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THE BOUNDARY-CONTACT PROBLEM OF ELASTICITY FOR HOMOGENEOUS ANISOTROPIC MEDIA WITH A CONTACT ON SOME PART OF THE BOUNDARIES

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ABSTRACT. The existence and uniqueness of solutions of the boundary-contact problem of elasticity for homogeneous anisotropic media with a contact on some part of their boundaries are investigated in the Besov and Bessel potential classes using the methods of the potential theory and the theory of pseudodifferential equations on manifolds with boundary. The smoothness of the solutions obtained is studied.

1. Introduction. The paper is dedicated to the investigation of the boundary-contact problem of the static theory of elasticity for homogeneous anisotropic media when the contact of two bounded domains occurs from the outside on some part of the boundaries. In what follows such problems will be called nonclassical.

Boundary-contact problems of this kind, i.e., when the contact of two bounded domains occurs from the outside on some part of the boundaries, were considered for one differential equation in the Sobolev spaces H_2^s by Schechter in [1].

Classical boundary-contact problems for isotropic homogeneous elastic media were completely investigated by the method of the potential theory and multidimensional singular integral equations on manifolds without boundary in the monograph by Kupradze, Gegelia, Basheleishvili, and Burchuladze [2].

Classical boundary-contact problems (i.e., problems with a contact all over the boundary) for anisotropic homogeneous elastic media were investigated by the method of the potential theory and pseudodifferential equations on compact manifolds without boundary in the monograph by Burchuladze and Gegelia [3] and in the papers by Chkadua [4] and Natroshvili [5].

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In this paper the methods of the potential theory and the general theory of pseudodifferential equations on manifolds with boundary are used to investigate the existence and uniqueness of solutions of the nonclassical boundary-contact problem in Besov and Bessel potential classes. The smoothness of solutions is studied. The solution possesses the C^α-smoothness for any $\alpha < \frac{1}{2}$.

2. Statement of the Problem. Let D_1 and D_2 be the bounded domains in the three-dimensional Euclidean space $\mathbb{R}^3$ with the boundaries $\partial D_1, \partial D_2 \in C^\infty$, $D_1 \cap D_2 = \varnothing$, $\partial D_1 \cap \partial D_2 = \overline{S}_0$; $\overline{S}_0$ is the closure of the nonempty open (in the topology of ∂D_1 and ∂D_2) set S_0. Then $\partial D_1 = S_1 \cup \overline{S}_0$, $\partial D_2 = S_2 \cup \overline{S}_0$, $\partial S_0 \in C^\infty$.

The basic static equations of elasticity for anisotropic homogeneous elastic media in terms of displacement components have the form (see [6], [7], [8])

$$A^{(q)}(\partial x)u^{(q)} + F^{(q)} = 0 \quad \text{in } D_q, \quad q = 1,2,$$

where $u^{(q)}=(u_1^{(q)}, u_2^{(q)}, u_3^{(q)})$ is the displacement vector, $F^{(q)}=(F_1^{(q)}, F_2^{(q)}, F_3^{(q)})$ is the mass force applied to D_q, and $A^{(q)}(\partial x)$ is the matrix differential operator

$$\begin{gathered} A^{(q)}(\partial x) = \|A_{jk}^{(q)}(\partial x)\|_{3\times 3}, \quad A_{jk}^{(q)}(\partial x) = a_{ijlk}^{(q)}\partial_i\partial_l, \\ \partial_i = \frac{\partial}{\partial x_i}, \quad q = 1,2. \end{gathered} \tag{1}$$

$a_{ijlk}^{(q)}$ are the elastic constants satisfying the conditions

$$a_{ijlk}^{(q)} = a_{lkij}^{(q)} = a_{ijkl}^{(q)}.$$

In (1) and below the repeated indices imply summation from 1 to 3.

Assume that the quadratic forms

$$a_{ijlk}^{(q)}\xi_{ij}\xi_{lk}, \quad \xi_{ij} = \xi_{ji}, \tag{2}$$

with respect to the variables ξ_{ij} are positively definite. Let us introduce the differential stress operator

$$\begin{aligned} &T^{(q)}(\partial z, n(z)) = \|T_{jk}^{(q)}(\partial z, n(z))\|_{3\times 3}, \\ &T_{jk}^{(q)}(\partial z, n(z)) = a_{ijlk}^{(q)}n_i(z)\partial_l, \quad q = 1,2, \end{aligned}$$

where $n(z) = (n_1(z), n_2(z), n_3(z))$ is the unit normal of the manifold $\partial D_1 \cup \partial D_2$ at the point $z \in \partial D_1$ (external with respect to D_1) and at the point $z \in \partial D_2$ (internal with respect to D_2).

