41_0)$$

$$u^{(j)}(t_0) = 0 \quad (j = 0, \dots, n-m-1)$$

has only a trivial solution.

Let u be an arbitrary solution of problem (2.40₀), (2.41₀). Then, since the function $(-1)^{n-m}p$ is nonpositive, we have

$$(-1)^{n-m}u^{(n)}(t)u(t) + |p(t)||u(t)|^2 = 0.$$

On integrating both sides of this equality from 0 to t_0 , by virtue of Lemma 2.2 and conditions (2.41₀) we obtain

$$\mu_m^n \int_0^{t_0} |u^{(m)}(t)|^2 dt + \int_0^{t_0} t^{n-2m} |p(t)||u(t)|^2 dt + l_0(u) = 0,$$

where $l_0(u) = 0$ for $n = 2m$ and $l_0(u) = \frac{1}{2}|u^{(m)}(0)|^2$ for $n = 2m + 1$. Hence it is clear that $u(t) \equiv 0$.

Since problem (2.40₀), (2.41₀) has only a trivial solution, problem (2.40), (2.41) is equivalent to the integral equation

$$u(t) = u_0(t) + \int_0^{t_0} G(t, s)q(u)(s)ds, \quad (2.43)$$

where u_0 is the solution of the homogeneous equation (2.40₀) under the boundary conditions (2.41) and G is the Green function of problem (2.40₀), (2.41₀).

By Shauder's principle [6] the continuity of the operator $q : C^{n-1}([0, t_0]) \rightarrow L([0, t_0])$ and condition (2.42) guarantees the existence of at least one solution of the integral equation (2.43). \square

§ 3. PROOFS OF THE EXISTENCE AND UNIQUENESS THEOREMS

Proof of Theorem 1.1. Let

$$\rho_0 = \max\{|c_i| \mid i = 0, \dots, m-1\}, \quad b_0(t, x, y) = b(t, x, y),$$

r be the number from Lemma 2.6, $r_0 = 2m^2\rho_0^2 + 2r + 1$, and

$$\chi(s) = \begin{cases} 1 & \text{for } 0 \leq s \leq r_0 \\ 2 - \frac{s}{r_0} & \text{for } r_0 < s < 2r_0 \\ 0 & \text{for } s \geq 2r_0 \end{cases} . \quad (3.1)$$

For any natural k put $q_k(u)(t) = \chi(\|\sigma_k(u)\|_{0,m})f(\sigma_k(u))(t)$ and consider the boundary value problem

$$u^{(n)}(t) = q_k(u)(t), \quad (3.2)$$

$$u^{(i)}(0) = c_i \quad (i = 0, \dots, m-1), \quad (3.3)$$

$$u^{(j)}(k) = 0 \quad (j = 0, \dots, n-m-1).$$

By Lemma 2.1 the continuity of the operator $f : C^{n-1} \rightarrow L$ implies the continuity of the operator $q_k : C([0, k]) \rightarrow L([0, k])$. Let $u \in C^{n-1}([0, k])$ be an arbitrary function satisfying the boundary conditions (3.3). Then in view of (1.5)

$$|q_k(u)(t)| \leq \chi(\|\sigma_k(u)\|_{0,m})b(t, |u(t)|, \|\sigma_k(u)\|_{0,m}) \quad \text{for } 0 \leq t \leq k.$$

On the other hand,

$$\begin{aligned} |u(t)| &= \left| \sum_{i=0}^{m-1} \frac{t^i}{i!} u^{(i)}(0) + \frac{1}{(m-1)!} \int_0^t (t-s)^{m-1} u^{(m)}(s) ds \right| \leq \\ &\leq m\rho_0(1+t)^{m-1} + t^{m-\frac{1}{2}} \left[\int_0^t |u^{(m)}(s)|^2 ds \right]^{\frac{1}{2}} \leq \\ &\leq (1+t)^{m-\frac{1}{2}} (m\rho_0 + \|\sigma_k(u)\|_{0,m}) \quad \text{for } 0 \leq t \leq k. \end{aligned}$$

