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LAGUERRE-TYPE EXPONENTIALS, AND THE RELEVANT L-CIRCULAR AND L-HYPERBOLIC FUNCTIONS

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Dedicated to the memory of Prof. V. D. Kupradze

Abstract. Some classes of entire functions which are eigenfunctions of generalizations of the Laguerre derivative operator are considered.

Since this property is an analog of the one characterizing the exponential function, we refer to such functions as Laguerre-type exponentials, or shortly L-exponentials. The definition of L-circular and L-hyperbolic functions easily follows.

Applications in the framework of generalized evolution problems are also mentioned.

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1. Introduction

The exponential function e^{ax} is an eigenfunction of the derivative operator since

$$De^{ax} = ae^{ax}, \tag{1.1}$$

where $D := d/dx$, and a denotes a real or complex arbitrary constant.

Another interesting differential operator exists in literature, namely the *Laguerre derivative* denoted in the following by D_L and defined by

$$D_L := DxD = \frac{d}{dx}x\frac{d}{dx}. \tag{1.2}$$

In the preceding articles, we have shown the role of the Laguerre derivative in the framework of the so-called *monomiality principle* and its application to the multidimensional Hermite (Hermite–Kampé de Fériet or Gould–Hopper polynomials, see [1], [2],[3]) or Laguerre polynomials [4], [5], [6].

It is easily seen, by induction, that the Laguerre derivative verifies

$$(DxD)^n = D^n x^n D^n. \tag{1.3}$$

Furthermore, introducing the Tricomi function of order zero or the relevant Bessel function

$$C_0(x) := \sum_{k=0}^{\infty}(-1)^k \frac{x^k}{(k!)^2} = J_0(2\sqrt{x}), \tag{1.4}$$

we obtain

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Theorem 1.1. *The function*

$$e_1(ax) := C_0(-ax) \tag{1.5}$$

is an eigenfunction of the Laguerre derivative operator, i.e.

$$D_L\, e_1(ax) = ae_1(ax). \tag{1.6}$$

Proof. Note that

$$D_L = D + xD^2, \tag{1.7}$$

and consequently

$$\begin{aligned} D_L\, e_1(ax) &= \left(D + xD^2\right) \sum_{k=0}^{\infty} a^k\, \frac{x^k}{(k!)^2} \\ &= \sum_{k=1}^{\infty} (k + k(k-1))\ a^k\, \frac{x^{k-1}}{(k!)^2} = \sum_{k=1}^{\infty} k^2\ a^k\, \frac{x^{k-1}}{(k!)^2} \\ &= a \sum_{k=0}^{\infty} a^k\, \frac{x^k}{(k!)^2} = ae_1(ax). \end{aligned} \tag{1.8}$$

□

Note that the preceding conclusion depends on the coefficients of the combination expressing the Laguerre derivative D_L in terms of D, and xD, so that it turns out the identity $(k + k(k-1)) = k^2$.

In the following we will show that the above technique can be iterated, producing Laguerre classes of exponential-type functions called *L-exponentials* and the relevant *L-circular*, *L-hyperbolic*, *L-Gaussian functions.*

Further extensions are given in the concluding section, and applications to the solution of generalized evolution problems is touched on.

2. Generalizations of the Laguerre Derivative and L-Exponential Functions

In this section, we generalize the Laguerre derivative and define the relevant L-exponential functions.

We start by considering the operator

$$D_{2L} := DxDxD = D\left(xD + x^2D^2\right) = D + 3xD^2 + x^2D^3, \tag{2.1}$$

and the function

$$e_2(x) := \sum_{k=0}^{\infty} \frac{x^k}{(k!)^3}\,. \tag{2.2}$$

The following theorem holds true:

Theorem 2.1. *The function* $e_2(ax)$ *is an eigenfunction of the operator* D_{2L}*, i.e.*

$$D_{2L}\, e_2(ax) = ae_2(ax) \tag{2.3}$$

Proof. Note that

$$
\begin{aligned}
D_{2L}\ e_2(ax) &= \left(D + 3xD^2 + x^2D^3\right)\ \sum_{k=0}^{\infty}\ a^k\ \frac{x^k}{(k!)^3} \\
&= \sum_{k=1}^{\infty} (k + 3k(k-1) + k(k-1)(k-2))\ a^k\ \frac{x^{k-1}}{(k!)^3} = \sum_{k=1}^{\infty} k^3\ a^k\ \frac{x^{k-1}}{(k!)^3} \\
&= a \sum_{k=0}^{\infty}\ a^k\ \frac{x^k}{(k!)^3} = ae_2(ax). \qquad (2.4)
\end{aligned}
$$

This completes the proof. □

Even in this case, the conclusion depends on the identity $k + 3k(k-1) + k(k-1)(k-2) = k^3$ so that it can be recognized that the coefficients of the combination expressing the $2L$-derivative D_{2L} in terms of D, xD^2, and x^2D^3 are the *Stirling numbers of second kind*, $S(3,1), S(3,2), S(3,3)$, (see [7], and [8], p. 835 for an extended table).

We can consequently extend the above results as follows.

Considering the operator

$$
\begin{aligned}
&D_{(n-1)L} := Dx \cdots DxDxD = D\left(xD + x^2D^2 + \cdots + x^{n-1}D^{n-1}\right) \\
&= S(n,1)D + S(n,2)xD^2 + \cdots + S(n,n)x^{n-1}D^n \qquad (2.5)
\end{aligned}
$$

and the function

$$
e_n(x) := \sum_{k=0}^{\infty} \frac{x^k}{(k!)^{n+1}}\ , \qquad (2.6)
$$

we can state the following theorem:

Theorem 2.2. *The function* $e_n(ax)$ *is an eigenfunction of the operator* D_{nL}*, i.e.*

$$
D_{nL}\ e_n(ax) = ae_n(ax). \qquad (2.7)
$$

Proof. Proceeding by induction, i.e. assuming equation (2.5) to be true, and recalling the above remarks, it is sufficient to prove that the coefficients of the combination expressing the nL-derivative D_{nL} in terms of D, xD^2, ..., and x^nD^{n+1}, verify the same induction property as the Stirling numbers of second kind, namely (see [7]):

$$
S(n+1,h) = S(n,h-1) + hS(n,h). \qquad (2.8)
$$

This is clearly true since, considering in the equation

$$
D_{nL} := D\left(S(n,1)xD + S(n,2)x^2D^2 + \cdots + S(n,n)x^nD^n\right), \qquad (2.9)
$$

the general terms, i.e.

$$
D\left(S(n,h-1)x^{h-1}D^{h-1} + S(n,h)x^hD^h\right), \qquad (2.10)
$$

we find

$$(h-1)S(n,h-1)x^{h-2}D^{h-1} + S(n,h-1)x^{h-1}D^h + hS(n,h)x^{h-1}D^h + S(n,h)x^hD^{h+1}$$

so that the coefficient of $x^{h-1}D^h$ is given by $S(n,h-1)+hS(n,h)$ and must coincide with $S(n+1,h)$ and then recursion (2.8) holds true. □

Remark 2.1. The above results show that, for every positive integer n, we can define a Laguerre-exponential function, satisfying an eigenfunction property, which is an analog of the elementary property (1.1) of the exponential. This function, denoted by $e_n(x) := \sum_{k=0}^{\infty} \frac{x^k}{(k!)^{n+1}}$, reduces to the exponential function when $n = 0$ so that we can set by definition:

$$e_0(x) := e^x, \qquad D_{0L} := D.$$

Obviously, $D_{1L} := D_L$.

For this reason we will refer to such functions as *L-exponential functions* or, shortly, *L-exponentials.*

Examples of the above functions are given in Fig. 1.