From the symmetry of coefficients $a_{ijlk}^{(q)}$ and the positive definiteness of the quadratic forms (2) it follows (see [7]) that the operators $A^{(q)}(\partial x)$,

$q = 1, 2$, are strongly elliptic formally self-adjoint differential operators and therefore for any real vector $\xi \in \mathbb{R}^3$ and any complex vector $\eta \in \mathbb{C}^3$ the relations

$$\operatorname{Re}\big(A^{(q)}(\xi)\eta, \eta\big) = \big(A^{(q)}(\xi)\eta, \eta\big) \geq P_0^{(q)}|\xi|^2|\eta|^2, \quad q = 1, 2,$$

are valid, where $P_0^{(q)} = \mathit{const} > 0$ depends only on the elastic constants. Thus, the matrices $A^{(q)}(\xi)$ which are the symbols of $A^{(q)}(\partial x)$ are positively definite for $\xi \in \mathbb{R}^3 \backslash \{0\}$.

In what follows the functional spaces will be denoted as in [8], [9]. For a sufficiently smooth surface M with boundary embedded into a sufficiently smooth compact surface M_0 without boundary we introduce the following Besov spaces:

$$B_{p,t}^s(M) = \big\{f\big|_M : f \in B_{p,t}^s(M_0)\big\},$$
$$\widetilde{B}_{p,t}^s(M) = \big\{g : g \in B_{p,t}^s(M_0), \operatorname{supp} g \subset \bar{M}\big\}.$$

The notations $H_p^s(M)$, $\widetilde{H}_p^s(M)$ have a similar meaning for the space of Bessel potentials.

We shall consider the following basic nonclassical boundary-contact problem:

In the domains D_q, $q = 1, 2$, find the vector-functions $u^{(q)} : D_q \to \mathbb{R}^3$ belonging to the class $W_p^1(D_q) = H_p^1(D_q)$, $q = 1, 2$, and satisfying the conditions

$$\begin{cases} A^{(q)}(\partial x)u^{(q)} = 0 \quad \text{in } D_q, \quad q = 1, 2, & (3) \\ \big\{u^{(1)}\big\}^+ = \varphi_1 \quad \text{on } S_1, & (4) \\ \big\{u^{(2)}\big\}^- = \varphi_2 \quad \text{on } S_2, & (5) \\ \big\{u^{(1)}\big\}^+ - \big\{u^{(2)}\big\}^- = g \quad \text{on } S_0, & (6) \\ \big\{T^{(1)}(\partial z, n(z))u^{(1)}\big\}^+ - \big\{T^{(2)}(\partial z, n(z))u^{(2)}\big\}^- = f \quad \text{on } S_0, & (7) \end{cases}$$

where $\varphi_1 \in B_{p,p}^{1/p'}(S_1)$, $\varphi_2 \in B_{p,p}^{1/p'}(S_2)$, $g \in B_{p,p}^{1/p'}(S_0)$, $f \in B_{p,p}^{-1/p}(S_0)$, $p' = \frac{p}{p-1}$, $1 < p < \infty$.

From the trace theorem it follows (see [8]) that the trace of any function $u^{(1)} \in W_p^1(D_1)$ $(u^{(2)} \in W_p^1(D_2))$ is defined on ∂D_1 (∂D_2) : $\{u^{(1)}\}^+ \in B_{p,p}^{1/p'}(\partial D_1)$ $(\{u^{(2)}\}^- \in B_{p,p}^{1/p'}(\partial D_2))$. Let $u^{(1)} \in W_p^1(D_1)$ $(u^{(2)} \in W_p^1(D_2))$ be such that $A^{(1)}(\partial x)u^{(1)} \in \mathbb{L}_p(D_1)$ $(A^{(2)}(\partial x)u^{(2)} \in \mathbb{L}_p(D_2))$. Then $\{T^{(1)}(\partial z, n(z))u^{(1)}\}^+$ $(\{T^{(2)}(\partial z, n(z))u^{(2)}\}^-)$ is well defined by the equality (see [10], [11])

$$\int\limits_{D_q} \big[v^{(q)}A^{(q)}(\partial x)u^{(q)} + E^{(q)}(u^{(q)}, v^{(q)})\big]\,dx =$$

