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ON SOME PROPERTIES OF COHOMOTOPY-TYPE FUNCTORS

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ABSTRACT. It is shown that if Π^n, $n > 2$, are Chogoshvili's cohomotopy functors [1, 2, 3], then

1) the isomorphism

$$\Pi^n(X_1 \times X_2) \approx \Pi^n(X_1) \oplus \Pi^n(X_2)$$

holds for topological spaces X_1 and X_2;

2) the relations

$$\operatorname{rank} \Pi^n(X) = \operatorname{rank} \pi_n(X),$$
$$\operatorname{Tor} \Pi^{n+1}(X) \approx \operatorname{Tor} \pi_n(X)$$

hold under certain restrictions for the space X.

In [2–5] we investigated Chogoshvili's cohomotopy functors $\Pi^n(-;-;G;H)$ (see [1]) from the category of pairs of topological spaces with base points into the category of abelian groups. The aim of this paper is to give full proofs of the results announced in [4, 5]. Like everywhere in our discussion below, we considered the case when H is the integral singular theory of cohomologies ($G = \mathbb{Z}$). In this paper all definitions, notions, and notation used in [3] are understood to be known and hence are not explained in the sequel. Note only that the auxiliary subcategories needed for defining functors Π^n will be denoted here by L_n (as distinct from [3] where they are routinely denoted by K_n). The base points of all spaces will be denoted by $*$ and, as a rule, will not be indicated if this does not cause any confusion. Moreover, as follows from [2], the base points may not be indicated at all for simply connected spaces which are mainly considered here.

We shall treat here mainly the absolute groups which were defined for $n > 2$ (see [3]). Therefore an arbitrary connected and simply connected space X with a finite-type module of homologies and $H^i(X) = 0$, $0 < i < n$,

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can be regarded as the object of L_n, $n > 2$. We shall also use the fact that for $X \in L_n$ the analogues of Gurevich homomorphisms in the Π theory

$$d_i : H^i(X) \to \Pi^i(X)$$

are isomorphisms for $2 < i \leq n$ (see [2]).

The paper consists of two sections. In Section 1 we state our results and prove some of them. In Section 2 we prove the statements that have been given in Section 1 without proof.

1. In this section we present our results and develop some of their proofs.

Proposition 1.1. *If a space X consists only of a base point $*$, then $\Pi^n(X) = 0$ for all $n > 2$.*

Proof. Let $p \in \Pi^n(X)$ and

$$\alpha = (X_n; f) \in \omega(X; n)$$

be an arbitrary index. Then $f \sim 0$ and from Corollary 1.3 of [3] it follows that $P_\alpha = 0$. Therefore $p = 0$ and the proposition is proved. □

By $K(\mathbb{Z}, m)$ denote any Eilenberg–MacLane space of the type $(\mathbb{Z}, m)$, having the homotopy type of a countable CW-complex.

Proposition 1.2. *For $n > 2$ and $m = 1, 2, \cdots$ we have*

$$\Pi^n(K(\mathbb{Z}, m)) = \begin{cases} 0, & \text{if } \ n \neq m; \\ \mathbb{Z}, & \text{if } \ n = m. \end{cases}$$

Proof. For $m \geq n$ we have $K(\mathbb{Z}, m) \in L_n$ and the proposition follows from the isomorphism

$$d_n : H^n(K(\mathbb{Z}, m)) \to \Pi^n(K(\mathbb{Z}, m)).$$

Let now $n > m$, $p \in \Pi^n(K(\mathbb{Z}, m))$, and

$$\alpha = (X; f) \in \omega(K(\mathbb{Z}, m); n), \quad X \in L_n,$$

be an arbitrary index. Then since $H^m(X) = 0$, the mapping $f : X \to K(\mathbb{Z}, m)$ is null-homotopic. Hence $p_\alpha = 0$. Therefore $p = 0$ and the proposition is proved. □

In Section 2 we shall prove

Theorem 1.3. *For $n > 2$ the isomorphism*

$$\Pi^n(X_1 \times X_2) \approx \Pi^n(X_1) \oplus \Pi^n(X_2).$$

holds for arbitrary spaces X_1 and X_2.

By K_m^t denote an arbitrary space of the homotopy type $\prod_{\alpha=1}^t K_\alpha$ where $K_\alpha = K(\mathbb{Z}, m)$, $\alpha = 1, 2, \dots, t$. By K_m^0 denote a space containing only one point $*$.

Proposition 1.2 and Theorem 1.3 imply

Corollary 1.4. *For $n > 2$, $m = 1, 2, \dots$, $t > 0$, we have*

$$\Pi^n(K_m^t) = \begin{cases} 0, & \text{if } m \neq n; \\ \underbrace{\mathbb{Z} \oplus \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{t}, & \text{if } m = n. \end{cases}$$

Let $B_n \in L_n$, $n > 4$. Choose some system $h_1, h_2, \dots, h_s$ of generators of the group $H^n(B_n)$ and consider the corresponding mappings

$$h_i : B_n \to K(\mathbb{Z}, n), \quad i = 1, 2, \dots, s.$$

Then we can construct the mapping

$$f = \prod_{i=1}^{s} h_i : B_n \to K_n^s.$$

The homomorphism f^* is obviously an epimorphism in the dimension n. Let further

$$K_{n-1}^s \to EK_n^s \to K_n^s$$

be the standard Serre path-space fibering ($K_{n-1}^s = \Omega K_n^s$) and

$$K_{n-1}^s \xrightarrow{i} E_{n-1} \xrightarrow{p_n} B_n$$

be the principal fibering induced by the mapping f (see [6]). Consider the diagram

$$\begin{array}{ccccc} K_{n-2}^t & \xrightarrow{i'} & E_{n+1} & & \\ & & \downarrow p_{n-1} & & \\ K_{n-1}^s & \xrightarrow{i} & E_{n-1} & \xrightarrow{f'} & K_{n-1}^t \\ & & \downarrow p_n & & \\ & & B_n & \xrightarrow{f} & K_n^s \quad , \end{array} \tag{1}$$

where the number t, the mapping f', and the fibering p_{n-1} are defined similarly to s, f, and p_n, respectively.

In Section 2 we shall prove

Lemma 1.5. *For $n > 4$ we have*

1) $E_{n-1} \in L_{n-1}$ *and* $\Pi^{n-1}(E_{n-1}) \approx H^{n-1}(E_{n-1})$ *are free groups;*

2) $\Pi^n(E_{n-1}) = H^n(E_{n-1}) = 0$*;*

3) *The homomorphism* $p_n^* : H^{n+1}(B_n) \to H^{n+1}(E_{n-1})$ *is an isomorphism;*

4) *Homomorphisms* $p_n^{\#} : \Pi^{n+k}(B_n) \to \Pi^{n+k}(E_{n-1})$ *are isomorphisms for* $k > 0$*;*

5) $E_{n+1} \in L_{n+1}$, $t \le s$, *and if* $H^n(B_n)$ *is a free group, then* $t = 0$ *and* $E_{n-1} = E_{n+1}$.

Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a fibering in the Serre sense. In [2] we obtained the semiexact sequence

$$\cdots \longrightarrow \Pi^{n-1}(B) \xrightarrow{p^{\#}} \Pi^{n-1}(E) \xrightarrow{i^{\#}} \Pi^{n-1}(F) \xrightarrow{\bar{\delta}^{\#}} \Pi^n(B) \longrightarrow \cdots . \quad (2)$$

Here the coboundary homomorphism $\bar{\delta}^{\#}$ is defined from the semiexact sequence of the pair (E, F) by means of the isomorphism (see [2])

$$p^{\#} : \Pi^n(B, *) \to \Pi^n(E, F), \quad n > 3.$$

Namely: $\bar{\delta}^{\#} = p^{\#-1}\delta^{\#}$, where $\delta^{\#}$ is the coboundary homomorphism of the pair (E, F). We shall now give another definition of the homomorphism $\bar{\delta}^{\#}$. For simplicity consideration will be given to fiberings in the Gurevich sense, which is sufficient for our purposes. These are, for example, all principal fiberings (see [6]). Let $f : (X_n, *) \to (B, *)$, where $X_n \in L_n$ is an arbitrary mapping. Consider the mapping

$$f_X : (C\Omega X_n, \Omega X_n) \to (B, *)$$

and the diagram

$$\begin{tikzcd}
 & & (E,F) \arrow[drr, "p"] & & \\
(C\Omega X_n, \Omega X_n) \arrow[urr, "\bar{f}"] \arrow[r, "e_X"] & (S\Omega X_n, *) \arrow[r, "\pi_X"] & (X_n, *) \arrow[rr, "f"] & & (B, *)
\end{tikzcd}$$

where CX and SX denote respectively the cone and the suspension over the space X in the category of spaces with base points, e_X and π_X are standard mappings, and $\bar{f}$ is the lifting of the mapping $f_X = f\pi_X e_X$. As is known, the mappings $\bar{f}$ and f_X define each other uniquely up to homotopy. We introduce the notation $\pi_X e_X = \psi_X$ and, moreover, consider the mapping

$$\bar{g} = \bar{f}|\Omega X_n : \Omega X_n \to F$$

and the indices

$$\alpha = (X_n; f) \in \omega(B; n),$$
$$\bar{\alpha} = (\Omega X_n; \bar{g}) \in \omega(F; n-1).$$

Note that $X_n \in L_n$ implies $\Omega X_n \in L_{n-1}$, $n > 3$, and $SX_n \in L_{n+1}$ (see [2]). Besides, as is known (see, for example, [2]), for the spaces $X_n \in L_n$, $n > 3$, the induced homomorphism π_X^* will be an isomorphism in cohomologies of dimension n. In the sequel we may not specially indicate this fact. Let $q \in \Pi^{n-1}(F)$. We assume

$$\bar{q}_\alpha = \psi_X^{*-1}(\delta(q_{\bar{\alpha}})),$$

where δ is a cohomological coboundary operator of the pair $(C\Omega X_n, \Omega X_n)$. In the sequel we may not indicate the corresponding pair for the operator δ.

We shall now show that the set $\{\bar{q}_\alpha\}$ defines an element of the group $\Pi^n(B)$. Consider the diagram

$$\begin{tikzcd}
& (E,F) \arrow[dr, "p"] & \\
(C\Omega X_n, \Omega X_n) \arrow[ur, "\bar{f}_1"] \arrow[r, "\psi_X"] \arrow[d, "\varphi_1"] & (X,*) \arrow[r, "f_1"] \arrow[d, "\varphi"] & (B,*) \arrow[d, equal] \\
(C\Omega Y_n, \Omega Y_n) \arrow[r, "\psi_Y"] \arrow[dr, "\bar{f}_2"'] & (Y,*) \arrow[r, "f_2"] & (B,*) \\
& (E,F) \arrow[ur, "p"'] &
\end{tikzcd}$$

where $f_2\varphi \sim f_1$ and the mapping $\varphi_1 = C\Omega\varphi$ is defined in the standard manner. Then

$$p\bar{f}_2\varphi_1 = f_2\psi_Y\varphi_1 = f_2\varphi\psi_X \sim f_1\psi_X = p\bar{f}_1.$$

Accordingly, $\bar{f}_2 \sim \bar{f}_1$. Let

$$\bar{g}_1 = \bar{f}_1|\Omega X_n, \quad \bar{g}_2 = \bar{f}_2|\Omega Y_n, \quad \varphi_0 = \varphi_1|\Omega X_n.$$

Then $\bar{g}_2\varphi_0 \sim \bar{g}_1$. Consider the indices

$$\bar{\alpha}_1 = (\Omega X_n; \bar{g}_1) \in \omega(F; n-1),$$
$$\bar{\alpha}_2 = (\Omega Y_n; \bar{g}_2) \in \omega(F; n-1),$$
$$\alpha_1 = (X_n; f_1) \in \omega(B; n),$$
$$\alpha_2 = (Y_n; f_2) \in \omega(B; n).$$

We have $\alpha_1 < \alpha_2$ and $\bar{\alpha}_1 < \bar{\alpha}_2$. Then

$$\varphi^*(\bar{q}_{\alpha_2}) = \varphi^*(\psi_Y^{*-1}(\delta(q_{\bar{\alpha}_2}))) = \psi_X^{*-1}(\varphi_1^*(\delta(q_{\bar{\alpha}_2}))) = \\ = \psi_X^{*-1}(\delta(\varphi_0^*(q_{\bar{\alpha}_2}))) = \psi_X^{*-1}(\delta(q_{\bar{\alpha}_1})) = \bar{q}_{\alpha_1}.$$

This means (see [3]) that we have defined the element $\bar{q} \in \Pi^n(B,*)$ and thereby the mapping

$$\widetilde{\delta}^{\#} : \Pi^{n-1}(F,*) \to \Pi^n(B,*).$$

Consider the diagram

$$\begin{tikzcd} \Pi^{n-1}(F,*) \arrow[rr, "\delta^{\#}"] \arrow[dr, "\widetilde{\delta}^{\#}"'] & & \Pi^n(E,F) \\ & \Pi^n(B,*) \arrow[ur, "p^{\#}"'] & \end{tikzcd} \tag{3}$$

Proposition 1.6. *Diagram* (3) *is commutative and the mapping* $\widetilde{\delta}^{\#} = p^{\#-1}\delta^{\#}$ *is a homomorphism.*

Proof. Let $q \in \Pi^{n-1}(F)$. It is sufficient to check the equality $p^{\#}(\widetilde{\delta}^{\#}(q)) = \delta^{\#}(q)$ for indices of the form (see [2])

$$\alpha = (CX_{n-1}, X_{n-1}; f) \in \omega(E,F;n), \quad X_{n-1} \in L_{n-1}.$$

Consider the commutative diagram

$$\begin{array}{ccccc} (C\Omega SX_{n-1}, \Omega SX_{n-1}) & \xleftarrow{k_1} & (CX_{n-1}, X_{n-1}) & \xrightarrow{f} & (E,F) \\ \downarrow e_1 & & \downarrow e & & \downarrow p \\ (S\Omega SX_{n-1}, *) & \xrightarrow{\pi} & (SX_{n-1}, *) & \xrightarrow{\varphi} & (B,*) \end{array},$$

where the mapping φ is induced by the mapping f, the mappings e_1, e and π are standard mappings, and the mapping $k_1 = ck$ is obtained from the standard mapping

$$k : X_{n-1} \to \Omega SX_{n-1}.$$

We introduce the notation $\pi e_1 = \psi_1$. Let

$$\Phi : (C\Omega SX_{n-1}, \Omega SX_{n-1}) \to (E,F)$$

be the lifting of the mapping $\varphi\psi_1$. Consider the indices

$$\begin{aligned} \alpha_1 &= (X_{n-1}; f|X_{n-1}) \in \omega(F; n-1), \\ P(\alpha) &= (CX_{n-1}, X_{n-1}; pf) \in \omega(B,*;n), \\ \beta &= (SX_{n-1}; \varphi) \in \omega(B;n), \\ \beta_1 &= (\Omega SX_{n-1}; \Phi|\Omega SX_{n-1}) \in \omega(F; n-1). \end{aligned}$$