Therefore

$$|q_k(u)(t)| \leq q^*(t) \quad \text{for } 0 \leq t \leq k,$$

where $q^*(t) = b(t, m\rho_0 + 2r_0)(1+t)^{m-\frac{1}{2}}, 2r_0$ and $q^* \in L([0, k])$. Since all the conditions of Lemma 2.11 are fulfilled for problem (3.2), (3.3), it is solvable. Let u_k be some solution of this problem. By inequalities (1.4) and (1.5)

$$\begin{aligned} &(-1)^{n-m-1} u_k^{(n)}(t) u_k(t) = \\ &= (-1)^{n-m-1} \chi(\|\sigma_k(u_k)\|_{0,m}) \sigma_k(u_k)(t) f(\sigma_k(u_k))(t) \geq \\ &\geq -a_1(t) \|\sigma_k(u_k)\|_{0,m}^2 - a_2(t) \quad \text{for } 0 \leq t \leq k, \\ &|u_k^{(n)}(t)| \leq b_0(t, |u_k(t)|, \|\sigma_k(u_k)\|_{0,m}) \quad \text{for } 0 \leq t \leq k. \end{aligned} \quad (3.4)$$

Therefore by virtue of Lemmas 2.1 and 2.6 we have the estimates

$$\begin{aligned} |u_k^{(i)}(t)| &\leq r(1+t)^{m-i-\frac{1}{2}} \quad (i = 0, \dots, m-1), \\ |u_k^{(i)}(t)| &\leq r(1+t)^{n-1-i} + \int_0^t (t-s)^{n-1-i} b_0(s, r(1+s)^{m-\frac{1}{2}}, r) ds \\ &\quad (i = m, \dots, n-1) \quad \text{for } 0 \leq t \leq k, \end{aligned} \quad (3.5)$$

$$\int_0^k |u_k^{(m)}(s)|^2 ds \leq r, \quad \|\sigma_k(u_k)\|_{0,m} \leq r_0. \quad (3.6)$$

Let us extend u_k to the entire R_+ using the equality

$$u_k(t) = \sigma_k(u_k)(t) \quad \text{for } t \geq k. \quad (3.7)$$

Then due to (3.1) and (3.6) we have

$$u_k^{(n)}(t) = f(u_k)(t) \quad \text{for } 0 \leq t \leq k. \quad (3.8)$$

From (3.4)–(3.6) it is clear that the sequences $(u_k^{(i)})_{k=1}^{+\infty}$ ($i = 0, \dots, n-1$) are uniformly bounded and equicontinuous on each finite segment of R_+ . Therefore by the Arzella-Ascoli lemma there exists a subsequence $(u_{k_j})_{j=1}^{+\infty}$ of $(u_k)_{k=1}^{+\infty}$ such that $(u_{k_j}^{(i)})_{j=1}^{+\infty}$ ($i = 0, \dots, n-1$) uniformly converges on each finite segment of R_+ .

By the continuity of the operator $f : C^{n-1} \rightarrow L$ and equality (3.8) it is clear that the function $u(t) = \lim_{j \rightarrow +\infty} u_{k_j}(t)$ for $t \in R_+$ is a solution of equation (1.1). On the other hand, from (3.3) and (3.6) it follows that u satisfies conditions (1.2). \square

Proof of Theorem 1.2. Let u and \bar{u} be two arbitrary solutions of problem (1.1), (1.2). Putting $u_0(t) = u(t) - \bar{u}(t)$, we obtain

$$u_0^{(i)}(0) = 0 \quad (i = 0, \dots, m-1), \quad \int_0^{+\infty} |u_0^{(m)}(s)|^2 ds < +\infty.$$

On the other hand, by condition (1.8) we have

$$(-1)^{n-m}(1+t)^{n-2m} u_0(t) u_0^{(n)}(t) \leq (1+t)^{n-2m} a(t) \|u_0\|_{0,m}^2 \quad \text{for } t \in R_+.$$

After integrating this inequality from 0 to t , by Lemma 2.2 we obtain

$$\mu_m^n \int_0^t |u_0^{(m)}(s)|^2 ds \leq w(t) + \|u_0\|_{0,m}^2 \int_0^t (1+s)^{n-2m} a(s) ds,$$

where

$$\begin{aligned} w(t) = & (n-2m) \sum_{i=0}^{n-m-1} (-1)^{n-m-i} (i+1) u_0^{(i)}(t) u_0^{(n-2-i)}(t) - \\ & - (1+t)^{n-2m} \sum_{i=0}^{n-m-1} (-1)^{n-m-i} u_0^{(i)}(t) u_0^{(n-1-i)}(t). \end{aligned}$$

However, since by Lemma 2.8 the function w satisfies condition (2.38), from the latter inequality we find

$$\mu_m^n \|u_0\|_{0,m}^2 = \mu_m^n \int_0^{+\infty} |u_0^{(m)}(s)|^2 ds \leq \|u_0\|_{0,m}^2 \int_0^{+\infty} (1+s)^{n-2m} a(s) ds.$$

Hence on account of (1.9) we obtain $\|u_0\|_{0,m} = 0$, i.e., $u(t) \equiv \bar{u}(t)$. \square