$$= \pm\langle\{T^{(q)}(\partial z, n(z))u^{(q)}\}^{\pm}, \{v^{(q)}\}^{\pm}\rangle_{\partial D_q}, \tag{8}$$

for any $v^{(q)} \in W^1_{p'}(D_q)$, $E^{(q)}(u^{(q)}, v^{(q)}) = a^{(q)}_{ijlk}\partial_i u^{(q)}_j \partial_l \overline{v}^{(q)}_k$, $q = 1, 2$; the symbol $\langle\cdot,\cdot\rangle$ denotes the duality between the spaces $B^{-1/p}_{p,p}(\partial D_q)$ and $B^{1/p}_{p',p'}(\partial D_q)$. In (8) the sign $+$ is used when $q = 1$ and the sign $-$ when $q = 2$.

Consider the fundamental matrix-function (see [4])

$$H^{(q)}(x) = F^{-1}_{\xi'\to x'}\Big(\pm\frac{1}{2\pi}\int\limits_{\pm}\big(A^{(q)}(i\xi', i\tau)\big)^{-1}e^{i\tau x_3}d\tau\Big), \quad q = 1, 2,$$

where the sign $+$ refers to the case $x_3 > 0$ and the sign $-$ to the case $x_3 < 0$, $x = (x', x_3)$, $\xi' = (\xi_1, \xi_2)$; $\int_{\pm}$ denotes integration over the contour $L^{\pm}$ where L^{+} (L^{-}) has the positive orientation and covers all roots of the polynomial $\det A^{(q)}(i\xi', i\tau)$ with respect to τ in the upper (resp., lower) τ-halfplane; F^{-1} is the inverse Fourier transform.

Then the simple- and double-layer potentials will be written as

$$V^{(q)}(g_1)(x) = \int\limits_{\partial D_q} H^{(q)}(x - y)g_1(y)d_yS, \quad x \notin \partial D_q,$$

$$U^{(q)}(g_2)(x) = \int\limits_{\partial D_q} \big[T^{(q)}(\partial y, n(y))H^{(q)}(x - y)\big]'g_2(y)d_yS, \quad x \notin \partial D_q,$$

$$\partial D_q = S_q \cup \bar{S}_0, \quad q = 1, 2.$$

The symbol $[\ ]'$ denotes the matrix transposition.

For these potentials the theorems below are valid.

Theorem 1 (see [10], [11]). *Let $s \in \mathbb{R}$, $1 < p < \infty$, $1 \le t \le \infty$. Then the operators $V^{(q)}$, $U^{(q)}$, $q = 1, 2$, admit extensions to operators which are continuous in the following spaces:*

$$V^{(q)} : B^s_{p,t}(\partial D_q) \to B^{s+1+1/p}_{p,t}(D_q) \ \big(B^s_{p,p}(\partial D_q) \to H^{s+1+1/p}_p(D_q)\big),$$
$$U^{(q)} : B^s_{p,t}(\partial D_q) \to B^{s+1/p}_{p,t}(D_q) \ \big(B^s_{p,p}(\partial D_q) \to H^{s+1/p}_p(D_q)\big),$$
$$q = 1, 2.$$

Theorem 2 (see [10], [11]). *Let $1 < p < \infty$, $1 \le t \le \infty$, $\varepsilon > 0$, $g_1 \in B^{-1+\varepsilon}_{p,t}(\partial D_q)$, $g_2 \in B^{\varepsilon}_{p,t}(\partial D_q)$, $q = 1, 2$. Then*

$$\big\{V^{(q)}(g_1)(z)\big\}^{\pm} = \int\limits_{\partial D_q} H^{(q)}(z - y)g_1(y)d_yS, \quad z \in \partial D_q,$$

$$\left\{U^{(q)}(g_2)(z)\right\}^{\pm} = (-1)^{q+1}\frac{1}{2}g_2(z) +$$
$$+\int\limits_{\partial D_q}\left[T^{(q)}(\partial y, n(y))H^{(q)}(z-y)\right]' g_2(y)d_yS, \quad z\in\partial D_q, \quad q=1,2.$$