Since $p\Phi k_1 = \varphi\psi_1 k_1 = \varphi e = pf$, we have $\Phi k_1 \sim f$. Therefore

$$(\Phi|\Omega S X_{n-1})k = (\Phi k_1)|X_{n-1} \sim f|X_{n-1}.$$

Thus $\alpha_1 < \beta_1$. Then, on the one hand, we have (see [1,2])

$$[\delta^{\#}(q)]_\alpha = \delta(q_{\alpha_1})$$

and, on the other hand,

$$\begin{gathered}[p^{\#}(\widetilde{\delta}^{\#}(q))]_\alpha = [\widetilde{\delta}^{\#}(q)]_{P(\alpha)} = e^*([\widetilde{\delta}^{\#}(q)]_\beta) = \\ = e^*(\psi_1^{*-1}(\delta(q_{\beta_1}))) = (e^*\psi_1^{*-1})(\delta(k^{*-1}(q_{\alpha_1}))) = \\ = (e^*\psi_1^{*-1})(k_1^{*-1}(\delta(q_{\alpha_1}))) = \\ = (e^*\psi_1^{*-1}k_1^{*-1})(\delta(q_{\alpha_1})) = \delta(q_{\alpha_1}).\end{gathered}$$

Therefore $p^{\#}\widetilde{\delta} = \delta^{\#}$. □

In the sequel the homomorphisms $\widetilde{\delta}^{\#}$ and $\bar{\delta}^{\#}$ will both be denoted by $\delta^{\#}$.

Remark. The boundary homomorphism $\delta^{\#}$ has been defined here only for fiberings in the Gurevich sense because we wanted arbitrary indices to participate in the definition. For a fibering in the sense of Serre the definition will be more sophisticated, since we shall have to use CW-approximations of spaces. On the other hand, it should be noted that Theorem 2 from [3] could make it possible for the indices to limit our consideration to the suspended CW-complexes $X_n = SX_{n-1}$, $X_{n-1} \in L_{n-1}$. Then a simpler diagram

$$\begin{tikzcd} & (E,F) \arrow[dr, "p"] & \\ (CX_{n-1},X_{n-1}) \arrow[ur, "\bar{f}"] \arrow[r, "e"] & (SX_{n-1},*) \arrow[r, "f"] & (B,*) \end{tikzcd}$$

than ours would allow us to define the homomorphism $\delta^{\#}$ for all fiberings. As is known, using this diagram we can easily define the boundary homomorphism $\partial_{\#}$ in fiberings for homotopy groups.

Consider now the diagram

$$\begin{array}{ccccccccc} 0 & \to & H^{n-1}(E_{n-1}) & \xrightarrow{i^*} & H^{n-1}(K^s_{n-1}) & \xrightarrow{\tau} & H^n(B_n) & \to & 0 \\ & & d_{n-1}\downarrow\approx & & d_{n-1}\downarrow\approx & & d_n\downarrow\approx & & \\ 0 & \to & \Pi^{n-1}(E_{n-1}) & \xrightarrow{i^{\#}} & \Pi^{n-1}(K^s_{n-1}) & \xrightarrow{\delta^{\#}} & \Pi^n(B_n) & \to & 0, \end{array} \tag{4}$$

where the upper row is part of the cohomological exact Serre sequence for the fibering p_n.

In Section 2 we shall prove

Proposition 1.7. *Diagram* (4) *is commutative and its lower row is also an exact sequence.*

Corollary 1.4, Lemma 1.5, and Proposition 1.7 imply that the following proposition is valid.

Proposition 1.8. *The semiexact sequence* (2) *for the fibering* p_n *splits into short exact sequences*

$$0 \longrightarrow \Pi^{n-1}(E_{n-1}) \xrightarrow{i^{\#}} \Pi^{n-1}(K^s_{n-1}) \xrightarrow{\delta^{\#}} \Pi^n(B_n) \longrightarrow 0,$$
$$0 \longrightarrow \Pi^{n+k}(B_n) \xrightarrow{p^{\#}} \Pi^{n+k}(E_{n-1}) \longrightarrow 0, \quad k > 0.$$

Now consider segments of the exact sequences for the fiberings p_n and p_{n-1} from diagram (1) for the functors Π^n and π_n:

a) By Proposition 1.8, for cohomotopy functors Π^n we have

$$0 \longrightarrow \Pi^{n-1}(E_{n-1}) \xrightarrow{i^{\#}} \Pi^{n-1}(K^s_{n-1}) \xrightarrow{\delta^{\#}} \Pi^n(B_n) \longrightarrow 0,$$
$$0 \longrightarrow \Pi^{n-2}(K^t_{n-2}) \xrightarrow{\delta^{\#}} \Pi^{n-1}(E_{n-1}) \longrightarrow 0.$$

After replacing the group $\Pi^{n-1}(E_{n-1})$ by the group $\Pi^{n-2}(K^t_{n-2})$, we shall have the exact sequence

$$0 \longrightarrow \Pi^{n-2}(K^t_{n-2}) \xrightarrow{\delta_{n-1}} \Pi^{n-1}(K^s_{n-1}) \longrightarrow \Pi^n(B_n) \longrightarrow 0,$$

where the imbedding homomorphism δ_{n-1} is defined as the composition $\delta_{n-1} = i^{\#}\delta^{\#}$. We shall repeat our construction procedure this time for the space E_{n+1} (in the role of B_n). We have

$$\Pi^{n+1}(B_n) \approx \Pi^{n+1}(E_{n-1}) \approx \Pi^{n+1}(E_{n+1}).$$

After replacing the group $\Pi^{n+1}(E_{n+1})$ by the group $\Pi^{n+1}(B_n)$, we obtain an exact sequence

$$0 \longrightarrow \Pi^{n-1}(K^{t'}_{n-1}) \xrightarrow{\delta_n} \Pi^n(K^{s'}_n) \longrightarrow \Pi^{n+1}(B_n) \longrightarrow 0$$

and so forth.

b) For homotopy functors π_n we have

$$0 \longrightarrow \pi_n(E_{n-1}) \xrightarrow{p_{n\#}} \pi_n(B_n) \xrightarrow{\partial_{\#}} \pi_{n-1}(K^s_{n-1}) \xrightarrow{i_{\#}}$$
$$\xrightarrow{i_{\#}} \pi_{n-1}(E_{n-1}) \xrightarrow{p_{n\#}} \pi_{n-1}(B_n) \longrightarrow 0,$$
$$0 \longrightarrow \pi_{n-1}(E_{n-1}) \xrightarrow{\partial_{\#}} \pi_{n-2}(K^t_{n-2}) \longrightarrow 0,$$
$$\pi_n(E_{n-1}) \approx \pi_n(E_{n+1}).$$

