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ROOTS OF THE PHASE OPERATORS

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ABSTRACT. The kth root taken of the bosonic phase operator is considered. This leads to the extension of the Hilbert space. In the case $k = 2$ an ordinary fermionic extension arises. Particles whose statistics depends on k are introduced for other values of k.

1. PHASE OPERATORS

The problem of polar decomposition of the creation and annihilation operators ($\mathbf{a}^*$, $\mathbf{a}$) of a harmonic oscillator goes back to Dirac [1]. In this section some aspects of this problem will be reviewed [2].

Let h be the oscillator hamiltonian

$$h = (p^2 + q^2)/2 \tag{1.1}$$

on the phase plane Γ with coordinates p, q and Poisson brackets (PB)

$$\{q, p\} = 1. \tag{1.2}$$

If the polar angle φ is introduced by

$$p = \sqrt{2h}\sin\varphi, \qquad q = \sqrt{2h}\cos\varphi, \tag{1.3}$$

then for the complex variables a and a^*

$$a = \frac{q + ip}{\sqrt{2}}, \qquad a^* = \frac{q - ip}{\sqrt{2}}$$

(1.3) is equivalent to the representation

$$a = \sqrt{h}\exp(i\varphi), \qquad a^* = \sqrt{h}\exp(-i\varphi). \tag{1.3$'$}$$

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Although the variable φ is not global ($\varphi \in \mathrm{S}^1$), it is clear that the functions $\exp(\pm i\varphi)$ are correctly defined on the whole phase space Γ (except the origin), and for the PB with hamiltonian (1.1) we have

$$\{h, \exp(\pm i\varphi)\} = \pm i \exp(\pm i\varphi) \tag{1.4}$$

which in local coordinates corresponds to

$$\{h, \varphi\} = 1. \tag{1.4$'$}$$

In the quantum case for the operator $\mathbf{h}$ (we use boldface notation) we choose the normal ordering

$$\mathbf{h} = \mathbf{a}^*\mathbf{a} \equiv \frac{\mathbf{p}^2 + \mathbf{q}^2}{2} - \frac{1}{2}\mathbf{I}. \tag{1.5}$$

Then this $\mathbf{h}$ can be considereds a particle (boson) number operator. It is well known that the eigenvalues of $\mathbf{h}$ are nonnegative integers. The corresponding eigenvectors $|n\rangle$ $(n = 0, 1, 2, \dots)$

$$\mathbf{h}|n\rangle = n|n\rangle \tag{1.5$'$}$$

can be constructed by applying the creation operator to the vacuum state $|0\rangle$

$$|n\rangle = \frac{1}{\sqrt{n!}}\mathbf{a}^{*^n}|0\rangle$$

and we have

$$\mathbf{a}|n\rangle = \sqrt{n}|n-1\rangle, \quad \mathbf{a}^*|n\rangle = \sqrt{n+1}|n+1\rangle. \tag{1.5$''$}$$

The vectors $|n\rangle$ form the basis of the Hilbert space. For further convenience we denote this Hilbert space by H_B (bosonic space).

From the correspondence of PB with commutators, for the phase operators $\mathbf{exp}(\pm i\boldsymbol{\varphi})$ we get (see (1.4))

$$[\mathbf{h}, \mathbf{exp}(\pm i\boldsymbol{\varphi})] = \mp\, \mathbf{exp}(\pm i\boldsymbol{\varphi}).$$

So the operator $\mathbf{exp}(i\boldsymbol{\varphi})$ ($\mathbf{exp}(-i\boldsymbol{\varphi})$) now decreases (increases) by 1 the eigenvalues of the operator $\mathbf{h}$, and thus we can write

$$\mathbf{exp}(i\boldsymbol{\varphi})|n\rangle = \begin{cases} |n-1\rangle, & n > 0, \\ 0, & n = 0, \end{cases} \qquad \mathbf{exp}(-i\boldsymbol{\varphi})|n\rangle = |n+1\rangle \tag{1.6}$$

from which by virtue of (1.5$'$) we have

$$\mathbf{exp}(i\boldsymbol{\varphi}) = \frac{1}{\sqrt{\mathbf{h}+\mathbf{I}}}\mathbf{a}, \quad \mathbf{exp}(-i\boldsymbol{\varphi}) = \mathbf{a}^*\frac{1}{\sqrt{\mathbf{h}+\mathbf{I}}}. \tag{1.6$'$}$$

According to (1.3′) these expressions actually correspond to $\exp(\pm i\varphi)$ in the case of some operator ordering while the operator function $\sqrt{\mathbf{h}+\mathbf{I}}$ acts on the basis vectors as a diagonal operator

$$\frac{1}{\sqrt{\mathbf{h}+\mathbf{I}}}|n\rangle = \frac{1}{\sqrt{n+1}}|n\rangle.$$

Thus (1.6) and (1.6′) are equivalent and may be regarded as a definition of the phase operators $\mathbf{exp}(\pm i\boldsymbol{\varphi})$ in the Hilbert space H_B.

