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ON SEPARABLE SUPPORTS OF BOREL MEASURES

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Abstract. Some properties of Borel measures with separable supports are considered. In particular, it is proved that any σ-finite Borel measure on a Suslin line has a separable supports and from this fact it is deduced, using the continuum hypothesis, that any Suslin line contains a Luzin subspace with the cardinality of the continuum.

Let E be a topological space. We say that the space E has the property (S) if for every σ-finite Borel measure μ defined in this space there exists a separable support, i.e., a separable closed set $F(\mu) \subset E$ such that $\mu(E \backslash F(\mu)) = 0$.

Let us consider some examples of topological spaces having the property (S).

Example 1. It is obvious that any separable topological space E has the property (S).

Example 2. Let E be an arbitrary metric space whose topological weight is not measurable in a wide sense. Then according to the well-known result from the topological measure theory the space E has the property (S).

Example 3. Let E be the Alexandrov compactification of some discrete topological space. Then the following statements are equivalent:

a) the space E has the property (S);

b) card(E) is not measurable in a wide sense.

Example 4. Let E be a Hausdorff topological space. We say that E is a Luzin space if every σ-finite diffused (i.e., continuous) Borel measure defined in E is identically zero. The classical Luzin set on the real line R is a Luzin topological space (about Luzin sets see, for example, [1]). One can easily check that any σ-finite Borel measure defined in the Luzin topological space

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E is concentrated on a countable subset of E. The latter fact immediately implies that every Luzin space has the property (S). Notice here that the topological space E from Example 3 is a Luzin space if and only if the cardinal number $\mathrm{card}(E)$ is not measurable in a wide sense.

We shall show that the property (S) is not, generally speaking, a hereditary property in the class of all topological spaces.

Example 5. Denote by the symbol R^* the set of all real numbers equipped with the Sorgenfrey topology. Since the topological space R^* is separable, the product-space $R^* \times R^*$ is separable too. Thus we see that the space $R^* \times R^*$ has the property (S). On the other hand, the space $R^* \times R^*$ contains a closed discrete subspace D with the cardinality of the continuum. Assuming that the cardinality of the continuum is measurable in a wide sense, we can define in D a probability continuous Borel measure. But such a measure vanishes on all separable subsets of D. Therefore we find (under our assumption about the cardinality of the continuum) that the topological space $D \subset R^* \times R^*$ does not have the property (S).

Example 6. Let $(E, \cdot)$ be a topological group and let μ be a σ-finite Borel measure in the group E which is quasi-invariant with respect to some everywhere dense subgroup Γ of E. Assume also that there exists a closed separable set $F \subset E$ with $\mu(F) > 0$. Then we may assert that the given topological group E is separable too. Indeed, assume that the opposite is true: E is not a separable topological space. Using the method of transfinite recursion let us define an ω_1-sequence $(F_\xi)_{\xi<\omega_1}$ satisfying the following conditions:

a) the family $(F_\xi)_{\xi<\omega_1}$ is increasing by inclusion;
b) $(\forall\xi)(\xi < \omega_1 \Rightarrow F_\xi$ is a closed separable subgroup of $E)$;
c) if $\zeta < \xi < \omega_1$, then $\mu(F_\xi \backslash F_\zeta) > 0$.

Take as F_0 the closed subgroup of E generated by the set F. Assume that for an ordinal number ξ $(0 < \xi < \omega_1)$ we have already constructed a partial family of closed separable groups $(F_\zeta)_{\zeta<\xi}$. Denote by H the closure of the union of this partial family. It is obvious that H is also a closed separable subgroup of E. Since the group Γ is dense everywhere in E and E is not separable, there exists an element $g \in \Gamma \backslash H$. It is clear that

$$g \cdot H \cap H = \varnothing, \quad \mu(H) > 0, \quad \mu(g \cdot H) > 0.$$

Therefore we may take as F_ξ the closed subgroup of E generated by the set $H \cup g \cdot H$. We shall thereby define the needed ω_1-sequence $(F_\xi)_{\xi<\omega_1}$. Simultaneously, we shall obtain an uncountable family $(F_{\xi+1} \backslash F_\xi)_{\xi<\omega_1}$ of pairwise disjoint sets everyone of which has a strictly positive μ-measure. However the existence of such a family contradicts the σ-finiteness of the measure μ. Therefore the given topological group E is separable. By the foregoing arguments we actually establish the validity of a more general result. Namely, let E be again a topological group, μ be a σ-finite Borel measure in E, and Γ be some everywhere dense subset of E. If the measure μ is quasi-invariant with respect to Γ, then at least one of the following two assertions is valid:

a) the topological group E is separable;

b) the measure μ vanishes on all closed separable subsets of E.