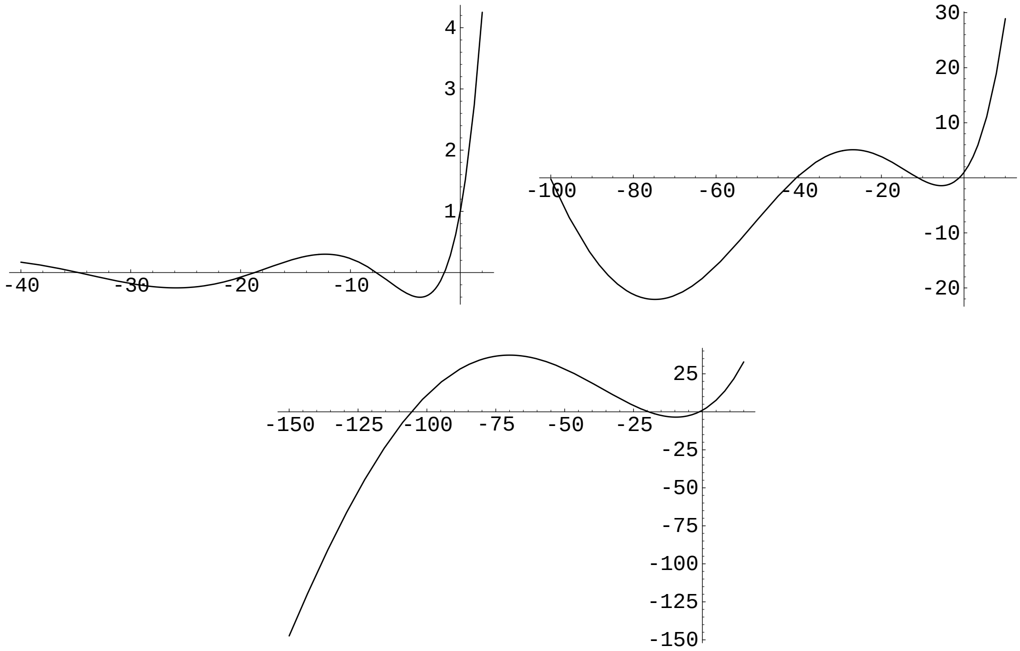

Fig. 1

The above definitions allow us to introduce an extension of the Gauss function e^{-x^2} by means of

Definition 2.1. The L-Gaussian functions are given by the entire even functions

$$e_1(-x^2) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(k!)^2}, \tag{2.11}$$

and, in general,

$$e_n(-x^2) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(k!)^{n+1}}. \tag{2.12}$$

Examples of the above functions are given in Fig. 2.

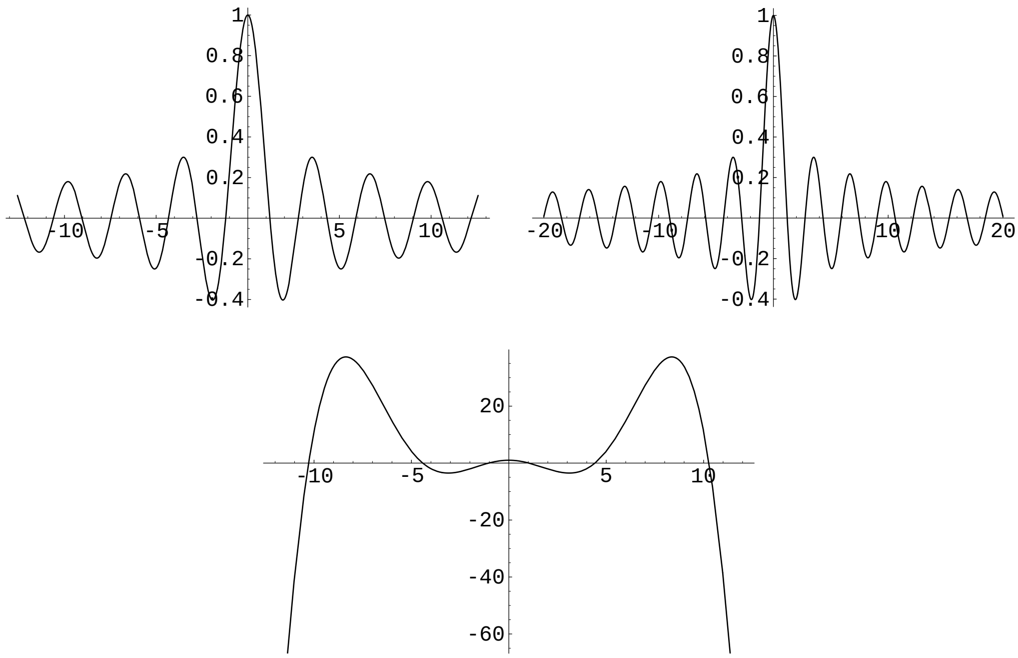

Fig. 2

3. L-Circular and L-Hyperbolic Functions

Write the $1L$-exponential as follows:

$$\begin{aligned} e_1(ix) &= \sum_{k=0}^{\infty} (i)^k \frac{x^k}{(k!)^2} \\ &= \sum_{h=0}^{\infty} (-1)^h \frac{x^{2h}}{((2h)!)^2} + i \sum_{h=0}^{\infty} (-1)^h \frac{x^{2h+1}}{((2h+1)!)^2}. \end{aligned} \tag{3.1}$$

Then we can formulate the definition

Definition 3.1. The *$1L$-circular functions* are given by

$$\cos_1(x) := \Re\left(e_1(ix)\right) = \mathcal{E}\left(e_1(ix)\right) = \sum_{h=0}^{\infty} (-1)^h \frac{x^{2h}}{((2h)!)^2}, \tag{3.2}$$

$$\sin_1(x) := \Im\left(e_1(ix)\right) = \frac{1}{i}\mathcal{O}\left(e_1(ix)\right) = \sum_{h=0}^{\infty} (-1)^h \frac{x^{2h+1}}{((2h+1)!)^2}, \tag{3.3}$$

where $\mathcal{E}(f)$ or $\mathcal{O}(f)$ are the *even part* or the *odd part* of f.
Obviously,

$$e_1(ix) = \cos_1(x) + i\sin_1(x), \qquad e_1(-ix) = \cos_1(x) - i\sin_1(x), \tag{3.4}$$

so that the Euler-type formulas

$$\cos_1(x) = \frac{e_1(ix) + e_1(-ix)}{2}, \qquad \sin_1(x) = \frac{e_1(ix) - e_1(-ix)}{2i} \tag{3.5}$$

hold true.

Recalling equation (1.3), we find the following result:

Theorem 3.1. *The* $1L$*-circular functions* (3.2), (3.3) *are solutions of the differential equation*

$$D_L^2\, v + v = \left(D^2 x^2 D^2\right) v + v = 0. \tag{3.6}$$

Proof. Equation (3.6) is an easy consequence of Theorem 1.1, since

$$D_L^2\, e_1(ix) = D_L\left(ie_1(ix)\right) = -e_1(ix), \tag{3.7}$$

so that

$$D_L^2\left(\cos_1(x) + i\sin_1(x)\right) = -\cos_1(x) - i\sin_1(x). \tag{3.8}$$

Then separating the real from the imaginary part in the above equation, the proclaimed result follows. □

Write now the nL-exponential in the form

$$\begin{aligned} e_n(ix) &= \sum_{k=0}^{\infty} (i)^k \frac{x^k}{(k!)^{n+1}} \\ &= \sum_{h=0}^{\infty} (-1)^h \frac{x^{2h}}{((2h)!)^{n+1}} + i \sum_{h=0}^{\infty} (-1)^h \frac{x^{2h+1}}{((2h+1)!)^{n+1}}. \end{aligned} \tag{3.9}$$

Then we can formulate the definition

Definition 3.2. The *nL-circular functions* are given by

$$\cos_n(x) := \Re\left(e_n(ix)\right) = \mathcal{E}\left(e_n(ix)\right) = \sum_{h=0}^{\infty} (-1)^h \frac{x^{2h}}{((2h)!)^{n+1}}, \tag{3.10}$$

$$\sin_n(x) := \Im\left(e_n(ix)\right) = \frac{1}{i}\mathcal{O}\left(e_n(ix)\right) = \sum_{h=0}^{\infty} (-1)^h \frac{x^{2h+1}}{((2h+1)!)^{n+1}}. \tag{3.11}$$