Proof of Theorem 1.3. Let

$$\rho_0 = \max\{c_i : i = 0, \dots, m - 1\}, \quad b_0(t, x, y) = b(t, x, y) + \gamma(1 + t)^{-n}x,$$

r be the number from Lemma 2.7, $r_0 = r + 1$, and χ be the function given by equality (3.1). For any natural k we put

$$q_k(u)(t) = \chi(\|\sigma_k(u)\|_m) \left[f(\sigma_k(u))(t) - (-1)^{n-m-1} \gamma(1 + t)^{-n} u(t) \right]$$

and consider the equation

$$u^{(n)}(t) = (-1)^{n-m-1} \gamma(1 + t)^{-n} u(t) + q_k(u)(t) \tag{3.9}$$

with the boundary conditions (3.3).

According to Lemma 2.1 the operator $q_k : C^{n-1}([0, k]) \rightarrow L([0, k])$ is continuous. On the other hand, by (1.11), for any $u \in C^{n-1}([0, k])$ we have

$$|q_k(u)(t)| \leq \chi(\|\sigma_k(u)\|_m) b_0(t, |u(t)|, \|\sigma_k(u)\|_m) \quad \text{for } 0 \leq t \leq k.$$

But

$$\begin{aligned} |u(t)| &= |\sigma_k(u)(t)| = \frac{1}{(m-1)!} \left| \int_t^{+\infty} (t-s)^{m-1} [\sigma_k(u)(s)]^{(m)} ds \right| \leq \\ &\leq \frac{1}{(m-1)!} \left| \int_t^{+\infty} (t-s)^{2m-2} (1+s)^{-2m} ds \right|^{\frac{1}{2}} \|\sigma_k(u)\|_m \leq \\ &\leq (1+t)^{-\frac{1}{2}} \|\sigma_k(u)\|_m \quad \text{for } 0 \leq t \leq k. \end{aligned}$$

Therefore

$$|q_k(u)(t)| \leq q^*(t) \quad \text{for } 0 \leq t \leq k,$$

where $q^*(t) = b_0(t, 2r_0(1+t)^{-\frac{1}{2}}, 2r_0)$ and $q^* \in L([0, k])$. Thus all the conditions of Lemma 2.11 are fulfilled for problem (3.9), (3.3). Therefore it has at least one solution. Let u_k be some solution of this problem. Then due to inequalities (1.10) and (1.11) we shall have

$$\begin{aligned} (-1)^{n-m-1} u_k(t) u_k^{(n)}(t) &= \gamma(1+t)^{-n} [1 - \chi(\|\sigma_k(u_k)\|_m)] |u_k(t)|^2 + \\ &\quad + (-1)^{n-m-1} \chi(\|\sigma_k(u_k)\|_m) \sigma_k(u_k)(t) f(\sigma_k(u_k)(t)) \geq \\ &\geq \gamma(1+t)^{-n} |u_k(t)|^2 - a_1(t) \|\sigma_k(u_k)\|_m^2 - a_2(t) \quad \text{for } 0 \leq t \leq k, \\ |u_k^{(n)}(t)| &\leq \gamma(1+t)^{-n} |u_k(t)| + b(t, |u_k(t)|, \|\sigma_k(u_k)\|_m) = \\ &= b_0(t, |u_k(t)|, \|\sigma_k(u_k)\|_m) \quad \text{for } 0 \leq t \leq k. \end{aligned} \tag{3.10}$$

Thus by Lemmas 2.1 and 2.7 we have the estimates

$$|u_k^{(i)}(t)| \leq r(1+t)^{-i-\frac{1}{2}} \quad (i = 0, \dots, m-1),$$

$$|u_k^{(i)}(t)| \leq r(1+t)^{n-1-i} + \int_0^t (t-s)^{n-i-1} b_0(s, r(1+s)^{-\frac{1}{2}}, r) ds$$

$$(i = m, \dots, n-1) \quad \text{for } 0 \leq t \leq k, \quad (3.11)$$

$$\sum_{i=0}^m \int_0^k (1+s)^{2i} |u_k^{(i)}(s)|^2 ds \leq r, \quad \|\sigma_k(u_k)\|_m \leq r_0. \quad (3.12)$$

Let us extend u_k to the entire R_+ using equality (3.7). Then identity (3.8) will be fulfilled by (3.1) and (3.12).