In particular, if the mentioned measure μ is not identically zero and the given topological group E is not separable, then E does not have the property (S).

In connection with the property (S) notice that we have

Proposition 1. *Let $(E_i)_{i\in I}$ be an arbitrary countable family of topological spaces everyone of which has the property (S). Then the product-space $\prod_{i\in I} E_i$ has the property (S) too.*

Proof. Let μ be a σ-finite Borel measure defined in the product-space $\prod_{i\in I} E_i$. It may be assumed without loss of generality that μ is a probability measure. Fix an index $i \in I$ and for any Borel set $X \subset E_i$ write

$$\mu_i(X) = \mu\big(X \times \prod_{j\in I\setminus\{i\}} E_j\big).$$

This formula defines the probability Borel measure μ_i in the topological space E_i. By the condition, there exists a separable support $F_i \subset E_i$ for μ_i. Now consider the product-space $\prod_{i\in I} F_i$. This space is certainly separable and, as one can easily check, represents a support for the original measure μ. □

Proposition 2. *Let I be a set of indices whose cardinality is not measurable in a wide sense. Assume $(E_i)_{i\in I}$ to be a family of topological spaces everyone of which has the property (S). Denote by the symbol E the topological sum of the family $(E_i)_{i\in I}$. Then the space E also has the property (S).*

Proof. Let μ be an arbitrary σ-finite Borel measure defined in the topological space E. We write

$$J = \{i \in I : \mu(E_i) = 0\}.$$

Since the measure μ is σ-finite, we have the inequality

$$\mathrm{card}(I\setminus J) \leq \omega_0.$$

Further, since $J \subset I$ and $\mathrm{card}(I)$ is not measurable in a wide sense, we find that $\mathrm{card}(J)$ is not measurable in a wide sense too. From the latter fact we readily conclude that the equality

$$\mu\big(\underset{i\in J}{\cup} E_i\big) = 0$$

is valid.

Let us now assume that for any index $i \in I\backslash J$ the set F_i is a separable support for the restriction of the measure μ onto the space E_i. Then it is easy to see that the set $\cup_{i\in I\backslash J} F_i$ is a separable support for the measure μ. □

Proposition 3. *Let E be a topological space and $(E_i)_{i\in I}$ be a countable family of subspaces of E everyone of which has the property (S). Then the space*

$$E' = \underset{i\in I}{\cup} E_i \subset E$$

has the property (S) too.

Proof. Let μ be an arbitrary σ-finite Borel measure in the topological space E'. For any index $i \in I$ denote by μ_i the trace of the measure μ on the set E_i. Recall that the measure μ_i is defined by the formula

$$\mu_i(X) = \mu^*(X)$$

where X is an arbitrary Borel subset of the space E_i and μ^* is the outer measure associated with μ. By the condition, for the measure μ_i there exists a separable support $F_i \subset E_i$. Hence it is easy to see that the closure of the set $\cup_{i\in I} F_i$ in the space E' is a separable support for the original measure μ. □

Example 7. Let α be a cardinal number strictly larger than the cardinality of the continuum and let T be the unit circle in the Euclidean plane R^2. Consider T as a compact commutative group and denote by μ the probability Haar measure in the compact commutative group T^α. It is easy to check that the topological group T^α is not separable. According to the result of Example 6 the Haar measure μ is not concentrated on a separable subset of T^α. Hence the topological space T^α does not have the property (S). We thus conclude that the topological product of spaces having the property (S) does not have, generally speaking, this property.

Example 8. Let E be a discrete topological space with the cardinality ω_1. According to the classical result of Ulam (see, for example, [1]) the cardinal number ω_1 is not measurable in a wide sense. Therefore E is a Luzin topological space and, in particular, has the property (S). Next consider the cardinal number ω_1 equipped with its order topology. It is obvious that the topological space ω_1 is locally compact and locally countable. Denote by the symbol μ the standard Dieudonne two-valued probability continuous measure defined on the Borel σ-algebra of the space ω_1. Notice that the measure μ is not concentrated on a separable subspace of ω_1 and therefore the space ω_1 does not have the property (S). At the same time the space ω_1 is an injective continuous image of the discrete space E. We have thereby shown that the property (S) is not, generally speaking, invariant even under injective continuous mappings.

The next proposition indicates a certain relationship between the topological spaces having the property (S) and Luzin subspaces of these spaces.

