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ON THE PERIODIC BOUNDARY-VALUE PROBLEM FOR SYSTEMS OF SECOND-ORDER NONLINEAR ORDINARY DIFFERENTIAL EQUATIONS

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ABSTRACT. The periodic boundary-value problem for systems of second-order ordinary nonlinear differential equations is considered. Sufficient conditions for the existence and uniqueness of a solution are established.

§ 1. STATEMENT OF THE MAIN RESULTS

Consider the periodic boundary-value problem

$$x'' = f(t, x, x'), \tag{1.1}$$

$$x(a) = x(b), \quad x'(a) = x'(b), \tag{1.2}$$

where the vector-function $f : [a, b] \times \mathbb{R}^{2n} \to \mathbb{R}^n$ ($\mathbb{R}^n$ denotes the n-dimensional Euclidean space with the norm $\|\cdot\|$) satisfies the local Caratheodory conditions, i.e., $f(\cdot, x, y) : [a, b] \to \mathbb{R}^n$ is measurable for each $(x, y) \in \mathbb{R}^{2n}$, $f(t, \cdot) : \mathbb{R}^{2n} \to \mathbb{R}^n$ is continuous for almost all $t \in [a, b]$, and the function

$$f_r(\cdot) = \sup\{\|f(\cdot, x, y)\| : \|x\| + \|y\| \le r\}$$

is Lebesgue integrable on $[a, b]$ for each positive r.

By a solution of the problem (1.1), (1.2) we mean a vector-function $x : [a, b] \to \mathbb{R}^n$ which has the absolutely continuous first derivative on $[a, b]$ and satisfies the differential system (1.1) almost everywhere in $[a, b]$, as well as the boundary conditions (1.2).

For the literature on (1.1),(1.2) we refer to [1,2] and the references cited therein. Note that [1] deals with the scalar variant of the boundary-value problem (1.1),(1.2) (i.e., when $n = 1$).

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Below, the sufficient conditions for solvability and unique solvability of the boundary value problem (1.1), (1.2) are given. They supplement some of those mentioned above.

We use the following notation:

$x \cdot y$ is the inner product of vectors $x, y \in \mathbb{R}^n$;

$\mathbb{R} = \mathbb{R}^1$, $\mathbb{R}_+ = [0, +\infty[$;

for each positive number r and vector $y \in \mathbb{R}^n$

$$\eta_r(y) = \begin{cases} 0 & \text{for } \|y\| \le r, \\ \frac{y}{\|y\|} & \text{for } \|y\| > r; \end{cases}$$

$U_\delta(t_0)$ is the δ-neighborhood of $t_0 \in \mathbb{R}$;

$\widetilde{C}^1([a,b];S)$ $(S \subset \mathbb{R}^n)$ is the set of vector-functions $x : [a,b] \to S$ which have an absolutely continuous first derivative on $[a,b]$;

$C(S_1;S_2)$ $(S_1 \subset \mathbb{R}, S_2 \subset \mathbb{R}^n)$ is the set of continuous vector-functions $x : S_1 \to S_2$;

$L([a,b];S)$ $(S \subset \mathbb{R})$ is the set of functions $x : [a,b] \to S$ which are Lebesgue integrable on $[a,b]$.

Definition 1.1 (see [3, Definition 1.1] or [4, Definition 1.2]). Suppose that the functions $\varphi :]a,b[\to]0,\infty[$ and $z :]a,b[\to \mathbb{R}^n$ have a first derivative which is absolutely continuous on every segment contained in $]a,b[$. A pair of functions (φ, z) is said to be a Nagumo pair of the differential system (1.1) if the condition

$$(x - z(t)) \cdot (f(t,x,y) - z''(t)) + \|y - z'(t)\|^2 - (\varphi'(t))^2 \ge \varphi(t)\varphi''(t)$$
$$\text{for } \ a < t < b, \ \|x - z(t)\| = \varphi(t) \text{ and } (x - z(t)) \cdot (y - z'(t)) = \varphi(t)\varphi'(t)$$

is satisfied, the function $\|z''(t)\| + \varphi(t)$ being essentially bounded from above on every segment contained in $]a,b[$.

Remark 1.1. The Nagumo pair of differential system (1.1) serves as a vector analog for the upper and lower functions of the scalar equation (1.1), which were introduced by Nagumo [5] and which since then have been widely adopted in the theory of boundary-value problems (see [1] and the references cited therein; also [4, Remark 1.2], [3, Remark 1.1]). Namely, if $n = 1$ and σ_1 and σ_2 are, respectively, the upper and lower solutions of the differential equation (1.1), then the pair (φ, z) defined by

$$\varphi(t) = \frac{\sigma_2(t) - \sigma_1(t)}{2} \quad \text{and} \quad z(t) = \frac{\sigma_2(t) + \sigma_1(t)}{2} \tag{1.3}$$

is the Nagumo pair of (1.1) (and vice versa).