Obviously,

$$e_n(ix) = \cos_n(x) + i\sin_n(x), \qquad e_n(-ix) = \cos_n(x) - i\sin_n(x) \tag{3.12}$$

so that we find again the Euler-type formulas:

$$\cos_n(x) = \frac{e_n(ix) + e_n(-ix)}{2}, \qquad \sin_n(x) = \frac{e_n(ix) - e_n(-ix)}{2i}. \tag{3.13}$$

The same method used in Theorem 3.1 yields the more general result

Theorem 3.2. *The nL-circular functions* (3.10), (3.11) *are solutions of the differential equation*

$$D_{nL}^2\ v + v = 0.$$

and satisfy the conditions

$$\cos_n(0) = 1, \qquad \sin_n(0) = 0.$$

Then, using the same proof as in Theorem 2.2, we obtain

Theorem 3.3. *The nL-circular functions satisfy*

$$D_{nL}\ \cos_n(x) = -\sin_n(x), \qquad D_{nL}\ \sin_n(x) = \cos_n(x). \tag{3.14}$$

Examples of the above functions are given in Fig. 3.

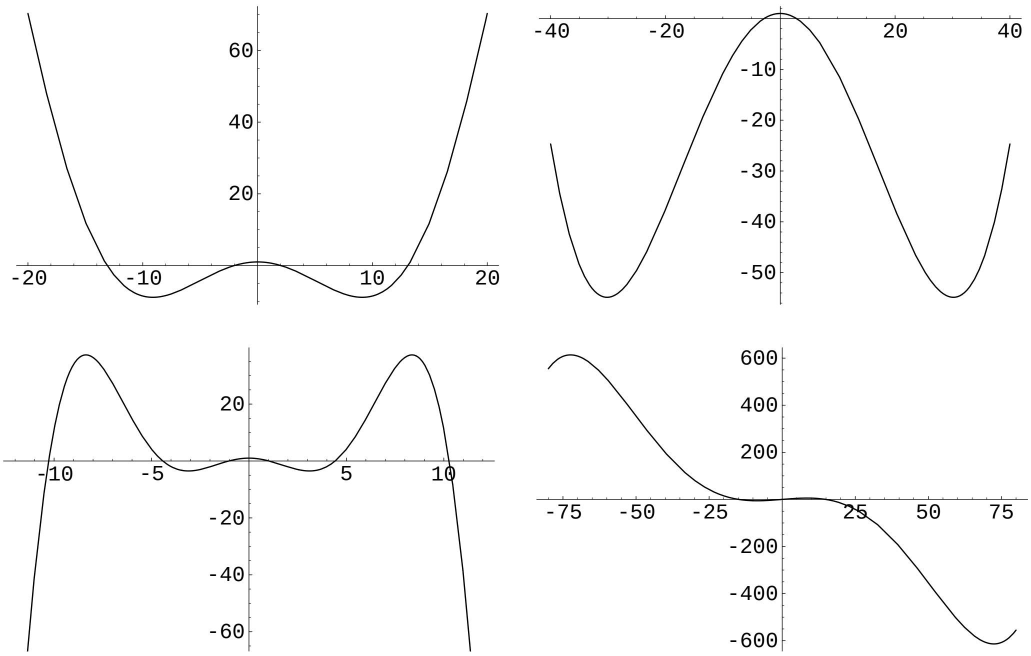

Fig. 3

We consider now the L-hyperbolic functions.

