GEORGIAN MATHEMATICAL JOURNAL: Vol. 2, No. 1, 1995, 79-92

EXISTENCE THEOREMS FOR NONLINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS OF NEUTRAL TYPE

YUICHI KITAMURA, TAKAŜI KUSANO,[1] AND B. S. LALLI

Abstract. Conditions are found upon satisfaction of which the differential equation

$$x^{(n)}(t) - \lambda x^{(n)}(t-\sigma) + f(t, x(g(t))) = 0$$

has solutions which are asymptotically equivalent to the solutions of the equation

$$x^{(n)}(t) - \lambda x^{(n)}(t-\sigma) = 0.$$

§ 0. Introduction

We consider the neutral functional differential equation

$$x^{(n)}(t) - \lambda x^{(n)}(t-\sigma) + f(t, x(g(t))) = 0 \tag{A}$$

under the assumptions that

(i) $n \geq 1$ is an integer; λ and σ are positive constants with $\lambda \leq 1$;

(ii) $g : [t_0, \infty) \to \mathbb{R}$ is continuous, $t_0 > 0$, and $\lim_{t\to\infty} g(t) = \infty$;

(iii) $f : [t_0, \infty) \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies

$$|f(t, x)| \leq F(t, |x|), \quad (t, x) \in [t_0, \infty) \times \mathbb{R}, \tag{0.1}$$

for some continuous function $F(t, u)$ on $[t_0, \infty) \times [0, \infty)$ which is nondecreasing in u for each fixed $t \geq t_0$.

1991 *Mathematics Subject Classification.* 34K15.

Key words and phrases. Neutral functional differential equation, existence of the solution.

[1]Work done while visiting the University of Saskatchewan as a visiting Professor of Mathematics.

1072-947X/95/0100-0079$07.50/0 © 1995 Plenum Publishing Corporation

We note that the associated unperturbed equation $x^{(n)}(t)-\lambda x^{(n)}(t-\sigma) = 0$ has the solutions

$$\begin{aligned} t, t^2, \dots t^n, w(t) &\text{ for the case } \lambda = 1, \\ 1, t, \dots, t^{n-1}, \lambda^{\frac{t}{\sigma}} w(t) &\text{ for the case } \lambda \neq 1, \end{aligned} \tag{0.2}$$

where $w(t)$ is an arbitrary σ-periodic function in $C^n(\mathbb{R})$.

Our objective here is to establish the existence of solutions $x(t)$ of (A) which are asymptotic to the functions (0.2) in the sense that

$$\begin{aligned} &x(t) = ct^j + \sigma(1) \text{ as } t \to \infty, \quad j \in \{1, 2, \dots, n\}, \\ &x(t) = cw(t) + \sigma(1) \text{ as } t \to \infty \end{aligned}$$

in the case $\lambda = 1$, and

$$\begin{aligned} &x(t) = ct^k + \sigma(1) \text{ as } t \to \infty, \quad t \in \{0, 1, \dots, n-1\}, \\ &x(t) = c\lambda^{\frac{t}{\sigma}}[w(t) + \sigma(1)] \text{ as } t \to \infty \end{aligned}$$

in the case $0 < \lambda < 1$, where $w(t) \not\equiv 0$ is a given σ-periodic function and c is a nonzero constant. A solution of (A) is naturally required to be n-times continuously differentiable and satisfy the equation for all sufficiently large t.

Our main results are stated and proved in Section 1 ($\lambda = 1$) and Section 2 ($0 < \lambda < 1$). There the desired solutions of (A) are obtained by solving, via a fixed point analysis, suitable (functional) integral equations involving an "inverse" of the difference operator $\Delta x(t) = x(t) - x(t - \sigma)$.

There has been increasing interest in the study of the qualitative behavior of neutral functional differential equations because of their importance in various theoretical and practical problems. Needless to say, the existence theory of solutions is a fundamental question to be investigated in depth for such equations. However, the existence results obtained so far have been concerned exclusively with neutral equations of the form

$$\big[x(t) - \lambda x(t-\sigma)\big]^{(n)} + f\big(t, x(g(t))\big) = 0 \tag{B}$$

and nothing is known about the existence of solutions, oscillatory or nonoscillatory, for neutral equations of the type (A); see the papers [1–6]. This observation motivated the present work.

We remark that the equations (A) and (B) are not equivalent. In fact, a solution of (A) is automatically a solution of (B), but not conversely. As was shown in the above references, a continuous function $x(t)$ which is not n-times differentiable can be a solution of (B) provided its "difference" $x(t) - \lambda x(t - \sigma)$ is n-times differentiable.