Theorem 3 (see [10], [11]). *Let* $1 < p < \infty$, $g_1 \in B_{p,p}^{-1/p}(\partial D_q)$, $g_2 \in B_{p,p}^{1/p'}(\partial D_q)$, $p' = \frac{p}{p-1}$, $q = 1,2$. *Then*

$$\left\{T^{(q)}(\partial z, n(z))V^{(q)}(g_1)(z)\right\}^{\pm} = (-1)^{q}\frac{1}{2}g_1(z) +$$
$$+\int\limits_{\partial D_q} T^{(q)}(\partial z, n(z))H^{(q)}(z-y)g_1(y)d_yS, \quad z \in \partial D_q,$$
$$\left\{T^{(q)}(\partial z, n(z))U^{(q)}(g_2)(z)\right\}^{+} = \left\{T^{(q)}(\partial z, n(z))U^{(q)}(g_2)(z)\right\}^{-}, \qquad (9)$$
$$z \in \partial D_q, \quad q = 1,2.$$

We introduce the notations

$$V_{-1}^{(q)}(g_1)(z) = \int\limits_{\partial D_q} H^{(q)}(z-y)g_1(y)d_yS,$$
$$V_0^{(q)}(g_2)(z) = \int\limits_{\partial D_q} \left[T^{(q)}(\partial y, n(y))H^{(q)}(z-y)\right]' g_2(y)d_yS,$$
$$\overset{*}{V}{}_0^{(q)}(g_1)(z) = \int\limits_{\partial D_q} T^{(q)}(\partial z, n(z))H^{(q)}(z-y)g_1(y)d_yS, \quad q = 1,2,$$
$$V_1^{(1)}(g_2)(z) = \left\{T^{(1)}(\partial z, n(z))U^{(1)}(g_2)(z)\right\}^{+},$$
$$V_1^{(2)}(g_2)(z) = \left\{T^{(2)}(\partial z, n(z))U^{(2)}(g_2)(z)\right\}^{-}.$$

Theorem 4 (see [10], [11]). *Let* $s \in \mathbb{R}$, $1 < p < \infty$, $1 \le t \le \infty$. *Then the operators* $V_{-1}^{(q)}$, $V_0^{(q)}$,$\overset{*}{V}{}_0^{(q)}$, $V_1^{(q)}$ *admit extensions onto operators which are continuous in the following spaces:*

$$V_{-1}^{(q)} : H_p^s(\partial D_q) \to H_p^{s+1}(\partial D_q) \ \left(B_{p,t}^s(\partial D_q) \to B_{p,t}^{s+1}(\partial D_q)\right),$$
$$V_0^{(q)}, \overset{*}{V}{}_0^{(q)} : H_p^s(\partial D_q) \to H_p^{s}(\partial D_q) \ \left(B_{p,t}^s(\partial D_q) \to B_{p,t}^{s}(\partial D_q)\right),$$
$$V_1^{(q)} : H_p^s(\partial D_q) \to H_p^{s-1}(\partial D_q) \ \left(B_{p,t}^s(\partial D_q) \to B_{p,t}^{s-1}(\partial D_q)\right),$$
$$q = 1,2.$$

The operators $V_{-1}^{(q)}$, $\pm\frac{1}{2}I+V_0^{(q)}$, $\pm\frac{1}{2}I+\overset{*}{V}{}_0^{(q)}$, $V_1^{(q)}$, $q=1,2$ (I is the identity operator) are the pseudodifferential operators of orders -1, 0, 0, 1, respectively. On the manifold ∂D_q their symbols have the form (see [4], [5], [12])

$$\sigma\big(V_{-1}^{(q)}\big)(z,\xi)=-\frac{1}{2\pi}\int\limits_{+}\big(A^{(q)}(\xi+n\tau)\big)^{-1}\,d\tau,\quad q=1,2,$$

$$\sigma\big(-\frac{1}{2}I+\overset{*}{V}{}_0^{(1)}\big)(z,\xi)=\frac{1}{2\pi i}\int\limits_{+}T^{(1)}(\xi+n\tau,n)\big(A^{(1)}(\xi+n\tau)\big)^{-1}\,d\tau,$$

$$\sigma\big(\frac{1}{2}I+V_0^{(1)}\big)(z,\xi)=-\frac{1}{2\pi i}\int\limits_{+}\big(T^{(1)}(\xi+n\tau,n)\big(A^{(1)}(\xi+n\tau)\big)^{-1}\big)'\,d\tau,$$

$$\sigma\big(\frac{1}{2}I+\overset{*}{V}{}_0^{(2)}\big)(z,\xi)=-\frac{1}{2\pi i}\int\limits_{-}T^{(2)}(\xi+n\tau,n)\big(A^{(1)}(\xi+n\tau)\big)^{-1}\,d\tau,$$

$$\sigma\big(-\frac{1}{2}I+V_0^{(2)}\big)(z,\xi)=\frac{1}{2\pi i}\int\limits_{-}\big(T^{(2)}(\xi+n\tau,n)\big(A^{(2)}(\xi+n\tau)\big)^{-1}\big)'\,d\tau,$$

$$\sigma\big(V_1^{(q)}\big)(z,\xi)=-\frac{1}{2\pi}\int\limits_{+}T^{(q)}(\xi+n\tau,n)\big(T^{(q)}(\xi+n\tau,n)\times$$
$$\times\big(A^{(q)}(\xi+n\tau)\big)^{-1}\big)'\,d\tau,\quad q=1,2,$$

where $n=n(z)$, ξ is the cotangential vector of the manifold ∂D_q at the point z.