Note also that the condition

$$x \cdot f(t,x,y) + \|y\|^2 \geq 0 \quad \text{for } \|x\| = r_0 \ \text{ and } \ x \cdot y = 0 \tag{1.4}$$

(see [2, Theorem 3.1]) is necessary and sufficient for (φ, z) to be a Nagumo pair of (1.1), where $z(t) \equiv 0$ and $\varphi(t) \equiv r_0 > 0$.

Definition 1.2. A Nagumo pair (φ, z) of the differential system (1.1) is said to be a Nagumo pair of the problem (1.1),(1.2) if $\varphi \in \widetilde{C}^1([a,b];\mathbb{R}_+)$, $z \in \widetilde{C}^1([a,b];\mathbb{R}^n)$ and the conditions

$$\varphi(a) = \varphi(b), \quad z(a) = z(b) \tag{1.5_1}$$

and

$$\|z'(a) - z'(b)\| \leq \varphi'(b) - \varphi'(a) \tag{1.5_2}$$

are satisfied.

Remark 1.2. In the scalar case, $(1.5_1) - (1.5_2)$ are equivalent to the conditions

$$\sigma_i(a) = \sigma_i(b), \quad (-1)^i(\sigma_i'(a) - \sigma_i'(b)) \leq 0 \quad (i = 1, 2),$$

assuming that (1.3) is satisfied. See these conditions in [1, § 16].

Definition 1.3 (see [3, Definition 2.1] or [4, Definition 1.1]). Suppose that $\varphi \in C([a,b];\mathbb{R}_+)$ and $z \in C([a,b];\mathbb{R}^n)$. A vector-function f is said to have the property $V([a,b],\varphi,z)$ if there exist positive constants r and r_1 such that if $a \leq t_1 < t_2 \leq b$, $\chi \in C(\mathbb{R}_+;[0,1])$ and $x \in \widetilde{C}^1([t_1,t_2];\mathbb{R}^n)$ is an arbitrary solution of the differential system

$$x'' = \chi(\|x'\|) f(t,x,x') \tag{1.6}$$

satisfying the inequalities

$$\|x(t) - z(t)\| \leq \varphi(t) \quad \text{for } \ t_1 \leq t \leq t_2 \tag{1.7}$$

and

$$\|x'(t)\| \geq r \quad \text{for } \ t_1 \leq t \leq t_2, \tag{1.8}$$

then x admits the estimate

$$\int_{t_1}^{t_2} \|x'(t)\| dt \leq r_1. \tag{1.9}$$

Remark 1.3. It is clear that each scalar function has the property $V([a,b],\varphi,z)$ taking an arbitrary positive number for r and $2\max\{\varphi(t)+\|z(t)\|: a\le t\le b\}$ for r_1. The class of vector-functions with the property $V([a,b],\varphi,z)$ is introduced just to unify the approach to the problem (1.1),(1.2) in both the scalar and the vector cases. Some other boundary-value problems were also studied using this approach (see [3,4] and the references cited therein).

Effective sufficient conditions for a vector-function f to have the property $V([a,b],\varphi,z)$ are contained in [4, Propositions 1.1, 1.2] and [3, Proposition 2.1]. For example, if

$$\begin{gathered}(f(t,x,y)\cdot y)(x\cdot y)-(x\cdot f(t,x,y))\|y\|^2\le l(t)\|y\|^3+k\|y\|^4\\ \text{for}\ \ a\le t\le b,\ \ \|x-z(t)\|\le\varphi(t)\ \text{ and }\ \|y\|>\rho,\end{gathered}\tag{1.10}$$

where $l\in L([a,b];\mathbb{R}_+)$, $k<1$ and $\rho>0$, then f has the property $V([a,b],\varphi,z)$.

Theorem 1.1$_1$. *Suppose that (φ,z) is a Nagumo pair of (1.1), (1.2), the vector-function f has the property $V([a,b],\varphi,z)$, and the inequality*

$$f(t,x,y)\cdot\eta_\rho(y)\le w(\|y\|)(l(t)+\|y\|)\tag{1.11}$$

is satisfied on the set

$$\{(t,x,y): a<t<b,\ \ \|x-z(t)\|\le\varphi(t)\},\tag{1.12}$$

where $\rho>0$, $l\in L([a,b];\mathbb{R}_+)$, $\omega\in C(\mathbb{R}_+;]0,+\infty[)$, and

$$\int\limits_0^{+\infty}\frac{ds}{\omega(s)}=+\infty.\tag{1.13}$$

Then the boundary-value problem (1.1), (1.2) has at least one solution $x\in\widetilde{C}^1([a,b];R^n)$ satisfying the estimate

$$\|x(t)-z(t)\|\le\varphi(t)\quad\text{for}\ \ a\le t\le b.\tag{1.14}$$

Theorem 1.1$_2$. *The conclusion of Theorem 1.1$_1$ remains valid if (1.11) is replaced by*

