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THE SUPERSTABILITY OF THE GENERALIZED D'ALEMBERT FUNCTIONAL EQUATION

ELHOUCIEN ELQORACHI AND MOHAMED AKKOUCHI

Abstract. We generalize the well-known Baker's superstability result for the d'Alembert functional equation with values in the field of complex numbers to the case of the integral equation

$$\int_G f(xty)d\mu(t) + \int_G f(xt\sigma(y))d\mu(t) = 2f(x)f(y) \;\; x, y \in G,$$

where G is a locally compact group, μ is a generalized Gelfand measure and σ is a continuous involution of G.

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1. Introduction

Let G be a locally compact group. We denote by $M(G)$ the Banach algebra of bounded measures on G with complex values. It is the topological dual of $C_0(G)$, the Banach space of continuous functions vanishing at infinity (cf. 13.1.2 of [5]). σ is a continuous involution of G, i.e. ,$(\sigma \circ \sigma)(x) = x$ and $\sigma(xy) = \sigma(y)\sigma(x)$ for all $x, y \in G$.

Let $\mu \in M(G)$ be a compactly supported measure on G. We say that μ is σ-invariant if $\prec \mu, f \circ \sigma \succ = \prec \mu, f \succ$ for all continuous functions f on G, where $\prec \mu,\ f \succ = \int_G f(x)d\mu(x)$.

Throughout this paper we assume that μ is a generalized Gelfand measure on G with compact support. This means that the following conditions are satisfied

(i) $\mu * \mu = \mu$ and

(ii) $\mu * M(G) * \mu$ is a commutative Banach algebra under the convolution product (see [1] for more information).

In a previous work [6], complex continuous solutions of the generalized d'Alembert functional equation

$$\int_G f(xty)d\mu(t) + \int_G f(xt\sigma(y))d\mu(t) = 2f(x)f(y), \quad x, y \in G, \tag{1}$$

are determined.

There is an important particular case of the integral equation (1) : $\mu = \delta_e$ and $\sigma(x) = -x$, where δ_e denotes the Dirac measure concentrated at the identity element of G. In this setting G is an abelian group and (1) reduces to the classical d'Alembert functional equation

$$f(x + y) + f(x - y) = 2f(x)f(y), \quad x, y \in G. \tag{2}$$

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In the paper [2] the superstability theorem of the d'Alembert functional equation (2) appears. More precisely, J. A. Baker proved the following result in [2] (Theorem 5).

Let G be an abelian group and $\delta > 0$. Let f be a complex function such that

$$|f(x+y)+f(x-y)-2f(x)f(y)| \leq \delta, \quad x, y \in G.$$

Then either

$$|f(x)| \leq \frac{1+\sqrt{1+2\delta}}{2}, \quad x \in G,$$

or

$$f(x+y)+f(x-y)=2f(x)f(y), \quad x, y \in G.$$

The aim of this note is to extend the above Baker's stability theorem to the case of the generalized d'Alembert functional equation (1) in which μ is a generalized σ-invariant Gelfand measure with compact support.

2. The Main Results

Theorem. *Let $\delta > 0$ and let f be a continuous complex-valued function on G such that*

$$\left| \int_G f(xty)d\mu(t) + \int_G f(xt\sigma(y))d\mu(t) - 2f(x)f(y) \right| \leq \delta, \;\; x, y \in G. \tag{3}$$

Then either

$$|f(x)| \leq \frac{\parallel \mu \parallel + \sqrt{\parallel \mu \parallel^2 + 2\delta}}{2}, \quad x \in G,$$

or

$$\int_G f(xty)d\mu(t) + \int_G f(xt\sigma(y))d\mu(t) = 2f(x)f(y), \quad x, y \in G.$$

The following lemma will be useful in the proof of the main results.

Lemma. *If f is a continuous and bounded solution of the functional inequality* (3) *then*

$$\sup |f| \leq \frac{\parallel \mu \parallel + \sqrt{\parallel \mu \parallel^2 + 2\delta}}{2}.$$

Proof. Let $M = \sup |f|$. Using the inequality of the lemma we find that

$$2|f(x)f(x)| \leq M \parallel \mu \parallel + M \parallel \mu \parallel + \delta,$$

from which we conclude that $M = \sup |f|$ satisfies $M^2 \leq M \parallel \mu \parallel + \frac{\delta}{2}$.

The rest of the proof consists in finding the roots of the second order polynomial $x^2 - x \parallel \mu \parallel - \frac{\delta}{2}$. □

Proof of Theorem. If f is bounded, then according to the lemma we are in the first case of the theorem. So we may from now on assume that f is unbounded.

Step one. First we show that

$$\int_G f(xt)d\mu(t) = \int_G f(tx)d\mu(t) = f(x),$$

for all $x \in G$ in the manner as follows.