Proposition 4. *Assume that the continuum hypothesis holds and let E be an arbitrary Hausdorff topological space with the cardinality of the continuum. If E has the property (S), then at least one of the next two assertions is valid:*

1) *E is a separable topological space;*
2) *E contains some Luzin subspace with the cardinality of the continuum.*

Proof. Assume that the given space E is not separable. Consider the injective family of all closed separable subsets of E. It is easy to check that the cardinality of this family is equal to the cardinality of the continuum. Therefore this family can be represented as an ω_1-sequence $(F_\xi)_{\xi<\omega_1}$. Further, using the method of transfinite recursion, define an injective ω_1-sequence $(x_\xi)_{\xi<\omega_1}$ of points of the space E. Assume that for an ordinal number $\xi < \omega_1$ we have already defined a partial ξ-sequence of points $(x_\zeta)_{\zeta<\xi}$. Consider the set

$$P = \Big(\cup_{\zeta<\xi} F_\zeta \Big) \cup \Big(\cup_{\zeta<\xi} \{x_\zeta\} \Big).$$

It is obvious that the set P is separable. Therefore we have $E \setminus P \neq \varnothing$. Take as x_ξ any point from the nonempty set $E \setminus P$. Thereby we shall construct the needed ω_1-sequence $(x_\xi)_{\xi<\omega_1}$. After that we write

$$X = \cup_{\xi<\omega_1} \{x_\xi\}.$$

Clearly, $\mathrm{card}(X)$ is equal to the cardinality of the continuum. Now, taking into consideration the fact that our topological space E has the property (S), it is not difficult to check that X is a Luzin subspace of E. $\square$

Remark. Assume again that the continuum hypothesis holds and let E be an arbitrary nonseparable topological space with the cardinality of the continuum. Then there exists a subspace X of E satisfying the following conditions:

a) $\mathrm{card}(X)$ is equal to the cardinality of the continuum;
b) every uncountable set $Y \subset X$ is nonseparable.

In particular, if the given topological space E is metrizable, then the mentioned space X has the property (S) although all uncountable subsets of X are nonseparable (moreover, in this case X is a Luzin space).

The proof of the existence of the mentioned space $X \subset E$ is quite similar to that of Proposition 4.

Example 9. Assume that the continuum hypothesis holds and consider any Sierpinski set E on the real line R (about Sierpinski sets see, for example, [1]). Since E is a separable metric space, it hereditarily has the property (S). At the same time, it is not difficult to show that the space E does not contain any uncountable Luzin subspace.

Example 10. Let E be an arbitrary nonseparable metric space. As is well known, E contains a discrete subspace D with the cardinality ω_1. Using the result of Ulam mentioned above, we see that D is an uncountable Luzin subspace of E.

Notice now that there may also exist nonseparable nonmetrizable topological spaces having the property (S). We wish to show here that such spaces can be encountered even among linearly ordered topological spaces. Let us recall a few simple notions from the theory of ordered sets. Let $(E, \leq)$ be an arbitrary linearly ordered set. We say that two nonempty open intervals in E are disjoint if the right end-point of one of these intervals is less than or equal to the left end-point of the other interval. In particular, any two disjoint intervals do not have the common points. Further, we say that a linearly ordered set $(E, \leq)$ has the Suslin property if every family of nonempty open disjoint intervals is at most countable. We wish to emphasize that this definition immediately implies that for arbitrary linearly ordered sets the Suslin property is the hereditary one (in this connection we remark here that the separability property and the Suslin property are not the hereditary ones for arbitrary topological spaces). Recall that the Suslin line is any linearly ordered set E having the Suslin property and being nonseparable with respect to the order topology in $(E, \leq)$. As is well known, the existence of the Suslin line is not provable in the modern set theory and does not contradict this theory (see, for example, [2]). We shall see below that any Suslin line equipped with its order topology has the property (S).

In the first place we need the following auxiliary statement.

Lemma. *Let $(E, \leq)$ be a linearly ordered set having the Suslin property and considered as a topological space with respect to its order topology. Further, let μ be an arbitrary Borel measure in E and let V be an arbitrary open subset of E locally negligible with respect to the measure μ (the latter means that for any point $x \in V$ there exists an open interval $V(x) \subset V$ containing* *the point x and being negligible with respect to μ). Then the whole set V is negligible with respect to μ, i.e., the equality $\mu(V) = 0$ is fulfilled.*

Proof. Let us define a binary relation $\sim$ in the given set V by the formula

$$x \sim y \Leftrightarrow \mu([x, y]) = 0 \quad \text{and} \quad \mu([y, x]) = 0.$$

Clearly, the relation $\sim$ is an equivalence relation in the set V. Let $(V_i)_{i \in I}$ be the partition of V canonically associated with $\sim$. Then it is not difficult to check that

a) every set V_i is open in E;

b) for any index $i \in I$ we have $\mu(V_i) = 0$;

c) $\mathrm{card}(I) \leq \omega_0$.