$$f(t,x,y)\cdot\eta_\rho(y)\ge-\omega(\|y\|)(l(t)+\|y\|).\tag{1.15}$$

Theorem 1.2. *Suppose that* (φ, z) *is a Nagumo pair of* (1.1), (1.2)*, the vector-function* f *has the property* $V([a,b],\varphi,z)$*, the inequality* (1.11) *is satisfied on the set*

$$\{(t,x,y) : a_0 \le t \le b,\ \ \|x - z(t)\| \le \varphi(t)\},$$

and the inequality (1.15) *on the set*

$$\{(t,x,y) : a < t < b_0,\ \ \|x - z(t)\| \le \varphi(t)\},$$

where $\rho > 0$*,* $a \le a_0 < b_0 \le b$*,* $l \in L([a,b];\mathbb{R}_+)$*,* $\omega \in C(\mathbb{R}_+;]0,+\infty[)$*, and* (1.13) *holds. Then the boundary-value problem* (1.1), (1.2) *has at least one solution* $x \in \widetilde{C}^1([a,b];\mathbb{R}^n)$ *satisfying the estimate* (1.14)*.*

Theorem 1.3. *Suppose that* (φ, z) *is a Nagumo pair of* (1.1), (1.2)*, the vector-function* f *has the property* $V([a,b],\varphi,z)$*, the inequality* (1.11) *is satisfied on the set*

$$\{(t,x,y) : t \in]a_1, t_0[\cup]b_2, b[,\ \ \|x - z(t)\| \le \varphi(t)\},$$

and the inequality (1.15) *on the set*

$$\{(t,x,y) : t \in]a, a_2[\cup]t_0, b_1[,\ \ \|x - z(t)\| \le \varphi(t)\},$$

where $\rho > 0$*,* $a < a_1 < a_2 < t_0 < b_2 < b_1 < b$*,* $l \in L([a,b];\mathbb{R}_+)$*,* $\omega \in C(\mathbb{R}_+;]0,+\infty[)$*, and* (1.13) *holds. Then the boundary-value problem* (1.1), (1.2) *has at least one solution* $x \in \widetilde{C}^1([a,b];\mathbb{R}^n)$ *satisfying the estimate* (1.14)*.*

Remark 1.4. Theorems 1.1–1.3 extend Theorem 3.1 [2] in the case of periodic boundary-value problem. As an example, define $f_i(t,x,y) = -y_i\|y\|^m + 1 - \|x\|$ and $f = (f_i)_{i=1}^n$, where m is an arbitrary natural number. Let us verify the conditions of, e.g., Theorem 1.1_1 assuming that $z(t) \equiv 0$, $\varphi \equiv 1$, $\rho = 1$, $l(t) \equiv 1$, and $\omega \equiv 1$. First, according to (1.4) where $r_0 = 1$, (φ, z) is the Nagumo pair of (1.1),(1.2). Further, according to (1.10) where $k = 0$, the vector-function f has the property $V([a,b],\varphi,z)$. Finally, the correctness of (1.11), as well as of (1.13), is evident. On the other hand, Theorem 3.1 [2] fails for this example when $m > 2$.

Theorem 1.2 can also be considered as a vector analog of Theorem 16.2 from [1].

Theorem 1.4. *Suppose that for each positive* r *there exist* $l_i(t,r) \in L([a,b];\mathbb{R}_+)$ $(i = 1,2)$ *such that* $l_1(t,r)$ *differs from zero on a subset of positive measure of the interval* $]a,b[$ *and*

$$\begin{gathered}[f(t,x_1,y_1) - f(t,x_2,y_2)](x_1 - x_2) \ge \\ \ge l_1(t,r)\|x_1 - x_2\|^2 - l_2(t,r)|(x_1 - x_2)\cdot(y_1 - y_2)| \\ \text{for } \|x_k\| \le r,\ \ \|y_k\| \le r \ \ (k = 1,2). \end{gathered} \tag{1.16}$$

Then the boundary-value problem (1.1), (1.2) *has at most one solution in the class* $\widetilde{C}^1([a,b];\mathbb{R}^n)$.

Theorem 1.4 can be considered as a vector analog of Theorem 16.4 from [1].

§ 2. Some Auxiliary Results

Lemma 2.1. *Suppose that a vector-function* $q:[a,b]\times\mathbb{R}^{2n}\to\mathbb{R}^n$ *satisfies the local Caratheodory conditions and the inequality*

$$\|q(t,x,y)\|\le l(t)$$

holds on $[a,b]\times\mathbb{R}^{2n}$ *where* $l\in L([a,b];\mathbb{R}_+)$. *Then the differential system*

$$x''=x+q(t,x,y)$$

has at least one solution $x\in\widetilde{C}^1([a,b];\mathbb{R}^n)$ *satisfying the boundary conditions* (1.2).