To conclude this section, note that although the operators $\mathbf{exp}(i\boldsymbol{\varphi})$ and $\mathbf{exp}(-i\boldsymbol{\varphi})$ are mutually conjugate, they satisfy only the one-side unitary relations

$$\mathbf{exp}(-i\boldsymbol{\varphi})\,\mathbf{exp}(i\boldsymbol{\varphi}) = \mathbf{I} - |0\rangle\langle 0|, \quad \mathbf{exp}(i\boldsymbol{\varphi})\,\mathbf{exp}(-i\boldsymbol{\varphi}) = \mathbf{I} \tag{1.7}$$

where $|0\rangle\langle 0|$ is the projection operator on the vacuum state.

2. Square root of the phase operators

In this section we consider a square root taken of the phase operators, i.e., expressions $\mathbf{exp}(\pm i\boldsymbol{\varphi}/2)$.

First note that the corresponding classical functions $\exp(\pm i\varphi/2)$ are not determined on the phase plane as single-valued continuous functions and it is expected that there will be problems in their definition. Formally for the PB we get (see (1.4′))

$$\{h, \exp(\pm i\varphi/2)\} = \pm(i/2)\exp(\pm i\varphi/2)$$

which in the quantum case has to be written in the form

$$[\mathbf{h}, \mathbf{exp}(\pm i\boldsymbol{\varphi}/2)] = \mp(1/2)\,\mathbf{exp}(\pm i\boldsymbol{\varphi}/2). \tag{2.1}$$

From this follows that the operators $\mathbf{exp}(\pm i\boldsymbol{\varphi}/2)$ must decrease (or increase) by 1/2 the levels of the oscillator hamiltonian. However, since such states are absent in the spectrum, relations (2.1) cannot be fulfilled. Therefore a meaningful determination of a square root taken of the phase operators is impossible on the Hilbert space H_B.

As is wellknown, since Dirac's equation describes fermions and Dirac's operator $i\gamma^\mu\partial_\mu$ is connected with the square root taken of the d'Alembert's operator ∂^2 [3], we shall try to connect the square root of the phase operators with the fermionic operators. For this we artificially extend the space H_B by introducing the fermionic operators $\mathbf{f}$ and $\mathbf{f}^*$

$$\mathbf{f}^2 = \mathbf{f}^{*^2} = \mathbf{0}, \quad \mathbf{f}\,\mathbf{f}^* + \mathbf{f}^*\mathbf{f} = \mathbf{I}. \tag{2.2}$$

These relations can be represented in the two-dimensional space H_F with basis vectors $|n\rangle_F$ $(n = 0, 1)$, where

$$\mathbf{f}|0\rangle_F = 0, \quad \mathbf{f}|1\rangle_F = |0\rangle_F, \quad \mathbf{f}^*|0\rangle_F = |1\rangle_F, \quad \mathbf{f}^*|1\rangle_F = 0 \tag{2.3}$$

and for the fermionic number operator $\mathbf{N}_F = \mathbf{f}^*\mathbf{f}$ we have

$$[\mathbf{N}_F, \mathbf{f}] = -\mathbf{f}, \quad [\mathbf{N}_F, \mathbf{f}^*] = \mathbf{f}^*, \quad \mathbf{N}_F|n\rangle_F = n|n\rangle_F.$$

Now we shall consider the exterior product of the spaces $\mathcal{H}_F = \mathrm{H}_B \otimes \mathrm{H}_F$ and modify the hamiltonian $\mathbf{h}$:

$$\mathbf{H} = \mathbf{h} \otimes \mathbf{I}_F + (1/2)\mathbf{I}_B \otimes \mathbf{N}_F \tag{2.4}$$

with $\mathbf{I}_B$ and $\mathbf{I}_F$ as identity operators on the spaces H_B and H_F, respectively.