§ 1. Existence Theorems for the Case $\lambda = 1$

.

We start with the case $\lambda = 1$ in (A), i.e.,

$$x^{(n)}(t) - x^{(n)}(t-\sigma) + f\big(t, x(g(t))\big) = 0. \tag{1.1}$$

Note that (1.1) can be written as

$$\Delta x^{(n)}(t) + f\big(t, x(g(t))\big) = 0$$

in terms of the difference operator

$$\Delta\xi(t) = \xi(t) - \xi(t-\sigma). \tag{1.2}$$

We make use of the observation that the unperturbed equation $\Delta x^{(n)}(t) = 0$ has the solutions $t, t^2, \dots, t^n, w(t)$, $w(t)$ being any σ-periodic function of class C^n, and intend to construct solutions of (1.1) which are asymptotic to these functions as $t \to \infty$.

Theorem 1. *Suppose that there is a constant $a > 0$ such that $F(t,a)$ is nonincreasing for $t \geq t_0$ and*

$$\int_{t_0}^{\infty} t^n F(t,a)dt < \infty. \tag{1.3}$$

Then, for an arbitrary σ-periodic C^n-function $w(t) \not\equiv 0$, the equation (1.1) has a solution $x(t)$ with the property that

$$x(t) = cw(t) + \sigma(1) \quad as \quad t \to \infty \tag{1.4}$$

for some nonzero constant c.

Theorem 2. *Let $j \in \{1, 2, \dots, n\}$. Suppose that there is a constant $a > 0$ such that $F\big(t, a[g(t)]^j\big)$ is nonincreasing for $t \geq t_0$ and*

$$\int_{t_0}^{\infty} t^n F\big(t, a[g(t)]^j\big)dt < \infty. \tag{1.5}$$

Then the equation (1.1) has a solution $x(t)$ with the property that

$$x(t) = ct^j + \sigma(1) \quad as \quad t \to \infty \tag{1.6}$$

for some nonzero constant c.

The proofs of these theorems require some elementary results (Lemmas 1–3 below) regarding an "inverse" of the difference operator Δ given by (1.2).

Let $S(T,\infty)$ denote the set of all functions $\xi \in C(T,\infty)$ such that the series

$$\eta(t) = \sum_{i=1}^{\infty} \xi(t+i\sigma), \quad t \geq T-\sigma, \tag{1.7}$$

converges uniformly on any compact subinterval of $[T-\sigma,\infty)$. We denote by Ψ the mapping which assigns to each $\xi \in S[T,\infty)$ a function $\eta \in C[T-\sigma,\infty)$ defined by (1.7).

Lemma 1. *If $\xi \in S[T,\infty)$, then $\Psi\xi$ is a solution of the difference equation $\Delta x(t) = -\xi(t)$ for $t \geq T$ and satisfies $\Psi\xi(t) \to 0$ as $t \to \infty$.*

Lemma 2. *If $v \in C[T-\sigma,\infty)$ is nonnegative, nonincreasing, and integrable on $[T-\sigma,\infty)$, then $v \in S[T,\infty)$ and*

$$\Psi v(t) \leq \frac{t}{\sigma} \int_t^{\infty} v(s)ds, \quad t \geq T-\sigma.$$

Lemma 3. *Let v be as in Lemma 2. Then $\int^{\infty} t^m v(t)dt < \infty$ for some $m \in \mathbb{N}$ implies $\int^{\infty} t^{m-1}\Psi v(t)dt < \infty$.*

Lemma 1 follows immediately from the definition of Ψ. The proofs of Lemmas 2 and 3 proceed in the following manner. In view of the nonincreasing property of $v(t)$ we see that

$$\Psi v(t) = \frac{1}{\sigma}\sum_{i=1}^{\infty} \sigma v(t+i\sigma) \leq \frac{1}{\sigma}\sum_{i=1}^{\infty} \int_{t+(i-1)\sigma}^{t+i\sigma} v(s)ds \leq \frac{1}{\sigma}\int_t^{\infty} v(s)ds$$

for $t \geq T-\sigma$, and using this inequality, we obtain

$$\int_{T-\sigma}^{\infty} t^{m-1}\Psi v(t)dt \leq \frac{1}{\sigma}\int_{T-\sigma}^{\infty} t^{m-1}\int_t^{\infty} v(s)dsdt \leq$$
$$\leq \frac{t}{\sigma}\int_{T-\sigma}^{\infty} \frac{1}{m}(s-T+\sigma)^m v(s)ds \leq \frac{1}{m\sigma}\int_{T-\sigma}^{\infty} s^m v(s)ds, \tag{1.8}$$

where we have supposed that $T-\sigma \geq 0$.