After replacing the groups $\pi_n(E_{n-1})$ and $\pi_{n-1}(E_{n-1})$ respectively by the groups $\pi_n(E_{n+1})$ and $\pi_{n-2}(K^t_{n-2})$, we obtain an exact sequence

$$\begin{gathered} 0 \longrightarrow \pi_n(E_{n+1}) \longrightarrow \pi_n(B_n) \longrightarrow \pi_{n-1}(K^s_{n-1}) \xrightarrow{\partial_{n-1}} \\ \xrightarrow{\partial_{n-1}} \pi_{n-2}(K^t_{n-2}) \longrightarrow \pi_{n-1}(B_n) \longrightarrow 0, \end{gathered} \tag{5}$$

where the homeomorphism ∂_{n-1} is defined as the composition $\partial_{n-1} = \partial_{\#} i_{\#}$. Note, besides, that the isomorphisms

$$(p_n p_{n-1})_{\#} : \pi_i(E_{n+1}) \longrightarrow \pi_i(B_n)$$

holds for $i \geq n+1$. On repeating our construction procedures this time for the space E_{n+1} (in the role of B_n), we obtain an exact sequence

$$\begin{gathered} 0 \longrightarrow \pi_{n+1}(E_{n+2}) \longrightarrow \pi_{n+1}(E_{n+1}) \longrightarrow \pi_n(K^{s'}_n) \xrightarrow{\partial_n} \\ \xrightarrow{\partial_n} \pi_{n-1}(K^{t'}_{n-1}) \longrightarrow \pi_n(E_{n+1}) \longrightarrow 0. \end{gathered} \tag{6}$$

Moreover, we have

$$\pi_{n+1}(E_{n+1}) \approx \pi_{n+1}(E_{n-1}) \approx \pi_{n+1}(B_n).$$

After replacing the group $\pi_{n+1}(E_{n+1})$ in (6) by the group $\pi_{n+1}(B_n)$, combining sequences (5) and (6) in the term $\pi_n(E_{n+1})$, and continuing our constructions, we obtain a gradually long exact sequence

$$\begin{gathered} \cdots \longrightarrow \pi_{n+1}(B_n) \longrightarrow \pi_n(K^{s'}_n) \xrightarrow{\partial_n} \pi_{n-1}(K^{t'}_{n-1}) \longrightarrow \pi_n(B_n) \longrightarrow \\ \longrightarrow \pi_{n-1}(K^s_{n-1}) \xrightarrow{\partial_{n-1}} \pi_{n-2}(K^t_{n-2}) \longrightarrow \pi_{n-1}(B_n) \longrightarrow 0. \end{gathered}$$

Note that because of the fact that the groups $\pi_{n-1}(B_n)$, $\pi_n(E_{n+1})$, and so forth are the finite ones the image of homomorphisms ∂_i, $i \geq n-1$, for an arbitrary element of the corresponding free abelian group contains some multiple of this element.

Let now X be an arbitrary space, $n > 2$, $p \in \Pi^n(X)$, and $f : S^n \to X$ be an arbitrary mapping where S^n is the standard sphere. Consider the index

$$\alpha = (S^n; f) \in \omega(X; n)$$

and assume

$$\varepsilon(p)([f]) = p_\alpha.$$

As follows from [3], we have correctly defined the mapping

$$\varepsilon(p) : \pi_n(X) \to H^n(S^n).$$

In Section 2 we shall prove

Proposition 1.9. *$\varepsilon(p)$ is a group homomorphism.*

For $p = 0$ we evidently have $\varepsilon(p)([f]) = p_\alpha = 0$. Besides, let $p, q \in \Pi^n(x)$. Then

$$\varepsilon(p+q)([f]) = [p+q]_\alpha = p_\alpha + q_\alpha = \varepsilon(p)([f]) + \varepsilon(q)([f]).$$

Thus we have defined the homomorphism

$$\varepsilon : \Pi^n(X) \to \operatorname{Hom}(\pi_n(X), H^n(S^n)), \quad n > 2.$$

Proposition 1.10. *If $X = K_n^t$, then ε is an isomorphism of dimension n.*

Proof. Consider the space $S^{(t)} = V_{k=1}^t S_k$, $S_k = S^n$, $k = 1, 2, \dots, t$. Let $i_k : S^n \to S^{(t)}$ be the standard imbeddings. Fix the system of generators of the group $\pi_n(K_n^t)$:

$$f_k : S^n \to K_n^t, \quad k = 1, 2, \dots, t.$$

Consider the commutative diagram

$$\begin{tikzcd} & K_n^t & \\ S^n \arrow[ur, "f_k"] \arrow[rr, "i_k"] & & S^{(t)} \arrow[ul, "f"'] \end{tikzcd}$$

where $f = Vf_k$. It is understood that the homomorphism f^* induced in cohomologies of dimension n will be an isomorphism. Let φ be an arbitrary homomorphism from

$$\operatorname{Hom}(\pi_n(K_n^t), H^n(S^n)).$$

Define an element $h \in H^n(S^{(t)})$ by the condition $i_k^*(h) = \varphi([f_k])$, $k = 1, 2, \dots, t$. Let $h_0 = f^{*-1}(h) \in H^n(K_n^t)$. Assume that

$$p = d_n(h) \in \Pi^n(K_n^t),$$

where d_n is the analogue of the Gurevich homomorphism in the Π theory. Then

$$\varepsilon(p)([f_k]) = p_{\alpha_k} = f_k^*(h_0) = (i_k^* f^*)(f^{*-1}(h)) = i_k^*(h) = \varphi([f_k]).$$

Therefore $\varepsilon(p) = \varphi$ and ε is an epimorphism. By applying the dimensional reasoning (Corollary 1.4) we now have $\operatorname{Ker} \varepsilon = 0$. $\square$

Let

$$\Sigma = e^{*-1}\delta : H^i(X) \to H^{i+1}(SX)$$

be the suspension isomorphism in cohomologies where δ is the coboundary homomorphism and

$$e : (CX, X) \to (SX, *)$$

is the standard mapping. Consider the m-sphere S^m as the space $S^m(S^0)$ where S^0 is the 0-dimensional sphere. Fix the isomorphism

$$\xi_1 : H^{n-1}(S^{n-1}) \to Z$$

and set $\xi_2 = \xi_1\Sigma$, where

$$\Sigma : H^{n-2}(S^{n-2}) \to H^{n-1}(S^{n-1})$$

is the suspension isomorphism.

Let $[f] \in \pi_{n-1}(K^s_{n-1})$ and $p \in \Pi^{n-1}(K^s_{n-1})$. Define the isomorphism

$$\varepsilon_1 : \Pi^{n-1}(K^s_{n-1}) \to \mathrm{Hom}(\pi_{n-1}(K^s_{n-1}), \mathbb{Z})$$

by the equality

$$\varepsilon_1(p)([f]) = \xi_1(\varepsilon(p)([f])).$$

Let now $n > 4$. Consider the diagram

$$\begin{array}{ccc} \mathrm{Hom}(\pi_{n-2}(K^t_{n-2}), \mathbb{Z}) & \xrightarrow{\widetilde{\partial}_{n-1}} & \mathrm{Hom}(\pi_{n-1}(K^s_{n-1}), \mathbb{Z}) \\ {\scriptstyle \varepsilon_2}\big\downarrow\approx & & {\scriptstyle \varepsilon_1}\big\downarrow\approx \\ \Pi^{n-2}(K^t_{n-2}) & \xrightarrow{\delta_{n-1}} & \Pi^{n-1}(K^s_{n-1}), \end{array} \tag{7}$$

where ∂_{n-1} and δ_{n-1} are the above-defined homomorphisms, $\widetilde{\partial}_{n-1}$ is the homomorphism dual to ∂_{n-1}, and the isomorphism ε_2 is defined analogously to ε_1.