These pseudodifferential operators are investigated in the Hölder spaces $C^{1,\beta}(S)$ (S is the compact manifold without boundary) in [4], [5], [12].

Equality (9) is a generalization of theorems of the Liapunov–Tauber type (see [4], [5]).

The principal homogeneous symbols of the operators $V_{-1}^{(q)}$ and $V_1^{(q)}$ are even matrix-functions. One can easily verify that this is so for the operator $V_{-1}^{(q)}$, while for the operator $V_1^{(q)}$ this follows from (9).

The pseudodifferential operators $V_{-1}^{(q)}$ and $V_1^{(q)}$ are also formally self-adjoint operators (see [4]).

Let us consider the question whether the Dirichlet boundary value problems are solvable in the classes $W_p^1(D_q)$, $q=1,2$.

The Dirichlet problem in the domain D_q is

$$\begin{cases} A^{(q)}(\partial x)u^{(q)}=0 & \text{in } D_q\,, \\ \{u^{(q)}\}^{\pm}=\Phi^{(q)} & \text{on } \partial D_q, \end{cases}\quad q=1,2, \tag{10_q}$$

where $\Phi^{(q)} \in B^{1/p'}_{p,p}(\partial D_q)$, $1 < p < \infty$; the sign $+$ is used when $q = 1$ and the sign $-$ when $q = 2$.

For problem $(10)_q$ we have

Lemma (see [11]). *The boundary value problem $(10)_q$ has a unique solution in the class $W^1_p(D_q)$, this solution being given by the formula $u^{(q)} = V^{(q)}(V^{(q)}_{-1})^{-1}\Phi^{(q)}$, $q = 1, 2$.*

The operators $\mathcal{A}_1 = (-\frac{1}{2}I + \overset{*}{V}{}^{(1)}_0)(V_{-1}{}^{(1)})^{-1}$ and $\mathcal{A}_2 = (\frac{1}{2}I + \overset{*}{V}{}^{(2)}_0)\times$ $\times(V^{(2)}_{-1})^{-1}$ are the pseudodifferential operators of order 1.

Theorem 5. *For the pseudodifferential operators $\mathcal{A}_1 = (-\frac{1}{2}I + \overset{*}{V}{}^{(1)}_0)\times$ $(V^{(1)}_{-1})^{-1}$ and $\mathcal{A}_2 = (\frac{1}{2}I + \overset{*}{V}{}^{(2)}_0)(V^{(2)}_{-1})^{-1}$ the following relations are valid:*

a) $\langle \mathcal{A}_1\varphi, \varphi\rangle_{\partial D_1} \geq 0$, *for* $\forall \varphi \in H^{1/2}_2(\partial D_1)$,
the equality being fulfilled only if $\varphi = [a^{(1)}_1 \times x] + b^{(1)}_1 + i([a^{(1)}_2 \times x] + b^{(1)}_2)$, *where* $a^{(1)}_1$, $b^{(1)}_1$, $a^{(1)}_2$, $b^{(1)}_2 \in \mathbb{R}^3$ *are arbitrary vectors;*

b) $\langle \mathcal{A}_2\psi, \psi\rangle_{\partial D_2} \leq 0$, *for* $\forall \psi \in H^{1/2}_2(\partial D_2)$,
the equality being fulfilled only if $\psi = [a^{(2)}_1 \times x] + b^{(2)}_1 + i([a^{(2)}_2 \times x] + b^{(2)}_2)$ *for* $\forall a^{(2)}_1$, $b^{(2)}_1$, $a^{(2)}_2$, $b^{(2)}_2 \in \mathbb{R}^3$.

In Theorem 5 the symbol $< \cdot, \cdot >_{\partial D_q}$ denotes the duality between the spaces $H^{\pm 1/2}_2(\partial D_q)$, $q = 1, 2$, defined by the formula

$$\langle f, g\rangle_{\partial D_q} = \int\limits_{\partial D_q} f \cdot \overline{g}\, dS \quad \text{for} \quad f, g \in C^1(\partial D_q)$$

and the symbol $[\cdot \times \cdot]$ denotes the vector product.