Proof of Theorem 1. Let $c > 0$ be such that $c(\max_t |w(t)| + 1) \le a$ and choose $T > t_0$ so that

$$T_* = \min\left\{T - \sigma, \inf_{t \ge T} g(t)\right\} \ge t_0$$

and

$$\int_{T-\sigma}^{\infty} t^n F(t,a)dt \le cn!\sigma. \tag{1.9}$$

Define the set $X \subset C[T_*, \infty)$ and the mapping $\mathcal{F} : X \to C[T_*, \infty$ by

$$X = \{x \in C[T_*, \infty) : |x(t)| \le a, \;\; t \ge T_*\}$$

and

$$\begin{cases} \mathcal{F}x(t) = c\omega(t) + (-1)^n \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \Psi\big[f(s, x(g(s)))\big] ds, & t \ge T - \sigma, \\ \mathcal{F}x(t) = \mathcal{F}x)(T - \sigma), \quad T_* \le t \le T - \sigma. \end{cases} \tag{1.10}$$

We want to show that $\mathcal{F}$ maps X continuously into a relatively compact subset of X. Let $x \in X$. Since $|f(t, x(gt)))| \le F(t,a)$, $t \ge T$, applying Lemma 3 (cf.(1.8)) and using (1.9), we see that

$$\left| \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \Psi\big[f(s, x(g(s)))\big] ds \right| \le \int_{T-\sigma}^\infty \frac{s^{n-1}}{(n-1)!} \Psi\big[F(s,a)\big] ds \le$$

$$\le \frac{1}{n!\sigma} \int_{T-\sigma}^\infty s^n F(s,a) ds \le c, \quad t \ge T - \sigma.$$

Consequently, we have

$$|\mathcal{F}x(t)| \le c|w(t)| + c \le c(\max_t |w(t)| + 1) \le a, \quad t \ge T - \sigma,$$

which implies that $\mathcal{F}x \in X$. Thus $\mathcal{F}$ maps X into itself. To prove the continuity of $\mathcal{F}$ let $\{x_\mu\}$ be a sequence in X converging to $x \in X$ in $C[T_*, \infty)$. It is clear that $f(\cdot, x_\mu \circ g) \to f(\cdot, x \circ g)$ in $C[T, \infty)$, and so using the continuity of Ψ in $C[T - \sigma, \infty)$, which is easy to verify, we conclude that

$$\Psi[f(\cdot, x_\mu \circ g)] \to \Psi[f(\cdot, x \circ g)] \quad \text{in} \quad C[T - \sigma, \infty).$$

This, combined with the Lebesque dominated convergence theorem, then implies that

$$\int_t^{\infty} \frac{(s-t)^{n-1}}{(n-1)!}\Psi\big[f(s,x_\mu(g(s)))\big]ds \to \int_t^{\infty} \frac{(s-t)^{n-1}}{(n-1)!}\Psi\big[f(s,x(g(s)))\big]ds,$$

the convergence being uniform on any compact subinterval of $[T-\sigma,\infty)$. It follows that $\mathcal{F}x_\mu \to \mathcal{F}x$ in $C[T_*,\infty)$, establishing the continuity of $\mathcal{F}$. Finally, the relative compactness of $\mathcal{F}(X)$ follows readily from the relations below holding for all $x \in X$ and $t \geq T-\sigma$: $|\mathcal{F}x(t)| \leq a$

$$\begin{aligned}|(\mathcal{F}x)'(t)| &= \big|cw'(t) + \Psi\big[f(t,x(g(t)))\big]\big| \leq \\ &\leq c|w'(t)| + \Psi[F(t,a)] \leq \\ &\leq c\max_t |w'(t)| + \frac{t}{\sigma}\int_{T-\sigma}^{\infty} F(s,a)ds\end{aligned}$$

for the case $n=1$;

$$\begin{aligned}|(\mathcal{F}x)'(t)| &= \left|cw'(t) + (-1)^{n-1}\int_t^{\infty} \frac{(s-t)^{n-2}}{(n-2)!}\Psi\big[f(s,x(g(s)))\big]ds\right| \leq \\ &\leq c|w'(t)| + \int_t^{\infty} \frac{s^{n-2}}{(n-2)!}\Psi\big[F(s,a)\big]ds \leq \\ &\leq c\max_t |w'(t)| + \frac{1}{(n-1)!\sigma}\int_{T-\sigma}^{\infty} s^{n-1}F(s,a)ds\end{aligned}$$

for the case $n \geq 2$.

