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EXISTENCE OF CONJUGATE POINTS FOR SECOND-ORDER LINEAR DIFFERENTIAL EQUATIONS

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ABSTRACT. The sufficient conditions are established under which the second-order linear differential equation is conjugate.

1. Consider the differential equation

$$u'' = p(t)u, \tag{1}$$

where $-\infty < a < b < +\infty$ and the function $p :]a,b[\to \mathbb{R}$ is Lebesgue integrable on each compact subset of $]a,b[$.

Definition 1. A function $p :]a,b[\to \mathbb{R}$ belongs to the class $\mathbb{O}(]a,b[)$ if the condition

$$\int_a^b (s-a)(b-s)|p(s)|ds < +\infty \tag{2}$$

is satisfied and the solution of equation (1), satisfying the initial conditions

$$u(a+) = 0, \quad u'(a+) = 1, \tag{3}$$

has at least one zero on $]a,b[$.

It is known [1] that if condition (2) is satisfied then any solution u of (1) has the finite left- and right-hand side limits $u(a+)$ and $u(b-)$ and problem (1),(3) is uniquely solvable.

Definition 2. A function $p :]a,b[\to \mathbb{R}$ belongs to the class $\mathbb{O}'(]a,b[)$ if

$$\int_a^b (s-a)|p(s)|ds < +\infty \tag{4}$$

and the derivative of the solution of problem (1),(3) has at least one zero on $]a,b]$.

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While investigating two-point singular boundary-value problems there arises a question as to an effective description of the classes $\mathbb{O}(]a,b[)$ and $\mathbb{O}'(]a,b[)$ (see, for example, [2]). Earlier attempts in this direction were undertaken in [3-7]. The statements given below complete the results of these papers.

2. On the Class $\mathbb{O}(]a,b[)$. In this section the function $p:]a,b[\to\mathbb{R}$ is assumed to satisfy (2).

Theorem 1. *Let there exist a point $t_0\in]a,b[$ and absolutely continuous functions $f_1:[a,t_0]\to[0,+\infty[$ and $f_2:[t_0,b]\to[0,+\infty[$ such that $f_1(a)=f_2(b)=0$, $f_1(t)>0$ for $a<t\le t_0$, $f_2(t)>0$ for $t_0\le t<b$,*

$$g_1(t_0)=\int_a^{t_0}\frac{[f_1{}'(s)]^2}{f_1(s)}ds<+\infty,\quad g_2(t_0)=\int_{t_0}^{b}\frac{[f_2{}'(s)]^2}{f_2(s)}ds<+\infty \tag{5}$$

and

$$f_2(t_0)\int_a^{t_0}f_1(s)p(s)ds+f_1(t_0)\int_{t_0}^{b}f_2(s)p(s)ds\le$$
$$\le-\frac{1}{4}\big(g_1(t_0)f_2(t_0)+g_2(t_0)f_1(t_0)\big). \tag{6}$$

Then $p\in\mathbb{O}(]a,b[)$.

Corollary 1. *Let there exist $t_0\in]a,b[$ and $\alpha\in]1,+\infty[$ such that*

$$(b-t_0)^\alpha\int_a^{t_0}(s-a)^\alpha p(s)ds+(t_0-a)^\alpha\int_{t_0}^{b}(b-s)^\alpha p(s)ds\le$$
$$\le-\frac{\alpha^2}{4(\alpha-1)}[(t_0-a)(b-t_0)]^{\alpha-1}.$$

Then $p\in\mathbb{O}(]a,b[)$.

When $t_0=\frac{a+b}{2}$ and $\alpha=2$ this proposition implies the result of R. Putnam ([4], p. 177).

Corollary 2. *Let one of the following three conditions hold:*

$$\int_a^b[(s-a)(b-s)]^{\frac{3}{2}}p(s)ds\le-\frac{9\pi(b-a)^2}{32},$$
$$\int_a^b[(s-a)(b-s)]^n p(s)ds\le-\frac{n!^2}{2(n-1)(2n-1)!}(b-a)^{2n-1},$$
$$n\ge2,\quad n\in\mathbb{N},$$
$$\int_a^b\sin^2\big(\frac{\pi(s-a)}{b-a}\big)p(s)ds\le-\frac{\pi^2}{2(b-a)}.$$

Then $p\in\mathbb{O}(]a,b[)$.