In Section 2 we shall prove

Proposition 1.11. *Diagram* (7) *is commutative.*

Theorem 1.12. *Let X be a linearly and simply connected space for which homology groups $H_i(X, \mathbb{Z})$ are finitely generated for all i and cohomology groups $H^i(X, \mathbb{Z}) = 0$ for $1 \leq i \leq 4$. Then for $n > 2$ we have*

$$\operatorname{rank} \Pi^n(X) = \operatorname{rank} \pi_n(X),$$
$$\operatorname{Tor} \Pi^{n+1}(X) \approx \operatorname{Tor} \pi_n(X).$$

Proof. Consider a sequence of fiberings constructed analogously to diagram (1) (by the conditions of the theorem $X \in L_5$)

$$X = X_5 \xleftarrow{k^{s5}_4} X_6' \xleftarrow{k^{t5}_3} X_6 \xleftarrow{k^{s6}_5} X_7' \xleftarrow{k^{t6}_4} X_7 \longleftarrow \cdots$$
$$\cdots \longleftarrow X_n \xleftarrow{k^{sn}_{n-1}} X'_{n+1} \xleftarrow{k^{tn}_{n-2}} X_{n+1} \longleftarrow \cdots,$$

where instead of the mappings we give the fibers of the corresponding fiberings. Note that this sequence is a certain Moore–Postnikov decomposition of the mapping $* \to X$ (see [6]). Then, as follows from the above discussion (items a) and b)), we obtain short exact sequences

$$0 \longrightarrow \Pi^{n-2}(K^{t_n}_{n-2}) \xrightarrow{\delta_{n-1}} \Pi^{n-1}(K^{s_n}_{n-1}) \longrightarrow \Pi^n(X) \longrightarrow 0,$$

where $n > 4$ and a long exact sequence

$$\cdots \longrightarrow \pi_n(X) \longrightarrow \pi_{n-1}(K^{s_n}_{n-1}) \xrightarrow{\partial_{n-1}} \pi_{n-2}(K^{t_n}_{n-2}) \longrightarrow \pi_{n-1}(X) \longrightarrow \cdots$$
$$\cdots \longrightarrow \pi_5(X) \longrightarrow \pi_4(K^{s_5}_4) \xrightarrow{\partial_4} \pi_3(K^{t_5}_3) \longrightarrow \pi_4(X) \longrightarrow 0.$$

Proposition 1.11 implies that the homomorphisms ∂_i and δ_i, $i \geq 4$, can be regarded as homomorphisms dual to each other. Now, recalling the well-known results of homological algebra and taking into account the last remark of item b), we obtain the statement of the theorem for $n > 4$. For $n = 4$ we have $\operatorname{Tor} \Pi^5(X) \approx \operatorname{Tor} \pi_4(X)$, while the equality $\operatorname{rank} \Pi^4(X) = \operatorname{rank} H^4(X) = \operatorname{rank} \pi_4(X) = 0$ is obvious. For $n = 3$ all groups from the statement of the theorem are trivial. $\square$

2. In this section we prove the statements that have been given in Section 1 without proof.

1. Proof of Theorem 1.3. It is more convenient for us to assume that the objects of subcategories L_n are finite CW-complexes without cells of dimension $1, 2, \ldots, n-2$ (see [3]). Let $X \in L_n$ and $h \in H^n(X \times X)$. Consider the diagram

$$\begin{array}{c} X \xrightarrow{i_1'} X \times X \xleftarrow{i_2'} X \\ \uparrow \Delta \\ X \end{array}$$

where i_1' and i_2' are the standard imbeddings and Δ is the diagonal mapping. Write h in the form

$$h = i_1^{'*}(h) \times 1 + 1 \times i_2^{*}(h),$$

where $1 \in H^0(X) = \mathbb{Z}$. Then

$$\Delta^*(h) = \Delta^*(i_1^{'*}(h) \times 1) + \Delta^*(1 \times i_2^{'*}(h)) =$$
$$= i_1^{'*}(h) \cdot 1 + 1 \cdot i_2^{'*}(h) =_1 i^{'*}(h) + i_2^{'*}(h).$$

Thus we have the equality

$$\Delta^*(h) = i_1^{'*}(h) + i_2^{'*}(h)$$

which we shall use below without indicating that we do so.

Let

$$j_t : X_1 \times X_2 \to X_t, \quad t = 1, 2,$$

be the standard projections and

$$i_t : X_t \to X_1 \times X_2, \quad t = 1, 2,$$

be the standard imbeddings. Consider the homomorphisms

$$j_t^{\#} : \Pi^n(X_t) \to \Pi^n(X_1 \times X_2), \ \ t = 1, 2.$$

Let us show that $j_t^{\#}$ are monomorphisms. Let $p \in \Pi^n(X_1)$ and $j_1^{\#}(p) = 0$. Consider the index

$$\alpha = (X; f) \in \omega(X_1; n)$$

and the commutative diagram

$$\begin{tikzcd} X_1 \times X_2 \arrow[rr, "j_1"] & & X_1 \\ & X \arrow[ul, "g_1"] \arrow[ur, "f"'] & \end{tikzcd}$$

where $g_1 = i_1 f$. Let

$$\beta = (X; g_1) \in \omega(X_1 \times X_2; n).$$

Then

$$0 = [j_1^{\#}(p)]_\beta = p_\alpha.$$

Therefore $p = 0$. Applying an analogous procedure, we obtain that $j_2^{\#}$ is a monomorphism.

Now we shall show that

$$Jmj_1^{\#} \cap Jmj_2^{\#} = 0.$$

Let

$$p = j_1^{\#}(p_1) = j_2^{\#}(p_2) \in \Pi^n(X_1 \times X_2),$$

where $p_t \in \Pi^n(X_t)$, $t = 1, 2$. Consider an arbitrary index

$$\alpha = (X; f) \in \omega(X_1 \times X_2; n)$$

and the mappings

$$f_1 = j_1 f : X \to X_1,$$
$$f' = i_1 j_1 f : X \to X_1 \times X_2.$$

We have the commutative diagram

$$\begin{tikzcd}
X_1 \times X_2 \arrow[rr, "j_i"] & & X_1 \\
& X \arrow[ul, shift right, "f"] \arrow[ul, shift left, "f'"'] \arrow[ur, "f_1"'] &
\end{tikzcd}.$$

Consider the indices

$$\alpha_1 = (X; f_1) \in \omega(X_1; n),$$
$$\alpha' = (X; f') \in \omega(X_1 \times X_2; n).$$

Then

$$p_\alpha = [j_1^{\#}(p_1)]_\alpha = [p_1]_{\alpha_1},$$
$$p_{\alpha'} = [j_1^{\#}(p_1)]_{\alpha'} = [p_1]_{\alpha_1}.$$

Therefore $p_\alpha = p_{\alpha'}$. Consider the commutative diagram

$$\begin{tikzcd}
X_1 \times X_2 \arrow[rr, "j_2"] & & X_2 \\
& X \arrow[ul, "f'"] \arrow[ur, "0"'] &
\end{tikzcd}$$

and the index

$$\alpha_0 = (X; 0) \in \omega(X_2; n),$$

where 0 is the constant mapping. Using Corollary 1.3 from [3], we obtain

$$P_{\alpha'} = [j_2^{\#}(p_2)]_{\alpha'} = [p_2]_{\alpha_0} = 0.$$

Therefore $p_\alpha = p_{\alpha'} = 0$ and $p = 0$.