Proof. It is easy to verify that the differential system

$$x''=x$$

has no nontrivial solution satisfying the boundary conditions (1.2). Thus Lemma 2.1 immediately follows from Proposition 2.3 [1]. □

The next result deals with the solvability of an auxiliary differential system

$$x''=g(t,x,x'). \tag{2.1}$$

Lemma 2.2. *Suppose that* (φ,z) *is a Nagumo pair of the boundary-value problem* (2.1), (1.2) *and on* $[a,b]\times\mathbb{R}^{2n}$

$$\|g(t,x,y)\|\le h(t,x), \tag{2.2}$$

where the vector-functions $g:[a,b]\times\mathbb{R}^{2n}\to\mathbb{R}^n$ *and* $h:[a,b]\times\mathbb{R}^n\to\mathbb{R}^n$ *satisfy the local Caratheodory conditions. Then the boundary-value problem* (2.1), (1.2) *has at least one solution* $x\in\widetilde{C}^1([a,b];\mathbb{R}^n)$ *satisfying the estimate* (1.14).

Proof. Put

$$\sigma(s,t)=\begin{cases}0 & \text{for } s\le 0 \text{ and } \tau\in\mathbb{R},\\ \tau & \text{for } |\tau|<s,\\ s\operatorname{sign}\tau & \text{for } |\tau|\ge s>0,\end{cases}$$

$$\gamma(t,x)=\begin{cases}1 & \text{for } \|x-z(t)\|\le\varphi(t),\\ \frac{\varphi(t)}{\|x-z(t)\|} & \text{for } \|x-z(t)\|>\varphi(t),\end{cases}$$

$$\sigma_1(t,x,y)=\sigma(\|x-z(t)\|-\varphi(t),\varphi'(t)\|x-z(t)\|-(x-z(t))\cdot(y-z'(t))),$$

$$\widetilde{y}(t,x,y)=\begin{cases} y & \text{for } \|x-z(t)\|\le\varphi(t),\\ y+\frac{\sigma_1(t,x,y)}{\|x-z(t)\|^2}(x-z(t)) & \text{for } \|x-z(t)\|>\varphi(t),\end{cases}$$

$$g_1(t,x,y)=x-z(t)+\gamma(t,x)[f(t,z(t)+\gamma(t,x)(x-z(t)),\\ \widetilde{y}(t,x,y))-x+z(t)]-(\gamma(t,x)-1)z''(t),$$

$$\sigma_2(t,x,y)=\sigma\Big[\|x-z(t)\|-\varphi(t),\quad \|\widetilde{y}(t,x,y)-z'(t)\|^2-\\ -\|y-z'(t)\|^2-(\varphi'(t))^2+\Big(\frac{(x-z(t))\cdot(y-z'(t))}{\|x-z(t)\|}\Big)^2\Big],$$

$$g_2(t,x,y)=\begin{cases} 0 \quad \text{for } \|x-z(t)\|\le\varphi(t),\\ \frac{\sigma_2(t,x,y)+(\|x-z(t)\|-\varphi(t))(\varphi''(t)+1)}{\|x-z(t)\|^2}(x-z(t)) \\ \qquad \text{for } \|x-z(t)\|>\varphi(t)\end{cases}$$

and

$$\widetilde{g}(t,x,y)=g_1(t,x,y)+g_2(t,x,y).$$

By the condition (2.2) on $[a,b]\times\mathbb{R}^{2n}$ we have

$$\|\widetilde{g}(t,x,y)-x\|\le h^*(t)+|\varphi''(t)|+\|z''(t)\|+2+\|z(t)\|+\varphi(t)$$

where

$$h^*(t)=\sup\{h(t,x):\|x-z(t)\|\le\varphi(t)\}.$$

Therefore according to Lemma 2.1 the differential system

$$x''=\widetilde{g}(t,x,x')$$

has at least one solution $x\in\widetilde{C}^1([a,b];\mathbb{R}^n)$ satisfying the boundary conditions (1.2). Due to the definition of $\widetilde{g}$ it remains to show that x admits the estimate (1.14). Assume, on the contrary, that (1.14) is violated. Then there exists $t_0\in[a,b]$ where the function

$$u(t)=\|x(t)-z(t)\|-\varphi(t)$$

reaches its positive maximum on $[a,b]$. Assume first that $t_0\in]a,b[$. Then by the Fermat theorem we have

$$u'(t_0)=\frac{(x(t_0)-z(t_0))\cdot(x'(t_0)-z'(t_0))}{\|x(t_0)-z(t_0)\|}-\varphi'(t_0)=0.$$