Corollary 3. *Let the inequality*

$$(b-t_0)\int_a^{t_0}(s-a)^{\frac{3}{2}}(b-s)^{\frac{1}{2}}p(s)ds+(t_0-a)\int_{t_0}^b(s-a)^{\frac{1}{2}}(b-s)^{\frac{3}{2}}p(s)ds\le$$
$$\le-\frac{5}{4}[(t_0-a)(b-t_0)]^{\frac{1}{2}}(b-a)$$

hold for some $t_0\in]a,b[$. *Then* $p\in\mathbb{O}(]a,b[)$.

Theorem 2. *Let the inequality*

$$(b-t)\int_a^t(s-a)^{\frac{3}{2}}(b-s)^{\frac{1}{2}}[p(s)]_-ds+$$
$$+(t-a)\int_t^b(s-a)^{\frac{1}{2}}(b-s)^{\frac{3}{2}}[p(s)]_-ds<[(t-a)(b-t)]^{\frac{1}{2}}(b-a)$$
$$\text{for}\quad a<t<b \tag{7}$$

hold, where $[p(t)]_-=\frac{|p(t)|-p(t)}{2}$. *Then* $p\notin\mathbb{O}(]a,b[)$.

3. On the Class $\mathbb{O}'(]a,b[)$**.** In this section the function $p:]a,b[\to\mathbb{R}$ is assumed to satisfy (4).

Theorem 3. *Let there exist* $t_0\in]a,b[$ *and absolutely continuous functions* $f_1:[a,t_0]\to[0,+\infty[$ *and* $f_2:[t_0,b]\to[0,+\infty[$ *such that* $f_1(a)=0$, $f_1(t)>0$ *for* $a<t<t_0$, $f_2(t)>0$ *for* $t_0<t<b$ *and conditions* (5) *and* (6) *are satisfied. Then* $p\in\mathbb{O}'(]a,b[)$.

Corollary 4. *Let there exist* $t_0\in]a,b[$ *and* $\alpha\in]1,+\infty[$ *such that*

$$\int_a^{t_0}(s-a)^\alpha p(s)ds+(t_0-a)^\alpha\int_{t_0}^b p(s)ds\le-\frac{\alpha^2}{4(\alpha-1)}(t_0-a)^{\alpha-1}.$$

Then $p\in\mathbb{O}'(]a,b[)$.

Corollary 5. *Let the inequality*

$$\int_a^{t_0}(s-a)^{\frac{3}{2}}p(s)ds+(t_0-a)\int_{t_0}^b(s-a)^{\frac{1}{2}}p(s)ds\le$$
$$\le-\frac{5}{4}(t_0-a)^{\frac{1}{2}}+\frac{t_0-a}{8(b-a)^{\frac{1}{2}}}$$

hold for some $t_0\in]a,b[$. *Then* $p\in\mathbb{O}'(]a,b[)$.

4. Proof of the Main results.

Proof of Theorem 1 (*Theorem* 3). Admit on the contrary that $p \notin \mathbb{O}(]a,b[)$ $(p \notin \mathbb{O}'(]a,b[))$. Then equation (1) has the solution u satisfying

$$u(a+) = 0, \;\; u'(a+) = 1, \;\; u(t) > 0 \;\text{ for }\; a < t < b.$$
$$\Big(\;\; u(a+) = 1, \;\; u'(b-) = 0, \;\; u(t) > 0 \;\text{ for }\; a \le t \le b \;\; \Big).$$

Denote

$$\rho(t) = \frac{u'(t)}{u(t)} \quad \text{for} \quad a < t < b.$$

It is clear that

$$\rho'(t) = p(t) - \rho^2(t) \quad \text{for} \quad a < t < b. \tag{8}$$

Multiplying both sides of this equality by f_1 and integrating from $a+\varepsilon$ to t_0 where $\varepsilon \in]0, t_0 - a[$, we have

$$-\int_{a+\varepsilon}^{t_0} f_1(s)p(s)ds + f_1(t_0)\rho(t_0) - f_1(a+\varepsilon)\rho(a+\varepsilon) =$$
$$= \int_{a+\varepsilon}^{t_0} \Big[{f_1}'(s)\rho(s) - f_1(s)\rho^2(s) \Big] ds < \frac{1}{4}\int_{a+\varepsilon}^{t_0} \frac{[{f_1}'(s)]^2}{f_1(s)} ds < \frac{1}{4} g_1(t_0).$$