Let now p be an arbitrary element of the group $\Pi^n(X_1 \times X_2)$. We shall show that

$$j_1^{\#}(i_1^{\#}(p)) + j_2^{\#}(i_2^{\#}(p)) = p.$$

Consider an arbitrary index

$$\alpha = (X; f) \in \omega(X_1 \times X_2; n)$$

and the commutative diagram

$$\begin{tikzcd}
& X_1 \times X_2 & \\
X \arrow[ur, "f_1"] \arrow[r, "i_1'"'] & X \times X \arrow[u, "f_1 \times f_2"'] & X \arrow[ul, "f_2"'] \arrow[l, "i_2'"] \\
& X \arrow[u, "\Delta"'] &
\end{tikzcd}$$

where $f_t = i_t j_t f$, $t = 1, 2$. It is clear that $(f_1 \times f_2)\Delta = f$ and $X \times X \in L_n$. Consider the indices

$$\alpha_0 = (X \times X; f_1 \times f_2),$$
$$\alpha_t = (X; f_t), \quad t = 1, 2,$$

where $\alpha_0, \alpha_1, \alpha_2 \in \omega(X_1 \times X_2; n)$. Now

$$p_{\alpha_1} + p_{\alpha_2} = i_1^{'*}(p_{\alpha_0}) + i_2'(p_{\alpha_0}) = \Delta^*(p_{\alpha_0}) = p_\alpha.$$

Consider also the commutative diagram

$$\begin{tikzcd}
X_1 \times X_2 \arrow[r, "j_1"] & X_1 \arrow[r, "i_1"] & X_1 \times X_2 \\
& X \arrow[ul, "f"] \arrow[u, "f'"'] \arrow[ur, "f_1"'] &
\end{tikzcd}$$

where $f' = j_1 f$, and the index

$$\alpha' = (X; f') \in \omega(X_1; n).$$

Now

$$[j_1^{\#}(i_1^{\#}(p))]_\alpha = [i_1^*(p)]_{\alpha'} = p_{\alpha_1}.$$

By an analogous procedure we obtain $[j_2^{\#}(i_2^{\#}(p))]_\alpha = p_{\alpha_2}$. Now

$$[j_1^{\#}(i_1^{\#}(p)) + j_2^{\#}(i_2^{\#}(p))]_\alpha = [j_1^{\#}(i_1^{\#}(p))]_\alpha +$$
$$+ [j_2^{\#}(i_2^{\#}(p))]_\alpha = p_{\alpha_1} + p_{\alpha_2} = p_\alpha. \quad \square$$

2. Proof of Lemma 1.5. Consider the homotopy sequence of the fibering p_n

$$\cdots \to \pi_{n-1}(K^s_{n-1}) \to \pi_{n-1}(E_{n-1}) \to \pi_{n-1}(B_n) \to 0.$$

Hence it follows that we have $\pi_i(E_{n-1}) = 0$, $i < n-1$, for the linear connected space E_{n-1}. The spaces B_n and K^s_{n-1} have finite-type modules of homologies. Therefore E_{n-1} also has a finite-type module of homologies. Thus $E_{n-1} \in L_{n-1}$. Since $\pi_{n-2}(E_{n-1}) = 0$, $H^{n-1}(E_{n-1})$ is a free group. Besides, the homomorphism

$$d_{n-1} : H^{n-1}(E_{n-1}) \to \Pi^{n-1}(E_{n-1})$$

is an isomorphism. We have thereby proved the first item.

Now let us consider the diagram of the term E_2 of a cohomological spectral sequence of the fibering p_n

$$\begin{tikzcd}
n+1 & 0 & & \\
n & 0 & & \\
n-1 & H^{n-1}(K^s_{n-1}) \arrow[ddr, "\tau"] & & \\
 & 0 & & \\
 & \mathbb{Z} & H^n(B_n) & H^{n+1}(B_n) \\
 & & n & n+1
\end{tikzcd}$$

In our case the transgression τ is an epimorphism. Therefore $H^n(E_{n-1}) = 0$. Besides, it follows from the diagram that

$$p_n^* : H^{n+1}(B) \to H^{n+1}(E_{n-1})$$

is an isomorphism. We have thereby proved the third item and half of the second item.

The number t is defined by the quantity of generators of the free group

$$H^{n-1}(E_{n-1}) \approx \operatorname{Ker}\tau \subset H^{n-1}(K^s_{n-1}).$$

If $H^n(B_n)$ is a free group, then $\operatorname{Ker}\tau = 0$ and hence $t = 0$. Therefore in that case $E_{n-1} \in L_{n+1}$. Applying now our above reasoning to the fibering p_{n-1}, we obtain that $E_{n+1} \in L_n$ and, moreover,

$$H^n(E_{n+1}) \approx H^n(E_{n-1}) = 0.$$

Therefore $E_{n+1} \in L_{n+1} \subset L_n$. We have proved the fifth item. Let

$$\alpha = (X_n; \varphi) \in \omega(E_{n-1}; n)$$

be an arbitrary index (here and in the sequel, when an analogous situation occurs, it will be assumed that the objects of auxiliary subcategories L_n are the finite CW-complexes (see [3])). The mapping

$$\varphi : X_n \to E_{n-1}$$

can be lifted to the mapping in the space of the fibering p_{n-1}

$$\widetilde{\varphi} : X_n \to E_{n+1}.$$

This follows from the fact that the obstruction to this lifting is zero (see [6])

$$c(\varphi) \in H^{n-1}(X_n, \oplus\mathbb{Z}) = 0.$$

Consider the index

$$\beta = (E_{n+1}, p_{n-1}) \in \omega(E_{n-1}; n).$$

Since $p_{n-1}\widetilde{\varphi} = \varphi$, we have $\alpha < \beta$. Let $p \in \Pi^n(E_{n-1})$. Then $p_\beta \in H^n(E_{n+1}) = 0$. Therefore

$$p_\alpha = \widetilde{\varphi}^*(p_\beta) = \widetilde{\varphi}^*(0) = 0.$$

Hence $p = 0$. We have thereby completely proved the second item. It remains to prove the fourth item.