Hence

$$\lim_{t\to t_0}\widetilde{y}(t,x(t),x'(t))=x'(t_0)$$

and for a certain positive δ the inequalities

$$|\varphi'(t)\|x(t)-z(t)\|-(x(t)-z(t))\cdot(x'(t)-z'(t))|<\\ <\|x(t)-z(t)\|-\varphi(t) \tag{2.3}$$

and

$$|\|\widetilde{y}(t,x(t),x'(t))-z'(t)\|^2-\|x'(t)-z'(t)\|^2-(\varphi'(t))^2+ \\ +\left(\frac{(x(t)-z(t))\cdot(x'(t)-z'(t))}{\|x(t)-z(t)\|}\right)^2|<\|x(t)-z(t)\|$$

hold in $U_\delta(t_0)$. Therefore due to the definition of $\widetilde{g}$ we obtain

$$u''(t)=\frac{(x(t)-z(t))\cdot(x''(t)-z''(t))+\|x'(t)-z'(t)\|^2}{\|x(t)-z(t)\|}- \\ -\frac{\left[\frac{(x(t)-z(t))\cdot(x'(t)-z'(t))}{\|x(t)-z(t)\|}\right]^2}{\|x(t)-z(t)\|}-\varphi''(t)= \\ =\frac{(x(t)-z(t))\cdot(g_1(t,x(t),x'(t))-z''(t))}{\|x(t)-z(t)\|}+ \\ +\frac{\|\widetilde{y}(t,x(t),x'(t))-z'(t)\|^2-\varphi(t)\varphi''(t)-(\varphi'(t))^2}{\|x(t)-z(t)\|}+ \\ +\frac{\|x(t)-z(t)\|-\varphi(t)}{\|x(t)-z(t)\|}\quad\text{for } t\in U_\delta(t_0).$$

Furthermore, by (2.3) for each $t\in U_\delta(t_0)$ we have

$$\|\gamma(t,x(t))(x(t)-z(t))\|=\varphi(t)$$

and

$$\gamma(t,x(t))(x(t)-z(t))\cdot(\widetilde{y}(t,x(t),x'(t))-z'(t))=\varphi(t)\varphi'(t).$$

Taking into account the last three equalities and Definition 1.1, it can be shown that $u''(t)$ is positive for each $t\in U_\delta(t_0)$. But this is impossible, since $t_0\in]a,b[$ and t_0 is a point of maximum for u. Thus $t_0\notin]a,b[$. In view of (1.2) and (1.5_1) both a and b are the points of maxima for the function u. Therefore $u'(a)\le 0$ and $u'(b)\ge 0$. Assuming $u'(a)=0$ or $u'(b)=0$, an argument similar to the one carried out above leads us to a contradiction. Thus

$$u'(a)=\frac{(x(a)-z(a))\cdot(x'(a)-z'(a))}{\|x(a)-z(a)\|}-\varphi'(a)<0$$

and

$$u'(b)=\frac{(x(b)-z(b))\cdot(x'(b)-z'(b))}{\|x(b)-z(b)\|}-\varphi'(b)>0.$$

But since $x'(a)=x'(b)$ and

$$\frac{x(a)-z(a)}{\|x(a)-z(a)\|}=\frac{x(b)-z(b)}{\|x(b)-z(b)\|},$$

the last two inequalities yield

$$\frac{x(a)-z(a)}{\|x(a)-z(a)\|}\cdot(z'(a)-z'(b))>\varphi'(b)-\varphi'(a),$$

which contradicts (1.5_2). Therefore the estimate (1.14) is proved. □

Lemma 2.3. *Suppose that* ρ, r_1, *and* ρ' *are positive constants,* $\omega \in C(\mathbb{R}_+;]0,+\infty[)$, $l \in L([t_1,t_2];\mathbb{R}_+)$ *and*

$$\int\limits_{\rho}^{\rho'}\frac{ds}{\omega(s)}>r_1+\int\limits_{t_1}^{t_2}l(t)dt. \tag{2.4}$$

Then an arbitrary $x \in \widetilde{C}^1([t_1,t_2];\mathbb{R}^n)$, *satisfying* (1.9) *and the inequalities*

$$\|x'(t_i)\|\le\rho \tag{2.5_i}$$

and

$$(-1)^{i-1}x''(t)\cdot\eta_\rho(x'(t))\le\omega(\|x'(t)\|)(l(t)+\|x'(t)\|)\quad \text{for}\ \ t_1\le t\le t_2 \tag{2.6_i}$$

with $i \in \{1,2\}$, *admits the estimate*

$$\|x'(t)\|\le\rho'\quad \text{for}\ \ t_1\le t\le t_2. \tag{2.7}$$

Proof. Assume for definiteness that $i = 1$. Admit to the contrary that (2.7) is violated, i.e., there exists $t^* \in]t_1,t_2]$ such that

$$\|x'(t^*)\|>\rho'. \tag{2.8}$$

By (2.5_1) there exists $t_* \in [t_1,t^*[$ such that

$$\|x'(t)\|=\rho\ \text{ and }\ \|x'(t)\|>\rho'\ \text{ for }\ t_*\le t\le t^*.$$

Hence, taking into account the definition of the function η_r, from (2.6_1) we get

$$\|x'(t)\|'\le\omega(\|x'(t)\|)(l(t)+\|x'(t)\|)\ \text{ for }\ t_*\le t\le t^*.$$

Dividing this inequality by $\omega(\|x'(t)\|)$, integrating from t_* to t^*, and using (1.9) and (2.5_1), we obtain

$$\int\limits_{\rho}^{\|x'(t^*)\|}\frac{ds}{\omega(s)}\le r_1+\int\limits_{t_1}^{t_2}l(t)dt,$$

which, on account of (2.8), contradicts (2.4). □

Definition 2.1. Suppose that r and r_1 are positive constants. A vector-function $x:[\alpha,\beta]\to\mathbb{R}^n$ is said to belong to the set $W_n([\alpha,\beta],r,r_1)$ if (1.8) implies the estimate (1.9) for arbitrary $t_1\in[\alpha,\beta]$ and $t_2\in]t_1,\beta]$.