Definition 3.3. The *nL-circular functions* are given by

$$\cosh_n(x) := \Re\left(e_n(x)\right) = \mathcal{E}\left(e_n(x)\right) = \sum_{h=0}^{\infty} \frac{x^{2h}}{((2h)!)^{n+1}}, \tag{3.15}$$

$$\sinh_n(x) := \Im\left(e_n(x)\right) = \mathcal{O}\left(e_n(x)\right) = \sum_{h=0}^{\infty} \frac{x^{2h+1}}{((2h+1)!)^{n+1}}. \tag{3.16}$$

Obviously,

$$e_n(x) = \cos_n(x) + \sin_n(x), \qquad e_n(-x) = \cos_n(x) - \sin_n(x) \tag{3.17}$$

so that we find again the Euler-type formulas:

$$\cos_n(x) = \frac{e_n(x)+e_n(-x)}{2}, \qquad \sin_n(x) = \frac{e_n(x)-e_n(ix)}{2}. \tag{3.18}$$

Furthermore, we have

Theorem 3.4. *The nL-hyperbolic functions* (3.15), (3.16) *are solutions of the differential equation*

$$D_{nL}^2\ w - w = 0, \tag{3.19}$$

and satisfy the conditions

$$\cosh_n(0) = 1, \qquad \sinh_n(0) = 0.$$

Theorem 3.5. *The nL-hyperbolic functions satisfy*

$$D_{nL}\ \cosh_n(x) = \sinh_n(x), \qquad D_{nL}\ \sinh_n(x) = \cosh_n(x). \tag{3.20}$$

We omit graphics of the L-hyperbolic functions, whose shapes are similar to those of ordinary hyperbolic functions.

3.1. **Pseudo-L-circular and pseudo-L-hyperbolic functions.** According to the results stated in [9], [10] it is easy to find the functions decomposing the L-exponentials with respect to the cyclic group of a given (integral) order r.

The relevant functions are denoted by [10]

$$\Pi_{[r,k]}e_n(x) := \sum_{h=0}^{\infty} \frac{x^{rh+k}}{((rh+k)!)^{n+1}}, \qquad (k=0,1,\dots,r-1), \tag{3.21}$$

and called *pseudo-nL-hyperbolic functions* of order r, while the *pseudo-nL-circular functions* of order r are defined by the position

$$\sigma_0^{-k}\Pi_{[r,k]}e_n(\sigma_0\, x) = \sum_{h=0}^{\infty} (-1)^h \frac{x^{rh+k}}{((rh+k)!)^{n+1}}, \quad (k=0,1,\dots,r-1), \tag{3.22}$$

where σ_0 denotes any complex r-th root of -1.

The following statement is valid.

Theorem 3.6. *The pseudo-nL-circular functions of order* r, (3.22) *are solutions of the differential equation*

$$D_{nL}^r\ w + w = 0. \tag{3.23}$$

The pseudo-nL-hyperbolic functions of order r, *(3.21) are solutions of the differential equation*

$$D_{nL}^r\ w - w = 0. \tag{3.24}$$

Remark 3.1. Note that equation (1.3) can be easily generalized as follows:

$$D_{nL}^r = \underbrace{(DxDx\cdots DxD)^r}_{(n+1)\ Derivatives} = D^r x^r D^r x^r \cdots D^r x^r D^r.$$

4. Consideration of the Above Results, and Further Properties

We prove the following

Theorem 4.1. *The only polynomial solutions of the differential equation* $D_{nL}\, p = 0$ *are given by constants.*

Proof. Note that $D_L\, p(x) = D\,(xD\, p(x)) = 0$ implies $xD\, p(x) = \mathit{constant}$ and consequently $p(x) = c_1 + c_2 \log x$, $c_h = \mathit{constants}$ $(h = 1, 2)$ so that the unique polynomial solution is a constant. Furthermore, $D_{2L}\ p(x) = D_L xD\ p(x) = 0$ implies $xD\ p(x) = c_1 + c_2 \log x$ so that $D\ p(x) = c_1 + c_2 \log x + c_3 \int \frac{\log x}{x} dx$, and the unique polynomial solution is again a constant. Proceeding by induction, we find that $D_{nL}\, p(x) = D_{(n-1)L}\ xD\ p(x) = 0$ implies that $xD\ p(x)$ is a linear combination of a set of functions obtained by adding to the preceding set the primitive of each function divided by x. In any case, the only polynomial solution derived in such a way is always a constant. $\square$

It was previously noted (see, e.g., [5]) that, considering in the space of polynomial functions the correspondence

$$D \to D_L, \qquad x\cdot \to D_x^{-1}, \tag{4.1}$$

where

$$D_x^{-n}(1) := \frac{x^n}{n!}, \tag{4.2}$$

a *differential isomorphism* is determined.

