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CONTINUOUS TRANSFORMATIONS OF DIFFERENTIAL EQUATIONS WITH DELAYS

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Abstract. The aim of this paper is to find the class of continuous pointwise transformations (as general as possible) in the framework of which Kummer's transformation $z(t) = g(t)y(h(t))$ represents the most general pointwise transformation converting every linear homogeneous differential equation of the nth order into an equation of the same type. Further, some forms of these equations having certain subspaces of solutions aer cobstructed.

Let $I = [a, b)$, $J = [c, d)$ be intervals, where b, d may be infinite. Further, let

$$t = f_1(x, y), \qquad z = f_2(x, y)$$

where $(x, y) \in I \times \mathbb{R}$ and denote by $F = (f_1, f_2)$ a pointwise transformation of $I \times \mathbb{R}$ into $U \subset \mathbb{R}^2$.

In this article we shall study transformations F of a linear homogeneous differential equation of the nth order with m delays

$$y^{(n)}(x) + \sum_{i=0}^{n-1} p_i(x) y^{(i)}(x) + \sum_{i=0}^{n-1} \sum_{j=1}^{m} q_{ij}(x) y^{(i)}(\tau_j(x)) = 0 \tag{1}$$

on $[x_0, b)$, where the initial set $E_{x_0} = [a, x_0]$, p_i, q_{ij}, $\tau_j \in C^0([x_0, b))$, $\tau_j(x) < x$ on $[x_0, b)$, and $q_{ij} \not\equiv 0$ on $[x_0, b)$ for a pair (i, j) $(i = 0, \dots, n, j = 1, \dots, m)$. We wish to derive the form of such a transformation F which converts (in the sense of a pointwise transformation of solutions) every equation (1) into an equation of the same type, i.e.,

$$z^{(n)}(t) + \sum_{i=0}^{n-1} r_i(t) z^{(i)}(t) + \sum_{i=0}^{n-1} \sum_{j=1}^{m} s_{ij}(t) z^{(i)}(\mu_j(t)) = 0 \tag{2}$$

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on $[t_0, d)$, where $E_{t_0} = [c, t_0]$, r_i, s_{ij}, $\mu_j \in C^0([t_0, d))$, $\mu_j(t) < t$ on $[t_0, d)$, and $s_{ij} \not\equiv 0$ on $[t_0, d)$ for a pair (i, j) $(i = 0, \dots, n, j = 1, \dots, m)$. Notice that solutions $y(x)$ (resp. $z(t)$) of equation (1) (resp.(2)) are functions defined on the whole I (resp.J) and the space of solutions of both equations has an infinite dimension.

Next, let $W(y_1, \dots, y_k)(x)$, where $y_1, \dots, y_k \in C^{k-1}(I)$, be the Wronski determinant of functions $y_1, \dots, y_k$ at $x \in I$.

We denote the following hypotheses concerning F:
$(\mathbf{H_1})$ F is a C^n-diffeomorphism of $I \times \mathbb{R}$ onto $J \times \mathbb{R}$;
$(\mathbf{H_1'})$ F is a homeomorphism of $I \times \mathbb{R}$ onto $J \times \mathbb{R}$;
$(\mathbf{H_2})$ for every equation (1) there exists an equation (2) such that F converts, pointwise,

(i) every nontrivial solution $y(x)$ of (1) into a nontrivial solution $z(t)$ of (2);

(i') every nontrivial solution $y(x)$ of (1) into a C^n function $z(t)$ defined on J;

(ii) every k-tuple $y_1, \dots, y_k$ of solutions of (1) satisfying $W(y_1, \dots, y_k)(x) \neq 0$ on I into a k-tuple $z_1, \dots, z_k$ of solutions of (2) satisfying $W(z_1, \dots, z_k)(t) \neq 0$ on J, where $k \in \{2, \dots, n+1\}$ is a suitable number;

(iii) every function $y \circ \tau_j$, where y is a solution of (1), into a function $z \circ \mu_j$, where z is a solution of (2) $(j = 1, \dots, m)$.