Lemma 2.4$_1$. *Suppose that r and r_1 are positive constants, $l\in L([t_1,t_2];\mathbb{R}_+)$, $\omega\in C(\mathbb{R}_+;]0,+\infty[)$, and (1.13) holds. Then there exists a positive constant r' such that if $\delta\in]0,\frac{b-a}{4}[$ and an arbitrary $x\in\widetilde{C}^1([a,b];\mathbb{R}^n)$ $\cap W_n([a+\delta,b-\delta],r,r_1)$ satisfies the boundary conditions (1.2), the inequality*

$$\|x''(t)\|\le l(t)\quad for\ \ t\in]a,a+\delta[\cup]b-\delta,b[, \tag{2.9}$$

and furthermore the inequality

$$x''(t)\cdot\eta_r(x'(t))\le\omega(\|x'(t)\|)(l(t)+\|x'(t)\|) \tag{2.10}$$

on the set $[a+\delta,b-\delta]$, then x admits an estimate

$$\|x'(t)\|\le r'\quad for\ \ a\le t\le b. \tag{2.11}$$

Proof. Due to Definition 2.1, without loss of generality we may assume that $r(b-a)>2r_1$. In view of (1.13) there exist positive numbers r^* and r' such that

$$\int_r^{r^*}\frac{ds}{\omega(s)}>r_1+\int_a^b l(t)dt \tag{2.12}$$

and

$$\int_\mu^{r'}\frac{ds}{\omega(s)}>r_1+\int_a^b l(t)dt \tag{2.13}$$

where

$$\mu=r^*+\int_a^b l(t)dt.$$

Let us show that r' is a suitable constant.

First note that for a certain $t_0\in\left[\frac{3a+b}{4},\frac{a+3b}{4}\right]$ we have

$$\|x'(t_0)\|\le r. \tag{2.14}$$

Indeed, assuming the contrary implies that

$$\|x'(t)\|>r\quad\text{for}\ \ \frac{3a+b}{4}\le t\le\frac{a+3b}{4}.$$

Therefore, taking into account the inequality $r(b-a) \geq 2r_1$, we obtain

$$\int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} \|x'(t)\| dt > r_1.$$

But this is impossible, since $x \in W_n([a+\delta, b-\delta], r, r_1)$. Thus (2.14) is proved.

Now let us show that

$$\|x'(t)\| \leq r^* \quad \text{for } \ t_0 \leq t \leq b-\delta. \tag{2.15}$$

Suppose to the contrary that for arbitrary $t_2 \in]t_0, b-\delta]$ we have

$$\|x'(t_2)\| > r^*. \tag{2.16}$$

Then by (2.14) there exists $t_1 \in [t_0, t_2[$ such that

$$\|x'(t_1)\| = r \ \text{ and } \ \|x'(t)\| > r \ \text{ for } \ t_1 < t \leq t_2.$$

Hence, taking into account $x \in W_n([a+\delta, b-\delta], r, r_1)$, we get the estimate (1.9). Assuming $i = 1, \rho = r$, and $\rho' = r^*$, it is easy to verify that x satisfies the other conditions of Lemma 2.4_1 too. Therefore x admits the estimate (2.7), which contradicts (2.16). Thus (2.15) is proved. In view of (1.2) and (2.9) it implies that

$$\|x'(t)\| \leq \mu \quad \text{for } \ t \in [a, a+\delta] \cup [b-\delta, b]. \tag{2.17}$$

In particular, $\|x'(a+\delta)\| \leq \mu$. Applying Lemma 2.3 where $\rho = \mu$, $\rho' = r'$ and $i = 1$, an argument similar to the one carried out above yields the estimate

$$\|x'(t)\| \leq r' \quad \text{for } \ a+\delta \leq t \leq t_0. \tag{2.18}$$

Finally, from (2.15), (2.17), and (2.18) we obtain the estimate (2.11). □

In a similar manner we can prove

Lemma 2.4_2. *Suppose that the conditions of Lemma* 2.4_1 *are satisfied, except that the inequality* (2.10) *is replaced by*

$$x''(t) \cdot \eta_r(x'(t)) \geq -\omega(\|x'(t)\|)(l(t) + \|x'(t)\|). \tag{2.19}$$

Then x admits the estimate (2.11).