Therefore we are able to apply the Schauder–Tychonoff fixed point theorem, concluding that there exists an $x \in X$ such that $x = \mathcal{F}x$, which means that $x(t)$ satisfies the (functional) integral equation

$$x(t) = cw(t) + (-1)^n \int_t^{\infty} \frac{(s-t)^{n-1}}{(n-1)!}\Psi\big[f(s,x(g(s)))\big]ds, \quad t \geq T-\sigma. \tag{1.11}$$

Letting the difference operator Δ operate on

$$x^{(n)}(t) = cw^{(n)}(t) + \Psi\big[f(t,x(g(t)))\big], \quad t \geq T-\sigma,$$

which follows from (1.11) by differentiation, we find in view of Lemma 1 that

$$x^{(n)}(t) - x^{(n)}(t-\sigma) = -f(t, x(g(t))), \quad t \ge T,$$

where use is made of the fact that $cw^{(n)}(t)$ is σ-periodic. This shows that $x(t)$ is a solution of the neutral equation (1.1). From (1.1) it follows that $x(t) - cw(t) \to 0$ as $t \to \infty$, so that $x(t)$ has the required asymptotic property (1.4). □

Proof of Theorem 2. Take $c > 0$ and $T > t_0$ so that $2c \le a$,

$$T_* = \min\Big\{T - \sigma, \inf_{t \ge T} g(t)\Big\} \ge \max\{t_0, 1\}$$

and

$$\int_{T-\sigma}^{\infty} t^n F(t, a[g(t)]^j)dt \le cn!\sigma.$$

Define

$$Y = \{y \in C[T_*, \infty) \ : \ |y(t)| \le at^j, \quad t \ge T_*\},$$

$$\begin{cases} \mathcal{G}y(t) = ct^j + (-1)^n \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \Psi\big[f(s, y(g(s)))\big]ds, & t \ge T - \sigma, \\ \mathcal{G}y(t) = \mathcal{G}y(T-\sigma)\frac{t^j}{(T-\sigma)^j}, & T_* \le t \le T - \sigma. \end{cases}$$

Then, proceeding exactly as in the proof of Theorem 1 one can verify that $\mathcal{G}$ is continuous on Y and maps Y onto a relatively compact subset of Y. The Schauder–Tychonoff theorem ensures the existence of a $y \in Y$ such that $y = \mathcal{G}y$, which satisfies

$$y(t) = ct^j + (-1)^n \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \Psi\big[f(s, y(g(s)))\big]ds, \quad t \ge T - \sigma. \tag{1.12}$$

It follows that $y(t)$ is a solution of the equation (1.1) for $t \ge T$ satisfying (1.6). This sketches the proof of Theorem 2. The details are left to the reader. □

Remark 1. The solutions constructed in Theorem 2 are all nonoscillatory, whereas those obtained in Theorem 1 are oscillatory or nonoscillatory according to whether the periodic function $w(t)$ is oscillatory or nonoscillatory. Since $w(t)$ does not appear explicitly in (1.3), Theorem 1 asserts that under the integral condition (1.3) the equation (1.1) possesses both oscillatory and nonoscillatory solutions. Thus one can easily speak of the phenomenon of coexistence of oscillatory and nonoscillatory solutions for neutral equations. This is an aspect which is not shared by non-neutral equations of the form $x^{(n)}(t) + f(t, x(g(t))) = 0$.

Remark 2. Suppose that there is a constant $a > 0$ such that

$$\int_{t_0}^{\infty} t^n F\big(t, a[g(t)]^n\big)\,dt < \infty.$$

Then, from Theorems 1 and 2 it follows that (1.1) possesses oscillatory solutions $x^0(t)$ which are asymptotic to an arbitrary σ-periodic function $w \not\equiv 0$ of class C^n in the sense that

$$x^0(t) = cw(t) + \sigma(1) \quad \text{as} \ \ t \to \infty$$

for some $c \neq 0$ as well as nonoscillatory solutions $x_j(t)$ which are asymptotic to t^j, $j = 0, 1, \cdots, n$, in the sense that

$$x_j(t) = ct^j + o(1) \quad \text{as} \ \ t \to \infty, \quad j = 0, 1, \cdots, n.$$