Assuming $q_{ij} \equiv 0$ on $[x_0, b)$ for each pair (i, j) in (1) we obtain a differential equation without any delay. The problem of the most general pointwise transformation converting any such equation (1) into an equation (2) (with $s_{ij} \equiv 0$ on $[t_0, d)$) was first solved by P.Stäckel in [1]. He proved that under hypotheses $(\mathbf{H_1})$ and $(\mathbf{H_2})(i)$ F has the form

$$t = f(x),\ z = g(t)y \quad \text{for } n \geq 2 \quad t = f(x),\ z = g(t)y^{\lambda},\ \lambda > 0 \quad \text{for } n = 1$$

where f is a C^n-diffeomorphism of I onto J, $g \in C^n(J)$, $g(t) \neq 0$ on J. Recently M.Čadek has shown (see [2]) that the assumption of differentiability of F is not necessary and the form of F remains preserved also under $(\mathbf{H_1'})$, $(\mathbf{H_2})(i')$, and $(\mathbf{H_2})(ii)$, where $k = n$ (for more details see [3]).

Provided $q_{ij} \not\equiv 0$ on $[x_0, b)$ for a pair (i, j) V.Tryhuk proved in [4] that assuming $(\mathbf{H_1}), (\mathbf{H_2})(i)$, and $(\mathbf{H_2})(iii)$, F has the form $t = f(x)$, $z = g(t)y$ with g and f having the same properties as above and, moreover, $f'(x) > 0$ on I and $f \circ \tau_j = \mu_j \circ f$ on I for $j = 1, \dots, m$. The aim of this paper is to weaken the assumption of differentiability of F as M.Čadek did for equations without delays.

Proposition 1. *Let hypotheses $(\mathbf{H_1'})$ and $(\mathbf{H_2})(i')$ be fulfilled. Then f_1 is a homeomorphism between I and J not depending on y.*

Proof. Suppose on the contrary that there exist $t_0^* \in J$, z_1, $z_2 \in \mathbb{R}$ such that

$$(x_1, y_1) = F^{-1}(t_0^*, z_1), \qquad (x_2, y_2) = F^{-1}(t_0^*, z_2),$$

where $x_1 \neq x_2$. Choose $x_0 \in I$, $x_0 > \max(x_1, x_2)$ and let $y \in C^n(I)$ be a function such that $y(x_i) = y_i$ $(i = 1, 2)$ and $y(x) \neq 0$ on $[x_0, b)$. Now we can define $\tau\, [x_0, b) \xrightarrow{onto} I$ as an arbitrary continuously increasing function satisfying $\tau(x) < x$ on $[x_0, b)$ and denote $V := \tau^{-1}([a, x_0])$. Then put

$$p(x) = -\frac{1}{y(x)}(q(x)y(\tau(x)) + y^{(n)}(x)) \qquad \text{on } V$$

where q is an arbitrary continuous function on V and

$$q(x) = -\frac{1}{y(\tau(x))}(p(x)y(x) + y^{(n)}(x)) \qquad \text{on } [x_0, b) - V$$

where p is an arbitrary continuous function on $[x_0, b) - V$ satisfying $\lim\limits_{x \to \tau^{-1}(x_0)} p(x) = p(\tau^{-1}(x_0))$.

Thus p, q are well-defined continuous functions on $[x_0, b)$ and y is a solution of the equation

$$y^{(n)}(x) + p(x)y(x) + q(x)y(\tau(x)) = 0 \qquad \text{on}[x_0, b).$$

This contradicts the hypothesis $(\mathbf{H_2})(i')$ because the image of the function $y(x)$ is not a function. $\square$

Notation. Put $h := f_1^{-1}$, $g(t, y(h(t)) := f_2(h(t), y(h(t))$. Then a transformation F converting a solution $y(x)$ into a solution $z(t)$ can be rewritten as

$$z(t) = g(t, y(h(t)). \tag{3}$$

Proposition 2. *Let transformation* (3) *satisfy* $(\mathbf{H_2})(iii)$. *Then*
(*a*) $\mu_j(t) = h^{-1}(\tau_j(h(t))$ *on* $[t_0, d)$ $(j = 1, \dots, m)$;
(*b*) *if* $\lim\limits_{x \to b^-} \tau_j(x) = b$, *then* $\lim\limits_{t \to d^-} \mu_j(t) = d$ $(j = 1, \dots, m)$;
(*c*) h *is an increasing homeomorphism between* J *and* I.

Proof. The pointwise transformation (3) converts $y(\tau_j(x))$ into $z(\mu_j(t))$, i.e.,

$$g(\mu_j(t), y(h(\mu_j(t))) = z(\mu_j(t)) = g(h^{-1}(\tau_j(h(t)), \tau_j(h(t))),$$

which implies (a).