The solution of the considered boundary-contact problem (3)–(7) will be sought for in the form of simple-layer potentials in the domains D_q, $q = 1, 2$:

$$u^{(q)} = V^{(q)} g_q \quad \text{in} \quad D_q, \quad q = 1, 2.$$

From the boundary and boundary-contact conditions of the problem we obtain

$$\begin{cases} \pi_1 V^{(1)}_{-1} g_1 = \varphi_1 \quad \text{on} \quad S_1, & (11) \\ \pi_2 V^{(2)}_{-1} g_2 = \varphi_2 \quad \text{on} \quad S_2, & (12) \\ \pi_0 V^{(1)}_{-1}(g_1) - \pi_0 V^{(2)}_{-1} g_2 = g \quad \text{on} \quad S_0, & (13) \\ \pi_0\big(-\frac{1}{2}I + \overset{*}{V}{}^{(1)}_0\big)g_1 - \pi_0\big(\frac{1}{2}I + \overset{*}{V}{}^{(2)}_0\big)g_2 = f \quad \text{on} \quad S_0, & (14) \end{cases}$$

where π_i denotes the operator of restriction on S_i, $i = 0, 1, 2$.

Let $\Phi_0^{(q)} \in B_{p,p}^{1/p'}(\partial D_q)$ be an extention of the function $\varphi_q \in B_{p,p}^{1/p'}(S_q)$ on the entire boundary $\partial D_q = S_q \cup \bar{S}_0$, $q = 1,2$. It is easy to verify that any extension $\Phi^{(q)} \in B_{p,p}^{1/p'}(\partial D_q)$ of the function φ_q has the form $\Phi^{(q)} = \Phi_0^{(q)} + \varphi_0^{(q)}$, where $\varphi_0^{(q)} \in \widetilde{B}_{p,p}^{1/p'}(S_0)$, $q = 1,2$.

Now (11) and (12) imply

$$V_{-1}^{(q)} g_q = \Phi_0^{(q)} + \varphi_0^{(q)}, \quad q = 1,2.$$

Since $V_{-1}^{(q)}$, $q = 1,2$, are invertible operators (see [11], [12]), we have

$$g_q = (V_{-1}^{(q)})^{-1}\big(\Phi_0^{(q)} + \varphi_0^{(q)}\big), \quad q = 1,2. \tag{15}$$

The substitution of (15) in (13) and (14) gives a system of pseudodifferential equations with respect to $\varphi_0^{(1)}$ and $\varphi_0^{(2)}$ defined on the manifold with the boundary S_0:

$$\begin{cases} \varphi_0^{(1)} - \varphi_0^{(2)} = g^*, \\ \pi_0\big(-\frac{1}{2}I + \overset{*}{V}{}_0^{(1)}\big)(V_{-1}^{(1)})^{-1}\varphi_0^{(1)} - \\ \qquad -\pi_0\big(\frac{1}{2}I + \overset{*}{V}{}_0^{(2)}\big)(V_{-1}^{(2)})^{-1}\varphi_0^{(2)} = f^*, \end{cases}$$

where

$$\begin{aligned} g^* &= g - \pi_0\Phi_0^{(1)} + \pi_0\Phi_0^{(2)}, \\ f^* &= f - \pi_0\Big(-\frac{1}{2}I + \overset{*}{V}{}_0^{(1)}\Big)(V_{-1}^{(1)})^{-1}\Phi_0^{(1)} + \\ &\quad + \pi_0\Big(\frac{1}{2}I + \overset{*}{V}{}_0^{(2)}\Big)(V_{-1}^{(2)})^{-1}\Phi_0^{(2)}. \end{aligned}$$

In what follows, demanding $\varphi_k \in B_{p,t}^s(S_k)$, $k = 1,2$, $g \in B_{p,t}^s(S_0)$ (in particular, for $s = \frac{1}{p'}$, $t = p$) we will assume that the following condition holds:

$$\exists \Phi_0^{(k)} \in B_{p,t}^s(\partial D_k),\ k = 1,2\ :\ g - (\pi_0\Phi_0^{(1)} - \pi_0\Phi_0^{(2)}) \in \tilde{B}_{p,t}^s(S_0).$$

Note that in the case $\frac{1}{p} - 1 < s < \frac{1}{p}$ this condition holds automatically (see [8], Theorem 2.10.3(c)). In the case $\frac{1}{p} < s < \frac{1}{p} + 1$ it becomes $g|_{S_0} = \varphi_1|_{S_0} - \varphi_2|_{S_0}$ (see [8], Theorems 2.3.3(b), 2.10.3(b)). In the case $s = \frac{1}{p}$ it looks more complicated (see [8], Remark 4.3.2-2, and also 2.9.1, 4.2.2, and 4.2.3).