Example 1. Consider the neutral equation

$$x^{(n)}(t) - x^{(n)}(t-1) + q(t)|x(t-2)|^{\gamma} \operatorname{sgn} x(t-2) = 0, \tag{1.13}$$

where $\gamma > 0$ and $q : [t_0, \infty) \to \mathbb{R}$ is continuous, $t_0 > 2$, which is a special case of (1.1) in which $\sigma = 1$, $g(t) = t - 2$ and $f(t, x) = q(t)|x|^{\gamma} \operatorname{sgn} x$, and the conditions (0.1) are satisfied with $F(t, u) = |q(t)|u^{\gamma}$. Note that the conditions (1.3) and (1.5) written for (1.13) reduce, respectively, to

$$\int_{t_0}^{\infty} t^n |q(t)|\,dt < \infty \tag{1.14}$$

and

$$\int_{t_0}^{\infty} t^{n+\gamma j} |q(t)|\,dt < \infty, \quad j \in \{1, 2, \cdots, n\}. \tag{1.15}$$

From Theorems 1 and 2 it follows that if $|q(t)|$ is nonincreasing and satisfies (1.14), then, for any given σ-periodic function $w(t) \not\equiv 0$ in $C^n(\mathbb{R})$, there existes a solution $x(t)$ of (1.13) such that $x(t) = cw(t) + o(1)$ as $t \to \infty$ for some constant $c \neq 0$, and that if $t^{\gamma j}|q(t)|$ is nonincreasing and satisfies (1.15), then there exists a solution $x(t)$ of (1.13) such that $x(t) = ct^j + o(1)$ as $t \to \infty$ for some constant $c \neq 0$. If, in particular, $t^{\gamma n}|g(t)|$ is nonincreasing and satisfies

$$\int_{t_0}^{\infty} t^{1+\gamma)n} |q(t)|\,dt < \infty,$$

then (1.13) possesses all the solutions with the asymptotic properties specified above.

§ 2. Existence Theorems for the Case $0 < \lambda < 1$

Let us now turn to the case $0 < \lambda < 1$ in (A):

$$x^{(n)}(t) - \lambda x^{(n)}(t-\sigma) + f(t, x(g(t))) = 0, \quad 0 < \lambda < 1. \tag{2.1}$$

Using the generalized difference operator

$$\Delta_\lambda \xi(t) = \xi(t) - \lambda \xi(t-\sigma),$$

this equation is written as

$$\Delta_\lambda x^{(n)}(t) + f(t, x(g(t))) = 0,$$

which, in turn, can be expressed in the form

$$\Delta\big[\lambda^{-\frac{t}{\sigma}} x^{(n)}(t)\big] + \lambda^{-\frac{t}{\sigma}} f(t, x(g(t))) = 0 \tag{2.2}$$

with the use of the following relation connecting Δ_λ with Δ defined by (1.2)

$$\Delta_\lambda \xi(t) = \lambda^{\frac{t}{\sigma}} \Delta\big[\lambda^{-\frac{t}{\sigma}} \xi(t)\big].$$

Although (2.2) is not exactly of the same form as (1.1) because of the presence of the factor $\lambda^{-\frac{t}{\sigma}}$ in the leading part, a close look at the formation of the (functional) integral equations (1.11) and (1.12) solving (1.1) will lead us to the integral equations

$$x(t) = c\lambda^{\frac{t}{\sigma}} w(t) + (-1)^n \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \lambda^{\frac{s}{\sigma}} \Psi \times$$
$$\times \Big[\lambda^{-\frac{s}{\sigma}} f(s, x(g(s)))\Big] ds, \quad t \ge T - \sigma, \tag{2.3}$$

$$y(t) = ct^k + (-1)^n \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \lambda^{\frac{s}{\sigma}} \Psi \Big[\lambda^{-\frac{s}{\sigma}} f(s, y(g(s)))\Big] ds,$$
$$t \ge T - \sigma, \;\; k \in \{0, 1, \cdots, n-1\}, \tag{2.4}$$

for finding the desired solutions of (2.1) which are asymptotic to the solutions of the unperturbed equation $\Delta_\lambda x^{(n)}(t) = 0$. Our purpose here is to show that (2.3) and (2.4) are solvable with the aid of the Schauder–Tychonoff fixed point theorem.