Lemma 2.5. *Suppose that r and r_1 are positive constants, $a \le a_0 < b_0 \le b$, $l \in L([t_1,t_2];\mathbb{R}_+)$, $\omega \in C(\mathbb{R}_+;]0,+\infty[)$, and* (1.13) *holds. Then there exists a positive constant r' such that if $0<\delta<\min\{a_0-a,b-b_0\}$ and an arbitrary $x \in \widetilde{C}^1([a,b];\mathbb{R}^n)\cap W_n([a+\delta,b-\delta],r,r_1)$ satisfies the conditions* (2.9)*, the inequality* (2.10) *on the set $]a_0,b-\delta[$, and, furthermore, the inequality* (2.19) *on the set $]a+\delta,b_0[$, then x admits the estimate* (2.11)*.*

The *proof of Lemma* 2.5 is similar to the one carried out above for Lemma 2.4_1, so we shall note only the main points.

Due to Definition 2.1, without loss of generality we may assume that $r(b_0-a_0)\ge r_1$. In view of (1.13) there exists a positive number r^* such that (2.12) holds. The constant

$$r' = r^* + \int_a^b l(t)dt$$

is just the suitable one.

First, taking into account the conditions $x \in W_n([a+\delta,b-\delta],r,r_1)$, we conclude that for a certain $t_0\in[a_0,b_0]$ (2.14) is satisfied. Further, applying Lemma 2.3 ($i=1$) and the inequality (2.10), we get the estimate (2.15). Finally, from (2.9) and (2.15) it follows that we have $\|x'(t)\|\le r'$ on the set $[b-\delta,b]$. Thus the last estimate holds on $[t_0,b]$. Analogously, applying Lemma 2.3 ($i=2$) and the inequality (2.19), it can be proved on $[a,t_0]$.

In a similar manner we can prove

Lemma 2.6. *Suppose that r and r_1 are positive constants, $a<a_1<a_2<t_0<b_2<b_1<b$, $l\in L([t_1,t_2];\mathbb{R}_+)$, $\omega\in C(\mathbb{R}_+;]0,+\infty[)$, and* (1.13) *holds. Then there exists a positive constant r' such that if $0<\delta<\min\{a_1-a,b-b_1\}$ and an arbitrary $x\in\widetilde{C}^1([a,b];\mathbb{R}^n)\cap W_n([a+\delta,b-\delta],r,r_1)$ satisfies the conditions* (2.9)*, the inequality* (2.10) *on the set $]a_1,t_0[\cup]b_2,b[$, and furthermore the inequality* (2.19) *on the set $]a,a_2[\cup]t_0,b_1[$, then x admits the estimate* (2.11)*.*

§ 3. Proof of the Main Results

Proof of Theorem 1.1_1. Without loss of generality we can assume that

$$l(t)\ge\|z''(t)\|+|\varphi''(t)| \quad \text{for} \quad a<t<b.$$

Put

$$a_k = a+\frac{b-a}{4k}, \quad b_k = b-\frac{b-a}{4k}$$

and

$$\chi_\rho(s) = \begin{cases} 0 & \text{for } s \le \rho, \\ \frac{2\rho - s}{\rho} & \text{for } \rho < s < 2\rho, \\ 0 & \text{for } 2\rho \le s. \end{cases}$$

By Definition 1.1 there exists a sequence $(\rho_k)_{k=1}^{+\infty}$ such that $\lim\limits_{k\to+\infty} \rho_k = +\infty$ and for each $k \in \{1, 2, \dots\}$

$$(x - z(t)) \cdot (\chi_{\rho_k}(\|y\|) f(t, x, y) - z''(t)) + \|y - z'(t)\|^2 \ge \varphi(t)\varphi''(t)$$

for $a_k < t < b_k$, $\|x - z(t)\| = \varphi(t)$ and $(x - z(t)) \cdot (y - z'(t)) = \varphi(t)\varphi'(t)$.

Put

$$h(t, x, y) = \begin{cases} z''(t) + \frac{|\varphi''(t)|}{\varphi(t)}(x - z(t)) & \text{for } \|x - z(t)\| \le \varphi(t), \\ z''(t) + \frac{|\varphi''(t)|}{\varphi(t)}(x - z(t)) & \text{for } \|x - z(t)\| > \varphi(t) \end{cases}$$

and

$$f_k(t, x, y) = \begin{cases} h(t, x, y) & \text{for } t \notin [a_k, b_k], \\ \chi_{\rho_k}(\|y\|) f(t, x, y) & \text{for } t \in [a_k, b_k] \end{cases}$$

$(k = 1, 2. \dots)$. It is easy to verify that for each $k \in \{1, 2, \dots\}$ the vector-function $g(t, x, y) = f_k(t, x, y)$ satisfies all the conditions of Lemma 2.2. Therefore the differential system

$$x'' = f_k(t, x, x') \tag{3.1_k}$$

has at least one solution $x_k \in \widetilde{C}^1([a, b]; \mathbb{R}^n)$ satisfying the boundary conditions (1.2) and the estimate

$$\|x_k(t) - z(t)\| \le \varphi(t) \quad \text{for } a \le t \le b \ (k = 1, 2, \dots). \tag{3.2_k}$$