Further, if $\lim\limits_{x \to b^-} \tau_j(x) = b$, then $\lim\limits_{t \to d^-} h^{-1}(\tau_j(h(t)) = \lim\limits_{x \to b^-} h^{-1}(x) = d$. Finally, $\mu_j(t) < t$ and $h(\mu_j(t)) = \tau_j(h(t)) < h(t)$ holds for all $t \in J$, hence h, being a homeomorphism, is an increasing one. $\square$

Lemma. *Consider* $y_1, \dots, y_k \in C^n(I)$ *for some* $k \in \{1, \dots, n+1\}$ *such that* $W(y_1, \dots, y_k)(x) \neq 0$ *on* I*. Then there exists an equation* (1) *having* $y_1, \dots, y_k$ *as a* k*-tuple of linearly independent solutions.*

Proof. First let $1 \le k \le n$. Suppose that there exists $i \in \{1, \dots, k\}$ and $x_0 \in I$ such that $y_i^{(n)}(x_0) \neq 0$. Define $\tau : [x_0, b) \xrightarrow{onto} I$ as an arbitrary continuous function satisfying $\tau(x) < x$ on $[x_0, b)$ and consider the following linear system for unknown functions $q_j(x)$ in the form

$$\sum_{j=0}^{k-1} y_l^{(j)}(\tau(x)) q_j(x) = -y_l^{(n)}(x),$$

where $1 \le l \le k$, $x \in [x_0, b)$. The matrix of this system is regular for every $x \in [x_0, b)$ because $W(y_1, \dots, y_k)(x) \neq 0$ on I. Hence, the system has a unique solution $q_0(x), \dots, q_{k-1}(x)$ continuous on $[x_0, b)$ and, moreover, the relation $y_i^{(n)} \not\equiv 0$ implies $q_j \not\equiv 0$ on I for some $j \in \{0, 1, \dots, k-1\}$. Thus we get that $y_1, \dots, y_k$ are solutions of an equation with a delay τ

$$y^{(n)}(x) + \sum_{j=0}^{k-1} q_j(x) y^{(j)}(\tau(x)) = 0 \qquad \text{on } [x_0, b).$$

If $y_i^{(n)} \equiv 0$ on I for every $1 \le i \le k$, then the k-tuple $y_1, \dots, y_k$ obviously satisfies the equation

$$y^{(n)}(x) + y^{(n-1)}(x) - y^{(n-1)}(\tau(x)) = 0 \qquad \text{on } [x_0, b)$$

for every $x_0 \in I$ and any suitable delay τ.

Now consider the case $k = n+1$. Introduce the system of functions $\{f_x, x \in I\}$ by the relation

$$f_x(u) := c_1(x) y_1(u) + \ldots + c_{n+1}(x) y_{n+1}(u), \quad u \in I,$$

where

$$c_i(x) = (-1)^{n+i-1} \det \begin{pmatrix} y_1(x) & y_1'(x) & \dots & y_1^{(n-1)}(x) \\ \vdots & \vdots & \vdots & \vdots \\ y_{i-1}(x) & y_{i-1}'(x) & \dots & y_{i-1}^{(n-1)}(x) \\ y_{i+1}(x) & y_{i+1}'(x) & \dots & y_{i+1}^{(n-1)}(x) \\ \vdots & \vdots & \vdots & \vdots \\ y_{n+1}(x) & y_{n+1}'(x) & \dots & y_{n+1}^{(n-1)}(x) \end{pmatrix}$$

$(i = 1, \dots, n+1)$. Note that for every $x \in I$ there exists $i \in \{1, \dots, k\}$ such that $c_i(x) \neq 0$ and denote $V_x := \{u \in I, f_x(u) = 0\}$ for $x \in I$. We show that V_x has no accumulation point for any $x \in I$.

Let $u_m, u^* \in V_x$ for some $x \in I$ such that $u_m \to u^*$. Then $f_x(u_m) = f_x(u^*) = 0$ and with respect to Rolle's theorem we get

$$f_x(u^*) = \frac{\mathrm{d}f_x}{\mathrm{d}u}(u^*) = \ldots = \frac{\mathrm{d}^n f_x}{\mathrm{d}u^n}(u^*) = 0$$

what contradicts the assumption $W(y_1, \ldots, y_{n+1})(u^*) \neq 0$.