From (5) it easily follows that

$$\lim_{t \to a+} {f_1}'(t) = 0.$$

Therefore

$$\lim_{t \to a+} f_1(t)\rho(t) = 0.$$

In view of this the last inequality can be rewritten as

$$-\int_a^{t_0} f_1(s)p(s)ds + f_1(t_0)\rho(t_0) < \frac{1}{4} g_1(t_0). \tag{9}$$

Now multiplying both sides of (8) by f_2 and integrating from t_0 to $b-\varepsilon$ where $\varepsilon \in]0, b - t_0[$, we have

$$-\int_{t_0}^{b-\varepsilon} f_2(s)p(s)ds - f_2(t_0)\rho(t_0) + f_2(b-\varepsilon)\rho(b-\varepsilon) =$$
$$= \int_{t_0}^{b-\varepsilon} \Big[{f_2}'(s)\rho(s) - f_2(s)\rho^2(s) \Big] ds < \frac{1}{4}\int_{t_0}^{b-\varepsilon} \frac{[{f_2}'(s)]^2}{f_2(s)} ds < \frac{1}{4} g_2(t_0).$$

Taking into account

$$\lim_{t \to b-} {f_2}'(t) = 0 \;\text{ and }\; \lim_{t \to b-} (b-t)u'(t) = 0,$$

from the last inequality we obtain

$$-\int_{t_0}^{b} f_2(s)p(s)ds - f_2(t_0)\rho(t_0) < \frac{1}{4} g_2(t_0). \tag{10}$$

From (9) and (10) we have

$$-\left[f_2(t_0)\int_a^{t_0} f_1(s)p(s)ds + f_1(t_0)\int_{t_0}^{b} f_2(s)p(s)ds\right] <$$
$$< \frac{1}{4}\left[f_2(t_0)g_1(t_0) + f_1(t_0)g_2(t_0)\right],$$

which contradicts (6). □

Proof of Theorem 2. Admit on the contrary that $p \in \mathbb{O}(]a,b[)$. Then equation (1) has the solution u satisfying

$$u(a+) = u(b_1) = 0, \quad u(t) > 0 \ \text{ for } \ a < t < b_1 \le b.$$

According to the Green formula

$$u(t) =$$
$$= -\frac{1}{b_1 - a}\left[(b_1 - t)\int_a^t (s-a)p(s)u(s)ds + (t-a)\int_t^{b_1}(b_1 - s)p(s)u(s)ds\right]$$
$$\text{for } \quad a \le t \le b_1.$$

Hence we easily obtain

$$u(t) \le \frac{1}{b_1 - a}\left[(b_1 - t)\int_a^t (s-a)[p(s)]_- u(s)ds + \right.$$
$$\left. + (t-a)\int_t^{b_1}(b_1 - s)[p(s)]_- u(s)ds\right] \quad \text{for} \quad a \le t \le b_1,$$

i.e.,

$$v(t) \le$$
$$\le \frac{1}{[(t-a)(b-t)]^{\frac{1}{2}}}\left[\frac{b_1 - t}{b_1 - a}\int_a^t (s-a)^{\frac{3}{2}}(b-s)^{\frac{1}{2}}[p(s)]_- v(s)ds + \right.$$
$$\left. + \frac{t-a}{b_1 - a}\int_t^b (s-a)^{\frac{1}{2}}(b_1 - s)(b-s)^{\frac{1}{2}}[p(s)]_- v(s)ds\right] <$$
$$< \lambda \frac{1}{[(t-a)(b-t)]^{\frac{1}{2}}}\left[\frac{b-t}{b-a}\int_a^t (s-a)^{\frac{3}{2}}(b-s)^{\frac{1}{2}}[p(s)]_-)ds + \right.$$
$$\left. + \frac{t-a}{b-a}\int_t^b (s-a)^{\frac{1}{2}}(b-s)^{\frac{3}{2}}[p(s)]_- ds\right] \quad \text{for} \quad a < t < b_1,$$

where

$$v(t) = \frac{u(t)}{[(t-a)(b-t)]^{\frac{1}{2}}} \quad \text{for} \quad a < t < b_1$$

and

$$\lambda = \sup\{v(t) \quad t \in]a, b_1[\}.$$

Taking into account (7), we obtain the contradiction $\lambda < \lambda$. □

Corollaries 1-5 are obtained from Theorems 1 and 3 by an appropriate choice of functions f_1 and f_2.

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