This system of pseudodifferential equations can be rewritten as

$$\begin{cases} \varphi_0^{(1)} - \varphi_0^{(2)} = g^*, \\ \pi_0\mathcal{A}_1\varphi_0^{(1)} - \pi_0\mathcal{A}_2\varphi_0^{(2)} = f^*, \end{cases} \tag{16}$$

where $\pi_0\mathcal{A}_1 = \pi_0(-\frac{1}{2}I + \overset{*}{V}{}_0^{(1)})(V_{-1}^{(1)})^{-1}$, and $\pi_0\mathcal{A}_2 = \pi_0(\frac{1}{2}I + \overset{*}{V}{}_0^{(2)})(V_{-1}^{(2)})^{-1}$ are the pseudodifferential operators on the manifold with boundary S_0:

$$\pi_0\mathcal{A}_q : \widetilde{H}_p^s(S_0) \to H_p^{s-1}(S_0) \quad \big(\widetilde{B}_{p,t}^s(S_0) \to B_{p,t}^{s-1}(S_0)\big), \quad q = 1, 2.$$

For the operators $\pi_0\mathcal{A}_q$, $q = 1, 2$, we have

Theorem 6 (see [11]). *Let $1 < p < \infty$, $1 \le t \le \infty$, $\frac{1}{p} - \frac{1}{2} < s < \frac{1}{p} + \frac{1}{2}$. Then the operators*

$$\pi_0 A_q : \widetilde{H}_p^s(S_0) \to H_p^{s-1}(S_0) \ \big(B_{p,t}^s(S_0) \to B_{p,t}^{s-1}(S_0)\big), \ \ q = 1, 2,$$

are invertible.

We define $\varphi_0^{(2)}$ from the first equation of system (16)

$$\varphi_0^{(2)} = \varphi_0^{(1)} - g^*. \tag{17}$$

The substitution of (17) into the second equation of system (16) gives

$$\pi_0\mathcal{A}_1\varphi_0^{(1)} - \pi_0\mathcal{A}_2\varphi_0^{(1)} = f^* - \pi_0\mathcal{A}_2 g^*.$$

Thus we have to investigate the equation

$$(\pi_0\mathcal{A}_1 - \pi_0\mathcal{A}_2)\varphi_0^{(1)} = \Psi. \tag{18}$$

Recalling that a function of the form $[a_1 \times x] + b_1 + i([a_2 \times x] + b_2)$ equal to zero on S_q, $q = 1, 2$, is identically zero, from Theorem 5 it follows that the operator

$$\pi_0\mathcal{A}_1 - \pi_0\mathcal{A}_2 \ : \ \widetilde{H}_2^{1/2}(S_0) \to H_2^{-1/2}(S_0)$$

is invertible.

This pseudodifferential operator is an elliptic operator on the manifold with the boundary S_0.

We introduce the notation

$$\mathcal{P} = \pi_0\mathcal{A}_1 - \pi_0\mathcal{A}_2.$$

Assume that z is an arbitrary point of ∂S_0 and choose some local coordinate system in its neighborhood. Denote by $\sigma_\mathcal{P}(z, \xi')$, $\xi' = (\xi_1, \xi_2)$, the value, at the point z, of the principal homogeneous symbol of the operator $\mathcal{P}$ written in terms of the chosen local coordinate system.

The eigenvalues of the matrix

$$\big(\sigma_\mathcal{P}(z, 0, -1)\big)^{-1} \cdot \sigma_\mathcal{P}(z, 0, +1)$$

play an essential role in the investigation of the Noetherity of the operator $\mathcal{P}$ in the corresponding Besov and Bessel potential spaces.

Since the matrix $\sigma_\mathcal{P}(z, \xi')$ is positively definite, all the eigenvalues of the matrix $(\sigma_\mathcal{P}(z, 0, -1))^{-1} \cdot \sigma_\mathcal{P}(z, 0, +1)$ are positive.

Due to the foregoing arguments, using the general theorems on elliptic pseudodifferential operators on the manifold with boundary obtained in [13], [14], [15], we can prove the next statement (see also [16], [17]).