Further, choose $x_0 \in I$, $x_0 > a$, such that $a \notin V_{x_0}$ and introduce a continuous function $\tau : [x_0, b) \to I$ fulfilling $\tau(x_0) = a$, $\tau(x) < x$ and $\tau(x) \notin V_x$ for every $x \in [x_0, b)$. This is possible because V_x consists of only a finite number of points in $[a, x]$. Now consider the system with unknown functions $p_i(x), q(x)$

$$\sum_{i=0}^{n-1} y_l^{(i)}(x) p_i(x) + y_l(\tau(x)) q(x) = -y_l^{(n)}(x),$$

where $1 \leq l \leq n+1$ and $x \in [x_0, b)$. Since

$$W_\tau(y_1, \ldots, y_{n+1})(x) :=$$

$$= \det \begin{pmatrix} y_1(x) & \ldots & y_1^{(n-1)}(x) & y_1(\tau(x)) \\ \vdots & \vdots & \vdots & \vdots \\ y_{n+1}(x) & \ldots & y_{n+1}^{(n-1)}(x) & y_{n+1}(\tau(x)) \end{pmatrix} = f_x(\tau(x)) \neq 0$$

on $[x_0, b)$, the considered system has uniquely determined solutions $p_i(x)$, $q(x)$ continuous on $[x_0, b)$ and, moreover,

$$q(x) = -\frac{W(y_1, \ldots, y_{n+1})(x)}{W_\tau(y_1, \ldots, y_{n+1})(x)} \neq 0 \qquad \text{on } [x_0, b).$$

From here we get that functions $y_1, \ldots, y_{n+1}$ form linearly independent solutions of the equation

$$y^{(n)}(x) + \sum_{i=0}^{n-1} p_i(x) y^{(i)}(x) + q(x) y(\tau(x)) = 0 \quad \text{on } [x_0, b). \qquad \square \qquad (4)$$

Consequence. *Consider y_1, $y_2 \in C^1(I)$, $W(y_1, y_2)(x) \neq 0$ on I and let $V_x = \{u \in I,\, y_1(x)y_2(u) - y_2(x)y_1(u) = 0\}$ for every $x \in I$. Choose $x_0 \in I$, $x_0 > a$, and suppose that $\tau : [x_0, b) \xrightarrow{onto} I$ is a continuous function satisfying $\tau(x) < x$ on $[x_0, b)$. Then y_1, y_2 are solutions of the equation*

$$y'(x) + p(x)y(x) + q(x)y(\tau(x)) = 0 \qquad \text{on } [x_0, b) \qquad (5)$$

if and only if $\tau(x) \notin V_x$ for every $x \in [x_0, b)$. Moreover, τ with such a property always exists and the functions p, q are then uniquely determined *by the relations*

$$p(x) = \frac{y_1(\tau(x))y_2'(x) - y_2(\tau(x))y_1'(x)}{y_1(x)y_2(\tau(x)) - y_2(x)y_1(\tau(x))} \qquad on\ I,$$
$$q(x) = \frac{y_1'(x)y_2(x) - y_1(x)y_2'(x)}{y_1(x)y_2(\tau(x)) - y_2(x)y_1(\tau(x))} \qquad on\ I.$$

Proof. The system

$$y_l(x)p(x) + y_l(\tau(x))q(x) = -y_l'(x), \quad l = 1, 2, \quad \text{on } [x_0, b),$$

has (necessarily unique) continuous solution $p(x)$, $q(x)$ if and only if $y_1(x)y_2(\tau(x)) - y_2(x)y_1(\tau(x)) \neq 0$ for every $x \in [x_0, b)$, i.e., $\tau(x) \notin V_x$ for every $x \in [x_0, b)$. Since the form of equation (4) agrees with the form of (5) for $n = 1$, the existence of a suitable τ follows from the previous Lemma. □

Example. Consider the functions $y_1(x) = 1$, $y_2(x) = x$ with the nonzero Wronski determinant on an interval I. Then $V_x = \{x\}$ and according to the previous statement all the equations (5) having prescribed functions 1, x as solutions are of the form

$$y'(x) + \frac{1}{x - \tau(x)}y(x) - \frac{1}{x - \tau(x)}y(\tau(x)) = 0 \qquad \text{on } [x_0, b),$$

where τ is any continuous delay mapping $[x_0, b)$ onto I.

On the other hand, it is easy to see that some k-tuples of functions with a nonzero Wronski determinant on I are not solutions of any equation (5) for $k \geq 3$. For example, functions 1, x, x^2 have this property because the function $y(x) = x^2$ does not satisfy the previous equation. Of course, they can be solutions of an equation of the first order with more delays, e.g.,

$$y'(x) - \frac{\tau_1 + \tau_2}{\tau_1\tau_2}y(x) - \frac{\tau_2}{\tau_1(\tau_1 - \tau_2)}y(x - \tau_1) + \frac{\tau_1}{\tau_2(\tau_1 - \tau_2)}y(x - \tau_2) = 0,$$

where τ_1, $\tau_2 > 0$, $\tau_1 \neq \tau_2$.