Theorem 3. *Suppose that there is a constant $a > 0$ such that $\lambda^{-\frac{t}{\sigma}} F\big(t, a\lambda^{\frac{g(t)}{\sigma}}\big)$ is nonincreasing for $t \ge t_0$ and*

$$\int_{t_0}^\infty t^n \lambda^{-\frac{t}{\sigma}} F\big(t, a\lambda^{\frac{g(t)}{\sigma}}\big) dt < \infty. \tag{2.5}$$

Then, for any σ-periodic function $w(t) \not\equiv 0$ of class C^n, the equation (2.1) *possesses a solution $x(t)$ satisfying*

$$x(t) = c\lambda^{\frac{t}{\sigma}}\big[w(t) + o(1)\big] \quad as \ \ t \to \infty \tag{2.6}$$

for some nonzero constant c.

Theorem 4. *Let $k \in \{0, 1, \cdots, n-1\}$ and suppose that there is a constant $a > 0$ such that $\lambda^{-\frac{t}{\sigma}} F\big(t, a[g(t)]^k\big)$ is nonincreasing for $t \geq t_0$ and*

$$\int_{t_0}^{\infty} \lambda^{-\frac{t}{\sigma}} F\big(t, a[g(t)]^k\big)dt < \infty. \tag{2.7}$$

Then, the equation (2.1) *possesses a solution $x(t)$ satisfying*

$$x(t) = ct^k + o(1) \quad as \ \ t \to \infty \tag{2.8}$$

for some nonzero constant c.

Proof of Theorem 3. Choose $c > 0$ and $T > t_0$ so that $c(\max_t |w(t)|+1) \leq a$,

$$T_* = \min\{T - \sigma, \inf_{t \geq T} g(t)\} \geq \max\{t_0, 1\}, \tag{2.9}$$

and

$$\int_{T-\sigma}^{\infty} t^n \lambda^{-\frac{t}{\sigma}} F\big(t, a\lambda^{\frac{g(t)}{\sigma}}\big)dt \leq cn!\sigma. \tag{2.10}$$

Consider the set $X_\lambda \subset C[T_*, \infty)$ and the mapping $\mathcal{F}_\lambda : X_\lambda \to C[T_*, \infty)$ defined by

$$X_\lambda = \{x \in C[T_*, \infty) : |x(t)| \leq a\lambda^{\frac{t}{\sigma}}, \quad t \geq T_*\}$$

and

$$\begin{cases} \mathcal{F}_\lambda x(t) & = c\lambda^{\frac{t}{\sigma}} w(t) + (-1)^n \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \lambda^{\frac{s}{\sigma}} \times \\ & \times \Psi\Big[\lambda^{-\frac{s}{\sigma}} f(s, x(g(s)))\Big]ds, \quad t \geq T - \sigma, \\ \mathcal{F}_\lambda x(t) & = \mathcal{F}_\lambda x(T-\sigma)\lambda^{\frac{t}{\sigma}}, \quad T_* \leq t \leq T - \sigma. \end{cases} \tag{2.11}$$

That $\mathcal{F}_\lambda$ maps X_λ into itself is shown as follows. Let $x \in X_\lambda$. Using Lemma 3 and (2.10), we have

$$\left|(-1)^n \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \lambda^{\frac{s}{\sigma}} \Psi\Big[\lambda^{-\frac{s}{\sigma}} f(s, x(g(s)))\Big]ds\right| \leq$$

$$\leq \lambda^{\frac{t}{\sigma}} \int\limits_t^\infty \frac{s^{n-1}}{(n-1)!} \Psi\Big[\lambda^{-\frac{s}{\sigma}} F\big(s, a\lambda^{\frac{g(s)}{\sigma}}\big)\Big] ds \leq$$

$$\leq \frac{1}{n!\sigma} \lambda^{\frac{t}{\sigma}} \int\limits_{T-\sigma}^\infty s^n \lambda^{-\frac{s}{\sigma}} F\big(s, a\lambda^{\frac{g(s)}{\sigma}}\big) ds \leq c\lambda^{\frac{t}{\sigma}}, \quad t \geq T-\sigma.$$

From this and (2.11) it follows that

$$|\mathcal{F}_\lambda x(t)| \leq c\lambda^{\frac{t}{\sigma}} (\max_t |w(t)| + 1) \leq a\lambda^{\frac{t}{\sigma}}, \quad t \geq T-\sigma.$$