For any $x, y \in G$,

$$|2f(x)| \left| \int_G f(yt)d\mu(t) - f(y)| = |2f(x) \int_G f(yt)d\mu(t) - 2f(x)f(y) \right|$$
$$\leq \left| \int_G f(ytx)d\mu(t) + \int_G f(yt\sigma(x))d\mu(t) - 2f(x)f(y) \right|$$
$$+ \left| \int_G f(ytx)d\mu(t) + \int_G f(yt\sigma(x))d\mu(t) - 2f(x) \int_G f(yt)d\mu(t) \right|.$$

Since $\mu * \mu = \mu$, we get that

$$\left| \int_G f(ytx)d\mu(t) + \int_G f(yt\sigma(x))d\mu(t) - 2f(x) \int_G f(yt)d\mu(t) \right|$$
$$= \left| \int_G \int_G f(ytsx)d\mu(t)d\mu(s) + \int_G \int_G f(yts\sigma(x))d\mu(t)d\mu(s) - \right.$$
$$\left. -2f(x) \int_G f(yt)d\mu(t) \right|$$
$$\leq \int_G \left| \int_G f(ytsx)d\mu(s) + \int_G f(yts\sigma(x))d\mu(s) - 2f(x)f(yt) \right| d|\mu|(t) \leq \delta \parallel \mu \parallel .$$

It follows that

$$|2f(x)|| \int_G f(yt)d\mu(t) - f(y)| \leq \delta + \delta \parallel \mu \parallel .$$

Since f is unbounded, we have

$$\int_G f(yt)d\mu(t) = f(y), \quad y \in G.$$

On the other hand,

$$|2f(x)||f(y) - f(\sigma(y))| = |2f(x)f(y) - 2f(x)f(\sigma(y))|$$
$$\leq \left| \int_G f(xty)d\mu(t) + \int_G f(xt\sigma(y))d\mu(t) - 2f(x)f(y) \right|$$
$$+ \left| \int_G f(xt\sigma(y))d\mu(t) + \int_G f(xty)d\mu(t) - 2f(x)f(\sigma(y)) \right| \leq 2\delta.$$

Since f is unbounded, we have $f(\sigma(y)) = f(y)$, for all $y \in G$.

By using the above results we will prove that

$$\int_G f(ty)d\mu(t) = f(y), \quad y \in G.$$

Since μ is σ-invariant, we get for any $y \in G$ that

$$\int_G f(ty)d\mu(t) = \int_G f(\sigma(y)\sigma(t))d\mu(t) = \int_G f(\sigma(y)t)d\mu(t) = f(\sigma(y)) = f(y).$$

Now μ is a generalized Gelfand measure and therefore then we have

$$\begin{aligned}\int_G f(xty)d\mu(t) &= \int_G\int_G\int_G (kxtys)d\mu(k)d\mu(t)d\mu(s)\\ &= \prec \mu * \delta_x * \mu * \delta_y * \mu,\ f \succ = \prec \mu * \delta_x * \mu * \mu * \delta_y * \mu,\ f \succ\\ &= \prec \mu * \delta_y * \mu * \mu * \delta_x * \mu,\ f \succ = \prec \mu * \delta_y * \mu * \delta_x * \mu,\ f \succ\\ &= \int_G\int_G\int_G f(kytxs)d\mu(k)d\mu(t)\mu(s) = \int_G f(ytx)d\mu(t) \qquad (4)\end{aligned}$$

for all $x, y \in G$.

On the other hand, if we replace f by $\Psi(x) = \int_G f(zsx)d\mu(s)$ in the previous formula (4), we get

$$\int_G\int_G f(zsxty)d\mu(s)d\mu(t) = \int_G\int_G f(zsytx)d\mu(s)d\mu(t)$$

for all $x, y, z \in G$.

Step two. By using the ideas from the paper by Badora [3] we will prove that f is a solution of the integral equation (1). f is unbounded, so there exists a sequence $(a_n)_{n\in\mathbb{N}}$ in G such that

$$f(a_n) \neq 0 \text{ and } \lim_{n\to+\infty} |f(a_n)| = +\infty.$$

By inequality (3), for $x = a_n$ we have

$$\left| \frac{\int_G f(a_nty)d\mu(t) + \int_G f(a_nt\sigma(y))d\mu(t)}{f(a_n)} - 2f(y) \right| \leq \frac{\delta}{|f(a_n)|}$$

for all $y \in G$ and $n \in \mathbb{N}$.

It follows that the convergence of the sequence of functions

$$x \mapsto \frac{\int_G f(a_ntx)d\mu(t) + \int_G f(a_nt\sigma(x))d\mu(t)}{f(a_n)}, \quad n \in \mathbb{N}, \qquad (5)$$

to the function

$$x \mapsto 2f(x)$$

is uniform.