Theorem. *Let the hypotheses* $(\mathbf{H_1'})$, $(\mathbf{H_2})(i'), (ii), (iii)$ *be fulfilled. Then transformation* (3) *has the form*

$$z(t) = g(t)y(h(t)), \tag{3'}$$

where $g \in C^n(J)$, $g(t) \neq 0$ *on* J, h *is a* C^n*-diffeomorphism of* J *onto* I, $h'(t) > 0$ *on* J *and* $\tau_j(h(t)) = h(\mu_j(t))$ *on* J $(j = 1, \ldots, m)$.

Proof. According to the main result in [4] this statement holds under $(\mathbf{H_1})$ and $(\mathbf{H_2})(i), (iii)$. Transformation (3′) obviously satisfies $(\mathbf{H_2})(ii)$ as well because of the formula

$$W(z_1, \ldots, z_k)(t) = (g(t))^k(h'(t))^{\frac{k(k-1)}{2}}W(y_1, \ldots, y_k)(h(t))$$

(see [5]). We show that by weakening the assumptions $(\mathbf{H}_1)$ and $(\mathbf{H}_2)(i)$ to $(\mathbf{H}_1')$ and $(\mathbf{H}_2)(i')$ the class of transformations does not become larger.

Let the hypothesis $(\mathbf{H}_2)(ii)$ hold for a fixed $k \in \{2, \dots, n+1\}$. Consider a linear differential equation of the kth order without a delay in the form

$$y^{(k)}(x) + a_{k-1}(x)y^{(k-1)}(x) + \dots + a_0(x) = 0 \qquad \text{on } I, \tag{1'}$$

where $a_i(x) \in C^{n-k}(I)$ (if $k = n+1$ then $a_i(x) \in C^0(I)$), and let $y_1, \dots, y_k \in C^n(I)$ be a k-tuple of linearly independent solutions of (1′). According to the previous lemma there exists an equation (1) such that $y_1, \dots, y_k$ are its solutions. Let transformation (3) convert equation (1) into an equation (2) and denote $z_i(t) = g(t, y_i(h(t)))$ $(i = 1, \dots, k)$. Then with respect to $(\mathbf{H}_2)(ii)$ we get that $z_1, \dots, z_k \in C^n(J)$ are solutions of (2) and $W(z_1, \dots, z_k)(t) \neq 0$ on J; hence there exists an equation

$$z^{(k)}(t) + b_{k-1}(t)z^{(k-1)}(t) + \dots + b_0(t) = 0 \qquad \text{on } J \tag{2'}$$

having $z_1, \dots, z_k$ as a fundamental system of solutions. Since equation (1′) can be arbitrarily chosen, Čadek's result [2] shows that such a transformation has the form (3′), where $g \in C^n(J)$, $g(t) \neq 0$ on J, and h is a C^n-diffeomorphism of J on I. The remaining properties of h follow from Proposition 2. □

Remark 1. Assuming that the hypothesis $(\mathbf{H}_2)(ii)$ is fulfilled for $k = 1$ we obtain transformation (3) in the form $z(t) = g(t)y^{\lambda}(h(t))$, $\lambda > 0$, which does not agree with the form we wish to have.

Remark 2. We cannot represent each equation (1) by the space of its solutions as is possible for linear homogeneous equations without delays. Consider, e.g., all functions defined by the relation

$$y(x) = \begin{cases} \varphi(x) & \text{for } x \in [0,1], \\ \varphi(1) & \text{for } x \in [1,2], \\ \varphi(1)(3 - x - \frac{\sin 2\pi x}{2\pi}) & \text{for } x \in [2,3], \\ 0 & \text{for } x \geq 3, \end{cases}$$

where φ is any continuous function defined on $[0,1]$ (cf. [6]). Then this set forms the space of solutions of the equations

$$y'(x) + q(x)y(\tau(x)) = 0 \qquad \text{on } [1, \infty),$$

where

$$q(x) = \begin{cases} 0 & \text{for } x \in (2n-1, 2n) \\ 2(\sin \pi x)^2 & \text{for } x \in [2n, 2n+1] \end{cases}$$

and τ may be any continuous delay mapping every interval $[n; n+1]$ onto $[n-1; n]$, $n = 1, 2, \dots$. That is the reason why we cannot omit the assumption

$(\mathbf{H}_2)(iii)$ if we wish to preserve the validity of the relation $\tau_j \circ h = h \circ \mu_j$ on J in the case of transformations of such an equation.

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