Below, for simplicity, $\mathbf{h} \otimes \mathbf{I}_F$ and $\mathbf{I}_B \otimes \mathbf{N}_F$ will be replaced by $\mathbf{h}$ and $\mathbf{N}_F$, respectively, i.e.,

$$\mathbf{H} = \mathbf{h} + (1/2)\mathbf{N}_F = \mathbf{a}^*\mathbf{a} + (1/2)\mathbf{f}^*\mathbf{f}. \tag{2.4$'$}$$

The eigenstates of the new hamiltonian $\mathbf{H}$ are characterized by two numbers $|n_B, n_F\rangle$,

$$\mathbf{H}|n_B, n_F\rangle = (n_B + \frac{1}{2}n_F)|n_B, n_F\rangle \tag{2.4$''$}$$

where $n_B = 0, 1, 2, \dots,\ n_F = 0, 1$.

Introducing the operators $\mathbf{A}^\pm$

$$\mathbf{A}^+ = \mathbf{f}^* + \mathbf{exp}(-i\boldsymbol{\varphi})\mathbf{f}, \quad \mathbf{A}^- = \mathbf{f} + \mathbf{exp}(+i\boldsymbol{\varphi})\mathbf{f}^*, \tag{2.5}$$

it is easy to verify that the operator $\mathbf{A}^+$ increases by $1/2$ and the operator $\mathbf{A}^-$ decreases by $1/2$ every eigenvalue level of the operator $\mathbf{H}$ (except the vacuum state which is canceled by the action of $\mathbf{A}^-$). So the following relations are satisfied:

$$[\mathbf{H}, \mathbf{A}^\pm] = \pm(1/2)\mathbf{A}^\pm. \tag{2.5$'$}$$

Considering quadratic combinations of the operators $\mathbf{A}^\pm$, it is easy to verify (see (2.2) and (1.7)) that

$$\begin{gathered}(\mathbf{A}^+)^2 = \mathbf{exp}(-i\boldsymbol{\varphi}), \quad (\mathbf{A}^-)^2 = \mathbf{exp}(i\boldsymbol{\varphi}), \\ \mathbf{A}^+\mathbf{A}^- = \mathbf{I} - |0,0\rangle\langle 0,0|, \quad \mathbf{A}^-\mathbf{A}^+ = \mathbf{I}.\end{gathered} \tag{2.6}$$

From this (also compare (2.1) and (2.5$'$)) one concludes that the operators $\mathbf{A}^\pm$ may be considered as a definition of the operators $\mathbf{exp}(\mp i\boldsymbol{\varphi}/2)$:

$$\mathbf{exp}(\mp i\boldsymbol{\varphi}/2) \equiv \mathbf{A}^\pm. \tag{2.7}$$

Thus on the extended space $\mathcal{H}_F$ it is possible to define the square root of the phase operators which are connected with fermionic operators.

3. The kth root of phase operators

The arguments of Section 2 can be generalized to the case of arbitrary k ($k > 2$) if we consider the kth root taken of phase operators

$$\mathbf{exp}(\pm i\boldsymbol{\varphi}/k).$$

For this purpose we introduce the k-dimensional unitary space H_k with the orthonormal basis

$$|0\rangle_k,\ |1\rangle_k,\ |2\rangle_k, \dots, |k-1\rangle_k \tag{3.1}$$

and define the operators $\mathbf{f}_k, \mathbf{f}_k^*$:

$$\mathbf{f}_k|n\rangle_k = \begin{cases} |n-1\rangle_k, & n \geq 1, \\ 0, & n = 0; \end{cases} \tag{3.2}$$

$$\mathbf{f}_k^*|n\rangle_k = \begin{cases} |n+1\rangle_k, & n \leq k-2, \\ 0, & n = k-1. \end{cases} \tag{3.2$'$}$$

In the basis (3.1) the operators $\mathbf{f}_k$ and $\mathbf{f}_k^*$ have the representation

$$\mathbf{f}_k = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ . & . & . & \dots & . & . \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}, \quad \mathbf{f}_k^* = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ . & . & . & \dots & . & . \\ 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix}.$$