The continuity of $\mathcal{F}_\lambda$ and the relative compactness of $\mathcal{F}_\lambda(X_\lambda)$ are verified in a routine manner. Consequently, by the Schauder–Tychonoff theorem there exists a function $x \in X_\lambda$ satisfying the integral equation (2.3) for $t \geq T-\sigma$. Differentiating (2.3) n-times, we see that

$$\lambda^{-\frac{t}{\sigma}} x^{(n)}(t) = \Omega(t) + \Psi\Big[\lambda^{-\frac{s}{\sigma}} f(t, x(g(t)))\Big], \quad t \geq T-\sigma, \tag{2.12}$$

where $\Omega(t)$ is a continuous σ-periodic function, from which, by taking the difference Δ of both sides of (2.12), we conclude that

$$\Delta_\lambda x^{(n)}(t) = \lambda^{\frac{t}{\sigma}} \Delta\big[\lambda^{-\frac{t}{\sigma}} x^{(n)}(t)\big] = -f(t, x(g(t))), \quad t \geq T.$$

This means that $x(t)$ is a solution of (2.1) for $t \geq T$. From (2.3) it is clear that $x(t) - c\lambda^{\frac{t}{\sigma}} w(t) \to 0$ as $t \to \infty$. $\square$

Proof of Theorem 4. Choose $c > 0$ and $T > t_0$ so that $2c \leq a$, (2.9) holds and

$$\int\limits_{T-\sigma}^\infty \lambda^{-\frac{t}{\sigma}} F\big(t, a[g(t)]^k\big) dt \leq \frac{\big(\log(\frac{1}{\lambda})\big)^n c}{\sigma^{n-1}}. \tag{2.13}$$

Let us define

$$Y_\lambda = \{y \in C[T_*, \infty) : |y(t)| \leq at^k, \quad t \geq T_*\},$$

$$\begin{cases} \mathcal{G}_\lambda y(t) & = ct^k + (-1)^n \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \lambda^{\frac{s}{\sigma}} \times \\ & \times \Psi\Big[\lambda^{-\frac{s}{\sigma}} f(s, y(g(s)))\Big] ds, \ \ t \geq T-\sigma, \\ \mathcal{G}_\lambda y(t) & = \mathcal{G}_\lambda y(T-\sigma) \frac{t^k}{(T-\sigma)^k} \quad T_* \leq t \leq T-\sigma. \end{cases} \tag{2.14}$$

If $y \in Y_\lambda$, then by Lemma 2

$$\Big|\Psi\Big[\lambda^{-\frac{t}{\sigma}} f(t, y(g(t)))\Big]\Big| \leq \frac{1}{\sigma} \int\limits_t^\infty \lambda^{-\frac{s}{\sigma}} F(s, a[g(s)]^k) ds \quad t \geq T-\sigma.$$

Using this inequality together with (2.3), we estimate the integral in (2.14) as follows:

$$\left|(-1)^n \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!}\lambda^{\frac{s}{\sigma}}\Psi\Big[\lambda^{-\frac{s}{\sigma}} f(s,y(g(s)))\Big]ds\right| \le$$

$$\le \frac{1}{(n-1)!\sigma}\int_t^\infty s^{n-1}\lambda^{\frac{s}{\sigma}}\int_s^\infty \lambda^{-\frac{r}{\sigma}}F(r,a[g(r)]^k)drds =$$

$$= \frac{1}{(n-1)!\sigma}\int_t^\infty \Big(\int_t^r s^{n-1}\lambda^{\frac{s}{\sigma}}ds\Big)\lambda^{-\frac{r}{\sigma}}F(r,a[g(r)]^k)dr \le$$

$$\le \frac{1}{(n-1)!\sigma}\Big(\frac{\sigma}{\log(\frac{1}{\lambda})}\Big)^n (n-1)! \int_{T-\sigma}^\infty \lambda^{-\frac{r}{\sigma}}F(r,a[g(r)]^k)dr \le c,$$

$$t \ge T-\sigma,$$

where use is made of the inequality

$$\int_t^r s^{n-1}\lambda^{\frac{s}{\sigma}}ds \le \int_0^\infty s^{n-1}\lambda^{\frac{s}{\sigma}}ds = \Big(\frac{\sigma}{\log(\frac{1}{\lambda})}\Big)^n (n-1)!$$

Therefore, $y \in Y_\lambda$ implies that

$$|\mathcal{G}_\lambda y(t)| \le ct^k + c \le 2ct^k \le at^k, \quad t \ge T-\sigma,$$

which shows that $\mathcal{G}_\lambda$ is a self-map on Y_λ. Since $\mathcal{G}_\lambda$ is continuous and $\mathcal{G}_\lambda(Y_\lambda)$ is relatively compact in the $C[T_*,\infty)$-topology, $\mathcal{G}_\lambda$ has a fixed element $y \in Y_\lambda$ by the Schauder–Tychonoff theorem. This function $y(t)$ clearly satisfies the integral equation (2.4), and so gives a solution of the neutral equation (2.1) having the asymptotic behavior (2.8). This completes the proof of Theorem 4. □

Remark 3. The solutions given in Theorem 3 are oscillatory or nonoscillatory according to whether the periodic function $w(t)$ is oscillatory or nonoscillatory. The condition (2.5), which is independent of the choice of $w(t)$, guarantees the coexistence of both oscillatory and nonoscillatory solutions for the equation (2.1). It is clear that these solutions are all exponentially decaying to zero as $t \to \infty$. The solutions constructed in Theorem 4 are nonoscillatory.