Let $p \in \Pi^{n+k}(E_{n-1})$, $k > 0$, and let

$$\varphi : X_{n+1} \to B_n$$

be an arbitrary mapping, $X_{n+k} \in L_{n+k}$. Obstructions to the lifting of the mapping φ and to the homotopy between two such liftings belong, respectively, to the groups (see [6])

$$H^n(X_{n+k}, \oplus\mathbb{Z}) = 0,$$
$$H^{n-1}(X_{n+k}, \oplus\mathbb{Z}) = 0.$$

Therefore we can choose up to homotopy a unique lifting $\widetilde{\varphi}$ of the mapping φ

$$\widetilde{\varphi} : X_{n+k} \to E_{n+1},$$

$p_n\widetilde{\varphi} = \varphi$. Consider the indices

$$\alpha = (X_{n+k}; \varphi) \in \omega(B_n, n+k),$$
$$\widetilde{\alpha} = (X_{n+k}; \widetilde{\varphi}) \in \omega(E_{n-1}, n+k).$$

Assume that

$$[\varepsilon(p)]_\alpha = p_{\widetilde{\alpha}}.$$

As follows from [3], our definition is correct. Let us show that the set $[\varepsilon(p)]_\alpha$ defines an element $\varepsilon(p)$ of the group $\Pi^{n+k}(B_n)$. Consider the diagram

$$\begin{tikzcd}
& E_{n-1} \arrow[d, "p_n"] & \\
& B_n & \\
X_{n+k} \arrow[uur, "\widetilde{\varphi}"] \arrow[ur, "\varphi"] \arrow[rr, "g"] & & Y_{n+k} \arrow[uul, "\widetilde{\psi}"'] \arrow[ul, "\psi"']
\end{tikzcd},$$

where $\psi g \sim \varphi$, $p_n \widetilde{\psi} = \psi$, $\widetilde{\varphi} = \widetilde{\psi} g$, $Y_{n+k} \in L_{n+k}$. Then

$$p_n \widetilde{\varphi} = p_n \widetilde{\psi} g = \psi g \sim \varphi.$$

Also consider the indices

$$\beta = (Y_{n+k}; \psi) \in \omega(B_n, n+k),$$
$$\widetilde{\beta} = (Y_{n+k}; \widetilde{\psi}) \in \omega(E_{n-1}, n+k).$$

Then $\widetilde{\alpha} < \widetilde{\beta}$ and we have

$$g^*([\varepsilon(p)]_\beta) = g^*(p_{\widetilde{\beta}}) = p_{\widetilde{\alpha}} = [\varepsilon(p)]_\alpha.$$

Thus we have defined the mapping

$$\varepsilon : \Pi^{n+k}(E_{n-1}) \to \Pi^{n+k}(B_n).$$

Let now $p \in \Pi^{n+k}(B_n)$ and

$$\varphi : X_{n+K} \to B_n$$

be an arbitrary mapping. Consider the indices

$$\alpha = (X_{n+k}; \varphi) \in \omega(B_n; n+k),$$
$$\widetilde{\alpha} = (X_{n+k}; \widetilde{\varphi}) \in \omega(E_{n-1}; n+k),$$
$$p_n(\widetilde{\alpha}) = (X_{n+k}; p_n \widetilde{\varphi}) = (X_{n+k}; \varphi) = \alpha \in \omega(B_n; n+k).$$

Then

$$[\varepsilon(p_n^{\#}(p))]_\alpha = [p_n^{\#}(p)]_{\widetilde{\alpha}} = p_{p_n(\widetilde{\alpha})} = p_\alpha.$$

Therefore $\varepsilon p_n^{\#} = id$.

Let further $p \in \Pi^{n+k}(E_{n-1})$ and

$$\widetilde{\varphi} : X_{n+k} \to E_{n-1}$$

be an arbitrary mapping. Consider the indices

$$\widetilde{\alpha} = (X_{n+k}; \widetilde{\varphi}) \in \omega(E_{n-1}; n+k),$$
$$\alpha = p_n(\widetilde{\alpha}) = (X_{n+k}; p_n \widetilde{\varphi}) \in \omega(B_n; n+k).$$

We can use the mapping $\widetilde{\varphi}$ as the lifting of $p_n\widetilde{\varphi}$. Then

$$[p_n^{\#}(\varepsilon(p))]_{\widetilde{\alpha}} = [\varepsilon(p)]_{p_n(\widetilde{\alpha})} = [\varepsilon(p)]_{\alpha} = p_{\widetilde{\alpha}}.$$

Therefore $p_n^{\#}\varepsilon = id$. We have thereby completed the proof of the fourth item and Lemma 1.5. □

3. Proof of Proposition 1.7. The commutativity of the left square follows from the fact that homomorphisms di are natural. Consider the right square where $K_{n-1}^s = \Omega K_n^s$. Let $h \in H^{n-1}(\Omega K_n^s)$. Since $B_n \in L_n$, it suffices to check the equality

$$(\delta^{\#} d_{n-1})(h) = (d_n \tau)(h)$$

only for the index

$$\beta = (B_n; id) \in \omega(B_n; n).$$

The fibering

$$\Omega K_{n-1}^s \longrightarrow E_{n-1} \xrightarrow{p_n} B_n$$

is obtained here from the standard fibering

$$\Omega K_{n-1}^s \longrightarrow EK_n^s \xrightarrow{p} K_n^s$$

by the mapping

$$f : B_n \to K_n^s.$$

Therefore

$$\tau = f^*\tau_0 = f^*\pi^{*-1}\Sigma,$$

where the mapping

$$\pi : \Sigma\Omega K_n^s \to K_n^s$$

is the known mapping, the homomorphism

$$\Sigma : H^{n-1}(\Omega K_n^s) \to H^n(S\Omega K_n^s)$$

is an isomorphism of the suspension in cohomologies, and $\tau_0 = \pi_*^{-1}\Sigma$ is the transgression of the fibering p. Now consider the commutative diagram

$$\begin{tikzcd}
 & (E_{n-1}, \Omega K_n^s) \arrow[dr, "p_n"] & \\
(C\Omega B_n, \Omega B_n) \arrow[ur, "\Phi"] \arrow[r, "e_B"] \arrow[d, "C\Omega f"] & (S\Omega B_n, *) \arrow[r, "\pi_B"] \arrow[d, "S\Omega f"] & (B_n, *) \arrow[d, "f"] \\
(C\Omega K_n^s, \Omega K_n^s) \arrow[r, "e"] & (S\Omega K_n^s, *) \arrow[r, "\pi"] & (K_n^s, *)
\end{tikzcd}$$

where
$$E_{n-1}=\{(b,e)|b\in B_n,\ \ e\in EK^s_n,\ \ f(b)=p(e)\}.$$
Next define the lifting mapping Φ. But first note that when defining the cone CX, we contract the subspace
$$(X\times 0)\cup(*\times I)\subset X\times I$$
to the base point.