It is obvious that

$$(\mathbf{f}_k)^k = (\mathbf{f}_k^*)^k = \mathbf{0} \tag{3.3}$$

and it is also easy to verify that the k^2 number of "normally" ordered monomes $(\mathbf{f}_k^*)^j(\mathbf{f}_k)^m$, where $0 \leq j \leq k-1$, $0 \leq m \leq k-1$, give k^2 linearly independent matrices and form the basis in the space of linear operators on H_k. Therefore any monome can be written as a linear combination of normally ordered monomes. For example, $\mathbf{f}_k\mathbf{f}_k^*$ takes the form

$$\mathbf{f}_k\mathbf{f}_k^* = \mathbf{I} - (\mathbf{f}_k^*)^{k-1}(\mathbf{f}_k)^{k-1} \tag{3.4}$$

which is the generalization of the commutation relations (2.2) in the case of arbitrary k. It is possible to compute other operator relations; in particular we have

$$\begin{gathered} \sum_{j=0}^{k-1} (\mathbf{f})^j(\mathbf{f}^*)^{k-1}(\mathbf{f})^{k-j-1} = \mathbf{I}, \\ (\mathbf{f})^j(\mathbf{f}^*)^m = \mathbf{0}, \quad \text{if} \quad j - m \geq k. \end{gathered} \tag{3.5}$$

The particle number operator $\mathbf{N}_k$ can be introduced on the space H_k as

$$\mathbf{N}_k|n\rangle_k = n|n\rangle_k.$$

It is easy to verify that the operator $\mathbf{N}_k$ can be written in the form

$$\mathbf{N}_k = \mathbf{f}_k^*\mathbf{f}_k + \mathbf{f}_k^{*2}\mathbf{f}_k^2 + \cdots + (\mathbf{f}_k^*)^{k-1}(\mathbf{f}_k)^{k-1} \tag{3.6}$$

and satisfies the relations

$$[\mathbf{N}_k, \mathbf{f}_k] = -\mathbf{f}_k, \quad [\mathbf{N}_k, \mathbf{f}_k^*] = \mathbf{f}_k^*. \tag{3.6$'$}$$

After this let us consider the exterior product of the spaces H_B and H_k

$$\mathcal{H}_k = \mathrm{H}_B \otimes \mathrm{H}_k \tag{3.7}$$

and define the hamiltonian on $\mathcal{H}_k$

$$\mathbf{H} = \mathbf{h} + \frac{1}{k}\mathbf{N}_k \tag{3.8}$$

where we use the above-mentioned abbreviation (see (2.4)).

The eigenvectors of the new hamiltonian $\mathbf{H}$ are characterized by two quantum numbers $|n_B, n_k\rangle$, where $n_B = 0, 1, 2, \ldots,$ $n_k = 0, 1, 2, \ldots, k-1$, and we have

$$\mathbf{H}|n_B, n_k\rangle = (n_B + \frac{1}{k}n_k)|n_B, n_k\rangle. \tag{3.9}$$

The energy levels do not degenerate and are equidistant with an interval $1/k$.

Here, by analogy with (2.5), the operators $\mathbf{A}_k^{\pm}$ can be introduced in the following way:

$$\begin{aligned} \mathbf{A}_k^+ &= \mathbf{f}_k^* + \mathbf{exp}(-i\boldsymbol{\varphi})(\mathbf{f}_k)^{k-1}, \\ \mathbf{A}_k^- &= \mathbf{f}_k + \mathbf{exp}(i\boldsymbol{\varphi})(\mathbf{f}_k^*)^{k-1}. \end{aligned} \tag{3.10}$$

They change any energy level by $1/k$ ($\mathbf{A}_k^+$ increases and $\mathbf{A}_k^-$ decreases, except the vacuum state which is canceled by the action of $\mathbf{A}_k^-$) and satisfy the commutation relations

$$[\mathbf{H}, \mathbf{A}_k^{\pm}] = \pm\frac{1}{k}\mathbf{A}_k^{\pm}.$$

Now, using (3.4) and (3.5), we can verify that

$$(\mathbf{A}_k^+)^k = \mathbf{exp}(-i\boldsymbol{\varphi}), \quad (\mathbf{A}_k^-)^k = \mathbf{exp}(i\boldsymbol{\varphi}), \tag{3.11}$$

and also

$$\mathbf{A}_k^-\mathbf{A}_k^+ = \mathbf{I}, \quad \mathbf{A}_k^+\mathbf{A}_k^- = \mathbf{I} - |0,0\rangle\langle 0,0|.$$

So after all this our result can be formulated as

Theorem. *The kth root of the phase operators is defined on the Hilbert space* $l_2 \otimes H_k$ *and has form* (3.10).

Finally, note that the correct definition of nonsingle-valued functions $\exp(\pm i\varphi/k)$ can be made on the k-sheet Riemann surface. In [4] the case $k = 2$ was considered. As a result, one can conclude that the quantization on the two-sheet Riemann surface is connected with the introduction of fermionic degrees of freedom. It is easy to verify that this situation is valid for arbitrary k. On the other hand, the case of arbitrary k can be related with anyon physics and fractional statistics [5].

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