Now we immediately obtain

$$\mu(V) = \sum_{i \in I} \mu(V_i) = 0. \qquad \square$$

It is easy to see that in the formulation of the above lemma we can replace an arbitrary open locally negligible set $V \subset E$ by an arbitrary Borel locally negligible set $X \subset E$. Notice also that in the same formulation of the lemma the Suslin property of the considered linearly ordered set $(E, \leq)$ is quite essential. Indeed, let us take the ordinal number ω_1 equipped with its order topology. Denote by μ the Dieudonne probability continuous measure defined on the Borel σ-algebra of the space ω_1. Then it is obvious that the whole space ω_1 is locally negligible with respect to the measure μ but we have the equality $\mu(\omega_1) = 1$.

Using the above lemma, we obtain

Proposition 5. *Let $(E, \leq)$ be a linearly ordered set having the Suslin property and equipped with its order topology. Then the topological space E has the property (S).*

Proof. If our topological space E is separable, then there is nothing to prove. Therefore let us assume that E is nonseparable, i.e., that the linearly ordered set $(E, \leq)$ is the Suslin line. Let μ be any σ-finite Borel measure in E. We introduce the notation

$$V = \{x \in E : E \text{ is locally negligible at the point } x \text{ with respect to } \mu\}.$$

Obviously, V is an open subset of E locally negligible with respect to the measure μ. Therefore according to the above lemma we have $\mu(V) = 0$. Consider the closed set $F = E \backslash V$. We assert that this set is separable. Assume that the opposite is true: the set F is not separable. This means that the linearly ordered set $(F, \leq)$ is also the Suslin line. Let λ be the restriction of the measure μ onto the topological space F. The measure λ is certainly σ-finite and, moreover, has the following property: the λ-measure of any nonempty open interval in F is strictly positive. Now let us consider the topological product $F \times F$ equipped with the product-measure $\lambda \times \lambda$. According to the classical result of Kupera (see, for example, [2]) in the space $F \times F$ there exists an uncountable family of nonempty open pairwise disjoint rectangles. It is obvious that the $(\lambda \times \lambda)$-measure of everyone of these rectangles is strictly positive. But the product-measure $\lambda \times \lambda$ is σ-finite and therefore such a family of rectangles cannot exist. The obtained contradiction proves our proposition. $\square$

Notice that a result analogous to Proposition 5 and concerning Boolean algebras with measures was obtained by several authors (J.L.Kelley, D.Maharam, and others).

As a consequence of Proposition 5 we have

Proposition 6. *Assume that the continuum hypothesis holds. Then any Suslin line contains some Luzin subspace with the cardinality of the continuum.*

To prove this proposition it is enough to apply the results of Propositions 4 and 5 with regard to the fact that the cardinality of any Suslin line is less than or equal to the cardinality of the continuum. The latter fact readily follows, for example, from the well-known Erdös–Rado combinatorial theorem (about this theorem and its various applications see [2]).

Naturally, there arises the following question: does every uncountable topological space contain an uncountable subspace having the property (S)? Our next example shows that the answer to this question is negative.

Example 11. Assume that the continuum hypothesis holds. Let E be an arbitrary Sierpinski set on the real line R. Equip the set R with the usual density topology (about the density topology on R see, in particular, [1]). Denote by the symbol R^* the set R equipped with the density topology and consider the Sierpinski set E as a subspace of the topological space R^*. Now it is not difficult to check that every uncountable subspace of the topological space E does not have the property (S). Notice here that an analogous example can be constructed in the manner following. Assume again that the continuum hypothesis holds and E is again any Sierpinski set on the real line R. Consider the family of sets

$$\big\{V\backslash X : V \text{ is an open subset of } R \text{ and } X \\ \text{is at most a countable subset of } R\big\}.$$

In the set R this family of sets is a topology strictly stronger than the standard Euclidean topology in R and strictly weaker than the density topology in R. Denote by the symbol R^{**} the set R equipped with this topology and consider the Sierpinski set E as a subspace of the topological space R^{**}. Now it is not difficult to prove that any uncountable subspace of the topological space E does not have the property (S).

In connection with the last example let us recall that a topological space E is a Sierpinski space if it does not contain a Luzin subspace with the cardinality equal to the cardinality of E. Notice that Sierpinski spaces have some interesting topological properties. For instance, assume that the continuum hypothesis holds and take an arbitrary Sierpinski space E with the cardinality of the continuum. Then it can be proved that any analytic subset of E is a Borel set in E.

In conclusion let us formulate one unsolved problem concerning topological spaces having the property (S).

Problem. *Obtain a characterization of spaces having the property* (S) *in some purely topological terms.*

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