Theorem 7. *Let $1 < p < \infty$, $1 \leq t \leq \infty$. Then for the operator $\mathcal{P} : \widetilde{H}_p^s(S_0) \to H_p^{s-1}(S_0)$ to be Noetherian it is necessary that the inequality*

$$\frac{1}{p} - \frac{1}{2} < s < \frac{1}{p} + \frac{1}{2} \tag{19}$$

be fulfilled.

If (19) *is fulfilled, then the operator*

$$\mathcal{P} : \widetilde{H}_p^s(S_0) \to H_p^{s-1}(S_0) \quad \big(\widetilde{B}_{p,t}^s(S_0) \to B_{p,t}^{s-1}(S_0)\big)$$

is invertible.

Theorem 7 implies that there exist unique extensions $\Phi^{(q)} = \Phi_0^{(q)} + \varphi_0^{(q)}$ ($\Phi^{(q)} \in B_{p,p}^{1/p'}(\partial D_q)$) of the function φ_q ($\varphi_q \in B_{p,p}^{1/p'}(S_1)$) onto the entire boundary ∂D_q, $q = 1, 2$, such that solutions of the corresponding Dirichlet problems are solutions of the boundary-contact problem (3)-(7). Here $\varphi_0^{(1)}$ is the solution of equation (18), while $\varphi_0^{(2)}$ is defined by equality (17).

From the above discussion it follows that if $u^{(q)}$, $q = 1, 2$, is the solution of the boundary-contact prolem (3)-(7), then $u^{(1)}$ and $u^{(2)}$ are the solutions of the corresponding Dirichlet boundary value problems $(10)_q$, $q = 1, 2$. In the latter problems the boundary conditions are the unique extensions $\Phi^{(q)} = \Phi_0^{(q)} + \varphi_0^{(q)}$ of the function φ_q onto ∂D_q, $q = 1, 2$, where $\varphi_0^{(q)}$, $q = 1, 2$, is the solution of system (16).

Theorem 8. *Let $4/3 \leq p < 4$. Then the boundary-contact problem* (3)-(7) *has the unique solution in the classes $W_p^1(D_q)$, $q = 1, 2$, this solution being given by the formula*

$$u^{(q)} = V^{(q)}(V_{-1}^{(q)})^{-1}(\Phi_0^{(q)} + \varphi_0^{(q)})), \quad q = 1, 2, \tag{20}$$

where $\Phi_0^{(q)} \in B_{p,p}^{1/p'}(\partial D_q)$ is some extension of φ_q onto ∂D_q and $\varphi_0^{(q)} \in \widetilde{B}_{p,p}^{1/p'}(S_0)$ is the solution of system (16).

From Theorems 1, 7 and the embedding theorem (see [8]) we obtain

Theorem 9. *Let $4/3 \leq p < 4$, $1 < t < \infty$, $1 \leq r \leq \infty$, $\frac{1}{t} - \frac{1}{2} < s < \frac{1}{t} + \frac{1}{2}$, $u^{(q)} \in W_p^1(D_q)$, $q = 1, 2$, be the solution of the boundary-contact problem* (3)–(7). *Then:*

If $\varphi_1 \in B_{t,t}^s(S_1)$, $\varphi_2 \in B_{t,t}^s(S_2)$, $g \in B_{t,t}^s(S_0)$, $f \in B_{t,t}^{s-1}(S_0)$, then $u^{(q)} \in H_t^{s+1/t}(D_q)$, $q = 1, 2$;

If $\varphi_1 \in B^s_{t,r}(S_1)$, $\varphi_2 \in B^s_{t,r}(S_2)$, $g \in B^s_{t,r}(S_0)$, $f \in B^{s-1}_{t,r}(S_0)$, *then* $u^{(q)} \in B^{s+1/t}_{t,r}(D_q)$, $q = 1, 2$*;*

If $\varphi_1 \in C^\alpha(\bar{S}_1)$, $\varphi_2 \in C^\alpha(\overline{S}_2)$, $g \in C^\alpha(\bar{S}_0)$, $f \in B^{a-1}_{\infty,\infty}(S_0)$, $\alpha \in]0, 1/2]$, *then* $u^{(q)} \in \cap_{\alpha'<\alpha} C^{\alpha'}(\bar{D}_q)$, $q = 1, 2$.

Analogous theorems are proved in [11] and [18] for the mixed problems of elasticity and in [10] for the boundary value problems in the theory of cracks.

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