Example 2. For illustration consider the equation

$$x^{(n)}(t) - \frac{1}{e}x^{(n)}(t-1) + q(t)|x(t-2)|^\gamma \operatorname{sgn} x(t-2) = 0, \qquad (2.15)$$

where $\gamma > 0$ is a constant and $q : [t_0, \infty) \to \mathbb{R}$, $t_0 > 2$, is a continuous function. This is a special case of (2.1) in which $\lambda = \frac{1}{e}$ $\sigma = 1$, $g(t) = t - 2$ and $f(t, x) = q(t)|x|^{\gamma} \operatorname{sgn} x$. The function $F(t, u)$ bounding $f(t, x)$ can be taken to be $F(t, u) = |q(t)|u^{\gamma}$. It is easy to see that the conditions (2.5) and (2.7) written for (2.15) are equivalent to

$$\int_{t_0}^{\infty} t^n e^{(1-\gamma)t} |q(t)| dt < \infty \tag{2.16}$$

and

$$\int_{t_0}^{\infty} t^{\gamma k} e^t |q(t)| dt < \infty, \quad k \in \{0, 1, \cdots, n-1\}, \tag{2.17}$$

respectively.

From Theorem 3 it follows in particular that if $e^{(1-\gamma)t}|q(t)|$ is nonincreasing and (2.16) holds, then (2.15) has solutions of the form

$$x_m(t) = e^{-t}\Big(a_m \cos \frac{2m\pi t}{\sigma} + b_m \sin \frac{2m\pi t}{b} + o(1)\Big)$$
$$\text{as } t \to \infty, \quad m = 0, 1, 2, \cdots,$$

where a_m and b_m are constants with $a_m^2 + b_m^2 > 0$. If $m = 0$, then $x_0(t)$ is nonoscillatory; otherwise $x_m(t)$ are oscillatory.

Theorem 4 implies in particular that if $t^{\gamma(n-1)} e^t |q(t)|$ is nonincleasing and

$$\int_{t_0}^{\infty} t^{\gamma(n-1)} e^t |q(t)| dt < \infty, \tag{2.18}$$

then (2.15) has nonoscillatory solutions of the form

$$y_k(t) = c_k t^k = o(1) \text{ as } t \to \infty \text{ for all } k = 0, 1, \cdots, n-1.$$

Suppose that $\gamma \geq 1$. Then, (2.18) implies (2.16), and so (2.15) possesses all the solutions listed above provided (2.18) is satisfied.

References

1. J. Jaroš and T. Kusano, Asymptotic behavior of nonoscillatory solutions of nonlinear functional differential equations of neutral type. *Funkcial. Ekvac.* **32**(1989), 251-263.

2. J. Jaroš and T. Kusano, Existence of oscillatory solutions for functional differential equations of neutral type. *Acta Math. Univ. Comenian.* **90**(1991), 185-194.

3. Y. Naito, Nonoscillatory solutions of neutral differential equations. *Hiroshima Math. J.* **20**(1990), 233-258.

4. ——, Asymptotic behavior of decaying nonoscillatory solutions of neutral differential equations. *Funkcial. Ekvac.* **35**(1992), 95-110.

5. ——, Existence and asymptotic behavior of positive solutions of neutral differential equations. *(Preprint)*

6. J. Ruan, Types and criteria of nonoscillatory solutions of second order linear neutral differential equations. (Chinese) *Chinese Ann. Math. Ser. A* **8**(1987), 114-124.

(Received 14.09.1993)

Authors' addresses:

Y. Kitamura
Department of Mathematics, Faculty of Education,
Nagasaki University, Nagasaki 852, Japan

T. Kusano
Department of Mathematics, Faculty of Science,
Hiroshima University, 1-3-1 Kagamiyama,
Higashi-Hiroshima 724, Japan

B. S. Lalli
Department of Mathematics and Statistics,
University of Saskatchewan, Saskatoon S7N OWO, Canada