Choose the positive constants r and r_1 according to Definition 1.3 and the constant r' according to Lemma 2.4_1, assuming without loss of generality that $r \ge \rho$. Then by Lemma 2.4_1 we obtain

$$\|x'_k(t)\| \le r' \quad \text{for } a \le t \le b \ (k = 1, 2, \dots). \tag{3.3_k}$$

In view of (3.2_k) and (3.3_k) the sequences $(x_k)_{k=1}^{+\infty}$ and $(x'_k)_{k=1}^{+\infty}$ are uniformly bounded and equicontinuous on $[a, b]$. So due to the well-known Arzela–Ascoli theorem there exists a sequence $(k_j)_{j=1}^{+\infty}$ such that $(x_{k_j})_{j=1}^{+\infty}$ and $(x'_{k_j})_{j=1}^{+\infty}$ uniformly converge on $[a, b]$. Put

$$x(t) = \lim_{j\to\infty} x_{k_j}(t) \quad \text{for } a \le t \le b.$$

Due to the definition of the functions $f_k (k = 1, 2 \dots)$ x belongs to the set $\widetilde{C}^1([a, b]; \mathbb{R}^n)$ and is a solution of (1.1),(1.2). □

The *proof of Theorems* 1.1_2, 1.2, and 1.3 is similar to the one carried out above for Theorem 1.1_1. The only difference is that one has to apply Lemmas 2.4_2,2.5, and 2.6 respectively instead of Lemma 2.4_1.

Remark 3.1. Theorem 1.1_1 (Theorem 1.1_2) can be strengthened by a slight complication of Lemma 2.4_1 (Lemma 2.4_2). Namely, we can assume that the vector-function f has the property $V([\alpha,\beta],\varphi,z)$ for each segment $[\alpha,\beta]$ contained in the interval $]a,b]$ (in the interval $[a,b[$). In that case the vector-function f may fail to have the property $V([a,b],\varphi,z)$.

Proof of Theorem 1.4. Assume to the contrary that $x_i \in \widetilde{C}^1([a,b];\mathbb{R}^n)$ $(i = 1,2)$ are solutions of the boundary-value problem (1.1),(1.2) and $x_1(t) \not\equiv x_2(t)$. Put

$$x(t) = x_1(t) - x_2(t) \quad \text{for} \;\; a \le t \le b,$$
$$u(t) = \|x(t)\|$$

and

$$r = \max\{\sum_{i=1}^{2} \|x_i(t)\| + \|x_i'(t)\| : a \le t \le b\}.$$

Choose the functions $l_i(t,r)(i = 1,2)$ according to the condition of Theorem 1.4. First prove that $u'(t) \not\equiv 0$. Indeed, assuming the contrary we have

$$0 = u''(t) = \frac{x''(t)\cdot x(t) + \|x'(t)\|^2}{\|x(t)\|} - \frac{(x(t)\cdot x'(t))^2}{\|x(t)\|^3} \ge$$
$$\ge \frac{x''(t)\cdot x(t)}{\|x(t)\|} \quad \text{for} \;\; a < t < b.$$

Hence by (1.16) we obtain

$$l_1(t,r) \le 0 \quad \text{for} \;\; a < t < b.$$

But this is impossible, since $l_1(t,r)$ is nonnegative and differs from zero on a subset of positive measure of the interval $]a,b[$. Thus $u'(t) \not\equiv 0$. Therefore, there exists $t_0 \in]a,b[$ such that either

$$u(t_0) > 0, \;\; u'(t_0) > 0 \tag{3.4}$$

or

$$u(t_0) > 0, \;\; u'(t_0) < 0. \tag{3.5}$$

Without loss of generality assume that (3.4) holds. Then on $[t_0,b]$

$$u(t) > 0, \;\; u'(t) > 0. \tag{3.6}$$

Indeed, if this is not so, then there exists $t_1 \in]t_0, b]$ such that $u'(t_1) = 0$ and (3.6) holds on $[t_0, t_1[$. Applying (1.16) once more we obtain

$$u''(t) \geq \frac{x''(t) \cdot x(t)}{\|x(t)\|} \geq l_1(t,r)u(t) - l_2(t,r)|u'(t)| \geq$$
$$\geq -l_2(t,r)|u'(t)| \quad \text{for } \; t_0 < t < t_1.$$

According to the Gronwall–Bellman lemma (see e.g. [6]) the last inequality yields

$$u'(t_1) \geq u'(t_0) \exp\left[-\int_{t_0}^{t_1} l_2(t,r)dt\right] > 0.$$

The obtained contradiction shows that (3.6) holds on $[t_0, b]$. Hence, taking into account the equalities

$$u(a) = u(b), \quad u'(a) = u'(b) \tag{3.7}$$

which follow from the boundary conditions (1.2),we get

$$u(a) > 0, \quad u'(a) > 0.$$

Repeating the argument that was carried out above we can show the validity of (3.6) on $[a, b]$, but this contradicts (3.7). $\square$

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