For any $x, y \in G$ and $n \in \mathbb{N}$ it is easily seen that

$$\begin{aligned}&\Big| \int_G\int_G f(a_ntysx)d\mu(t)d\mu(s) + \int_G\int_G f(a_ntys\sigma(x))d\mu(t)d\mu(s)\\ &\qquad -2f(x)\int_G f(a_nty)d\mu(t)\Big|\\ &\leq \int_G \left| \int_G f(a_ntysx)d\mu(s) + \int_G f(a_ntys\sigma(x))d\mu(s) - 2f(x)f(a_nty) \right| d|\mu|(t)\\ &\leq \delta \parallel \mu \parallel .\end{aligned}$$

Similarly, we get

$$\Big| \int_G\int_G f(a_nt\sigma(y)sx)d\mu(t)d\mu(s) + \int_G\int_G f(a_nt\sigma(y)s\sigma(x))d\mu(t)d\mu(s)$$

$$-2f(x)\int_G f(a_n t\sigma(y))d\mu(t)\Bigg| \leq \delta \parallel \mu \parallel .$$

Combining this and

$$\int_G\int_G f(zsxty)d\mu(s)d\mu(t) = \int_G\int_G f(zsytx)d\mu(s)d\mu(t),$$

we obtain

$$\Bigg|\int_G\int_G f(a_n t\sigma(y)sx)d\mu(t)d\mu(s) + \int_G\int_G f(a_n t\sigma(x)sy)d\mu(t)d\mu(s)$$
$$+\int_G\int_G f(a_n t\sigma(y)s\sigma(x))d\mu(t)d\mu(s) + \int_G\int_G f(a_n txsy)d\mu(t)d\mu(s)$$
$$-2f(x)\left[\int_G f(a_n ty)d\mu(t) + \int_G f(a_n t\sigma(y))d\mu(t)\right]\Bigg| \leq 2\delta \parallel \mu \parallel .$$

After dividing both sides of this inequality by $|f(a_n)|$ we get

$$\Bigg|\int_G \frac{\int_G f(a_n t\sigma(y)sx)d\mu(t) + \int_G f(a_n t\sigma(x)\sigma(s)y)d\mu(t)}{f(a_n)}d\mu(s)$$
$$+\int_G \frac{\int_G f(a_n txsy)d\mu(t) + \int_G f(a_n t\sigma(y)\sigma(s)\sigma(x))d\mu(t)}{f(a_n)}d\mu(s)$$
$$-2f(x)\left[\frac{\int_G f(a_n ty)d\mu(t) + \int_G f(a_n t\sigma(y))d\mu(t)}{f(a_n)}\right]\Bigg| \leq \frac{2\delta \parallel \mu \parallel}{|f(a_n)|}.$$

In view of (5), we get by letting $n \to +\infty$ that

$$\lim_{n\to+\infty}\frac{\int_G f(a_n t\sigma(y)sx)d\mu(t) + \int_G f(a_n t\sigma(x)\sigma(s)y)d\mu(t)}{f(a_n)} = 2f(\sigma(y)sx),$$
$$\lim_{n\to+\infty}\frac{\int_G f(a_n txsy)d\mu(t) + \int_G f(a_n t\sigma(y)\sigma(s)\sigma(x))d\mu(t)}{f(a_n)} = 2f(xsy),$$

and

$$\lim_{n\to+\infty}\frac{\int_G f(a_n ty)d\mu(t) + \int_G f(a_n t\sigma(y))d\mu(t)}{f(a_n)} = 2f(y).$$

Moreover, since the convergence is uniform, we have

$$\left|2\int_G f(\sigma(y)sx)d\mu(s) + 2\int_G f(xsy)d\mu(s) - 4f(x)f(y)\right| \leq 0,$$

for all $x, y \in G$.

In view of (4) $\int_G f(xty)d\mu(t) = \int_G f(ytx)d\mu(t), \quad x, y \in G$, and thus we conclude that f is a solution of the functional equation (1). □

Corollary. *Let (G, K) be a compact Gelfand pair (see [4]) with $\sigma(K) \subset K$. Let $\delta > 0$ and let f be a continuous complex-valued function on G such that*

$$\left|\int_K f(xky)dk + \int_K f(xk\sigma(y))dk - 2f(x)f(y)\right| \leq \delta, \quad x, y \in G,$$

where dk denotes the normalized Haar measure on K.

Then either

$$|f(x)| \leq \frac{1+\sqrt{1+2\delta}}{2}, \quad x \in G,$$

or

$$\int_K f(xky)dk + \int_K f(xk\sigma(y))dk = 2f(x)f(y), \quad x, y \in G.$$

Remarks 1. (1) In the theorem, we can replace the condition that μ is a generalized Gelfand measure by a weaker condition that f satisfies the following version of Kannappan's condition :

$$\int_G \int_G f(zsxty)d\mu(s)d\mu(t) = \int_G \int_G f(zsytx)d\mu(s)d\mu(t), \quad x, y, z \in G.$$

(2) If μ is a complex measure with finite support, the complex function f in the theorem need not be assumed to be continuous.

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Author's addresses:

E. Elqorachi
Department of Mathematics, Faculty of Sciences
University of Ibnou Zohr, Agadir
Morocco
E-mail: h_elqorachi@caramail.com

M. Akkouchi
Department of Mathematics
Faculty of Sciences Semlalia
University of Cadi Ayyad, Marrakech
Morocco
E-mail: makkouchi@hotmail.com