Let
$$\sigma\in\Omega B_n,\quad 0\le t\le 1,\quad 0\le\xi\le 1.$$
Define the path $s_t\in EK^s_n$ by the equality
$$s_t(\xi)=((\Omega f)(\sigma))(\xi t)$$
and set
$$\Phi([(\sigma,t)])=(\sigma(t),s_t).$$
It is obvious that for $t=1$ we have $\Phi|\Omega B_n=\Omega f$. Consider the index
$$\beta_1=(\Omega B_n;\Omega f)\in\omega(\Omega K^s_n,n-1).$$
Then we have
$$[\delta^*(d_{n-1}(h))]_\beta=\psi_B^{*-1}(\delta[d_{n-1}(h)]_{\beta_1})=\psi_B^{*-1}(\delta((\Omega f)^*(h))),$$
where $\psi_B=\pi_Be_B$. On the other hand,
$$\begin{aligned}[d_n(\tau(h))]_\beta&=\tau(h)=f^*(\tau_0(h))=f^*(\pi^{*-1}(\Sigma(h)))=\\&=(f^*\pi^{*-1})(e^{*-1}(\delta(h)))=(\pi_B^{*-1}(S\Omega f)^*e^{*-1})(\delta(h))=\\&=(\pi_B^{*-1}e_B^{*-1})((C\Omega f)^*\delta)(h)=\psi_B^{*-1}(\delta((\Omega f)^*(h))).\qquad\square\end{aligned}$$
4. Proof of Proposition 1.9. It is obvious that $\varepsilon(p)(0)=0$. Let now
$$f_i:S^n\to X,\quad i=1,2,$$
be arbitrary mappings. Let, moreover,
$$i_k:S^n\to S^n\vee S^n,\quad k=1,2,$$
be standard imbeddings and
$$j_k:S^n\vee S^n\to S^n,\quad k=1,2,$$
be standard projections. Also consider the mapping
$$\nu:S^n\to S^n\vee S^n$$
which brings the H-cogroup structure onto the sphere. We have the commutative diagram

$$
\begin{tikzcd}
 & S^n \arrow[dl, "f_2"'] \arrow[dr, "i_2"] & & \\
X & & S^n \vee S^n \arrow[ll, "f_1 \vee f_2"'] & S^n \arrow[l, "\nu"'] \\
 & S^n \arrow[ul, "f_1"] \arrow[ur, "i_1"'] & &
\end{tikzcd}
.
$$

We introduce the notation $(f_1 \vee f_2)\nu = f_3$ and consider the indices

$$
\alpha_i = (S^n; f_i), \quad i = 1, 2, 3,
$$
$$
\beta = (S^n \vee S^n; \;\; f_1 \vee f_2).
$$

Then $\alpha_i < \beta$, $i = 1, 2, 3$. Let $e \in H^n(S^n)$ be the generator of the group $H^n(S^n)$. Then the elements $j_1^*(e)$ and $j_2^*(e)$ are the generators of the group $H^n(S^n \vee S^n)$. Let

$$
p_\beta = k_1 j_1^*(e) + k_2 j_2^*(e), \quad k_1, k_2 \in \mathbb{Z}.
$$

Then

$$
p_{\alpha_3} = \nu^*(p_\beta) = k_1 \nu^*(j_1^*(e)) + k_2 \nu^*(j_2^*(e)) = k_1 e + k_2 e,
$$
$$
p_{\alpha_1} = i_1^*(p_\beta) = k_1 i_1^*(j_1^*(e)) + k_2 i_1^*(j_2^*(e)) = k_1 e + k_2 0 = k_1 e.
$$

Analogously, $p_{\alpha_2} = k_2 e$. Therefore $p_{\alpha_3} = p_{\alpha_1} + p_{\alpha_2}$. Then

$$
\varepsilon(p)([f_1] + [f_2]) = p_{\alpha_3} = p_{\alpha_1} + p_{\alpha_2} = \varepsilon(p)([f_1]) + \varepsilon(p)([f_2]). \qquad \square
$$

5. Proof of Proposition 1.11. Let

$$
g : S^{n-1} \to K_{n-1}^s
$$

be an arbitrary mapping. Consider the commutative diagram

$$
\begin{tikzcd}
 & & (E_{n+1}, K_{n-2}^t) \arrow[dr, "p_{n-1}"] & \\
(C\Omega S^{n-1}, \Omega S^{n-1}) \arrow[urr, "\bar{g}"] \arrow[r, "e_s"] & (S\Omega S^{n-1}, *) \arrow[r, "\pi_s"] \arrow[d, "\pi_S"] & (S^{n-!}, *) \arrow[r, "ig"] \arrow[d, equal] & (E_{n-1}, *) \arrow[d, equal] \\
(CS^{n-2}, S^{n-2}) \arrow[u, "K_c"] \arrow[r, "e"] \arrow[drr, "\bar{\bar{g}}"'] & (SS^{n-2}, *) \arrow[r, "\pi = id"] & (S^{n-1}, *) \arrow[r, "ig"] & (E_{n-1}, *) \\
 & & (E_{n+1}, K_{n-2}^t) \arrow[ur, "p_{n-1}"'] &
\end{tikzcd}
$$

where the mapping

$$k : S^{n-2} \to \Omega S^{n-1} = \Omega S S^{n-2}$$

and the mappings π_s, e_s, e are the standard ones, the mapping $\bar{g}$ is obtained by lifting the mapping $(ig)\pi_s e_s$, and $\bar{\bar{g}} = \bar{g} k_c$. Consider the mappings

$$g_0 = \bar{\bar{g}}\, |S^{n-2} : S^{n-2} \to K^t_{n-2},$$
$$g_2 = \bar{g}|\Omega S^{n-1} : \Omega S^{n-1} \to K^t_{n-2}.$$

Then we obviously have

$$\partial_{n-1}([g]) = \partial_{\#}([id]) = [g_0], \ \ g_0 = g_2 k.$$

We introduce the indices

$$\beta = (S^{n-1}; g) \in \omega(K^s_{n-1}; n-1),$$
$$\beta_0 = (S^{n-2}; g_0) \in \omega(K^s_{n-2}; n-2),$$
$$\beta_1 = (S^{n-1}; ig) \in \omega(E_{n-1}; n-1),$$
$$\beta_2 = (\Omega S^{n-1}; g_2) \in \omega(K^t_{n-2}; n-2).$$

Then $\beta_0 < \beta_2$. Let $p \in \Pi^{n-2}(K^t_{n-2})$. Then

$$(\widetilde{\partial}_{n-1}(\varepsilon_2(p)))([g]) = \varepsilon_2(p)(\partial_{n-1}([g])) = \xi_2(\varepsilon(p)(\partial_{\#}([ig]))) =$$
$$= \xi_2(p_{\beta_0}) = \xi_1(\Sigma(p_{\beta_0})) = \xi_1(\Sigma(K^*(p_{\beta_2}))) = \xi_1((\Sigma K^*)(p_{\beta_2})).$$

But, on the other hand,

$$(\varepsilon_1(\delta_{n-1}(p)))([g]) = \xi_1((\varepsilon(\delta_{n-1}(p)))([g])) = \xi_1([\delta_{n-1}(p)]_\beta) =$$
$$= \xi_1([i^{\#}(\delta^{\#}(p))]_\beta) = \xi_1([\delta^{\#}(p)]_{\beta_1}) = \xi_1((\psi_s^{*-1}\delta)(p_{\beta_2})),$$

where $\psi_s = \pi_s e_s$. Since ξ_1 is an isomorphism, it is sufficient for us to check the equality $\Sigma K^* = \psi_s^{*-1}\delta$. Using the equality, $\pi_s e_s K_c = e$ we shall have

$$\Sigma K^* = (e^{*-1}\delta)(\delta^{-1}K_c^*\delta) = e^{*-1}K_c^*\delta =$$
$$= e^{*-1}e^* e_s^{*-1}\pi_s^{*-1}\delta = \psi_s^{*-1}\delta. \quad \square$$

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