According to estimates (3.10)–(3.12) the sequences $(u_k^{(i)})_{k=1}^{+\infty}$ ($i=0, \dots, n-1$) are uniformly bounded and equicontinuous on each finite segment of R_+ . Therefore there exists a subsequence $(u_{k_j})_{j=1}^{+\infty}$ of $(u_k)_{k=1}^{+\infty}$ such that $(u_{k_j}^{(i)})_{j=1}^{+\infty}$ ($i=0, \dots, n-1$) uniformly converges on each finite segment of R_+ .

Since the operator $f : C^{n-1} \rightarrow L$ is continuous, from conditions (3.3), (3.8), and (3.12) it follows that the function

$$u(t) = \lim_{j \rightarrow +\infty} u_{k_j}(t) \quad \text{for } t \in R_+$$

is a solution of problem (1.1), (1.3). \square

Proof of Theorem 1.4. Let u and \bar{u} be two arbitrary solutions of problem (1.1), (1.3). If we put $u_0(t) = u(t) - \bar{u}(t)$, then

$$u_0^{(i)}(0) = 0 \quad (i = 0, \dots, m-1), \quad \int_0^{+\infty} t^{2j} |u_0^{(j)}(t)|^2 dt < +\infty \quad (j = 0, \dots, m).$$

On the other hand by condition (1.14) we have

$$(-1)^{n-m} (1+t)^n u_0(t) u_0^{(n)}(t) + \gamma |u_0(t)|^2 \leq (1+t)^n a(t) \|u_0\|_m^2 \quad \text{for } t \in R_+.$$

After integrating this inequality from 0 to t , by Lemma 2.2 we obtain

$$\sum_{i=0}^m (-1)^{m-i} \frac{n!}{(2i)!} \mu_i^n \int_0^t (1+s)^{2i} |u_0^{(i)}(s)|^2 ds +$$

$$+ \gamma \int_0^t |u_0(s)|^2 ds \leq w(t) + \|u_0\|_m^2 \int_0^t (1+s)^n a(s) ds \quad \text{for } t \in R_+,$$

where

$$w(t) = \sum_{i=0}^{n-m-1} \sum_{j=i}^{n-1-i} (-1)^{m-j} \frac{n!}{(1+i+j)!} \mu_{ij}^n (1+t)^{1+i+j} u^{(i)}(t) u^{(j)}(t).$$

But by Lemma 2.9 the function w satisfies condition (2.38). Taking this fact and condition (1.15) into account, from the latter inequality we find

$$\delta \|u_0\|_m^2 + \left[\gamma + (-1)^m \frac{n!}{2} \right] \int_0^{+\infty} |u_0(s)|^2 ds \leq c_m(u_0), \quad (3.13)$$

where

$$c_m(u_0) = 0 \quad \text{for } m = 1,$$

$$c_m(u_0) = \sum_{i=1}^{m-1} (-1)^{m-i-1} \frac{n!}{(2i)!} \mu_i^n \int_0^{+\infty} (1+s)^{2i} |u_0^{(i)}(s)|^2 ds \quad \text{for } m > 1.$$

In view of (1.13), for a sufficiently small $\varepsilon \in]0, \delta[$ inequality (2.30) is fulfilled, where

$$\eta = \frac{\gamma_n}{\delta - \varepsilon} + \frac{(m-2)(4m^2 - m + 3)}{3}.$$

If $m > 1$, then by virtue of Lemma 2.10 we have

$$\begin{aligned} c_m(u_0) &\leq \\ &\leq \sum_{j=0}^{m_0-1} \frac{n!}{(2m-2-4j)!} \mu_{m-1-2j}^n \int_0^{+\infty} (1+s)^{2m-2-4j} |u_0^{(m-1-2j)}(s)|^2 ds \leq \\ &\leq \sum_{j=0}^{m_0-1} \frac{n!}{(2m-2-4j)!} \mu_{m-1-2j}^n \alpha_{m-1-2j}(\eta) \int_0^{+\infty} |u_0(s)|^2 ds + \\ &\quad + \sum_{j=0}^{m_0-1} \frac{n!}{(2m-2-4j)!} \mu_{m-1-2j}^n \beta_{m-1-2j}(\eta) \|u_0\|_m^2 \leq \\ &\leq \frac{m-1}{4} (\eta+4)^{m-1} \gamma_n \int_0^{+\infty} |u_0(s)|^2 ds + (\delta - \varepsilon) \|u_0\|_m^2. \end{aligned}$$

If along with this estimate we take into account inequality (2.30), then from (3.13) we obtain

$$\|u_0\|_m^2 + \int_0^{+\infty} |u_0(s)|^2 ds \leq 0.$$

Hence it is clear that $u_0(t) \equiv 0$, i.e., $u(t) \equiv \bar{u}(t)$. \square

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