In such an isomorphism, the exponential function is transformed into the function $e_1(x)$, the Hermite polynomials $H_n^{(1)}(x, y) := (x - y)^n$ become the Laguerre polynomials $\mathcal{L}_n(x, y) := n! \sum_{r=0}^{n} \frac{(-1)^r y^{n-r} x^r}{(n-r)!(r!)^2}$ and the *monomiality principle* ensures that all the relations proved in the initial polynomial space still hold after performing the substitutions stated in equations (4.1).

Note that an iterative application of equations (4.1) to the exponential function gives subsequently functions $e_1(x), e_2(x), e_3(x), \ldots$, and so on.

Accordingly, the derivative operator is transformed into

$$D_L = DxD, \qquad D_{2L} = D_L D_x^{-1} D_L, \qquad D_{3L} = D_L D_x^{-1} D_L D_x^{-1} D_L, \tag{4.3}$$

and so on.

Thus we can conclude that the L-exponentials (and the relevant L-circular and L-hyperbolic functions) are determined by an iterative application of the above mentioned-differential isomorphism.

This gives an explanation of the above results, since the subsequent derivatives are transformed into the powers of the Laguerre derivative, determining e.g., the validity of Theorems 3.1, 3.2, 3.3, 3.4, 3.5, 3.6.

5. Applications

5.1. **Diffusion equations.**

Theorem 5.1. *For any fixed integral* n, *consider the problem*

$$\begin{cases} D_{nL}\ S(x,t) = \frac{\partial}{\partial t}\ S(x,t), & \text{in the half}-\text{plane} \quad t>0, \\ S(0,t) = s(t), \end{cases} \tag{5.1}$$

with analytic boundary condition $s(t)$.

The operational solution of equation (5.1) *is given by*

$$S(x,t) = e_n\left(x\frac{\partial}{\partial t}\right) s(t) = \sum_{k=0}^{\infty} \frac{x^k}{(k!)^{n+1}} \frac{d^k}{dt^k} s(t). \tag{5.2}$$

Representing $s(t) = \sum_{k=0}^{\infty} a_k t^k$, from equation (5.2), we find, in particular:

$$S(x,0) = \sum_{k=0}^{\infty} a_k \frac{x^k}{(k!)^n}. \tag{5.2$_0$}$$

Note that the operational solution becomes an effective solution whenever the series in equation (5.2) is convergent. The validity of this condition depends on the growth of the coefficients a_k of the boundary data $s(t)$, but it is usually satisfied in physical problems.

More generally, the following results holds.

Theorem 5.2. *Let* $\hat{\Omega}_x$ *be a differential operator with respect to the variable* x, *and denote by* $\psi(x)$ *an eigenfunction of* $\hat{\Omega}_x$ *such that*

$$\hat{\Omega}_x\ \psi(ax) = a\ \psi(ax), \qquad \psi(0) = 1, \tag{5.3}$$

then the evolution problem

$$\begin{cases} \hat{\Omega}_x\ S(x,t) = \frac{\partial}{\partial t}\ S(x,t), & \text{in the half}-\text{plane} \quad t>0, \\ S(0,t) = s(t), \end{cases} \tag{5.4}$$

with analytic boundary condition $s(t)$ *admits an operational solution*

$$S(x,t) = \psi\left(x\frac{\partial}{\partial t}\right) s(t). \tag{5.5}$$

Proof. The eigenfunction property of ψ implies:

$$\hat{\Omega}_x\ S(x,t) = \hat{\Omega}_x\ \psi\left(x\frac{\partial}{\partial t}\right) s(t) = \frac{\partial}{\partial t}\ \psi\left(x\frac{\partial}{\partial t}\right) s(t) = \frac{\partial}{\partial t}\ S(x,t),$$

since $\frac{\partial}{\partial t}$ commutes with $\psi\left(x\frac{\partial}{\partial t}\right)$.

Furthermore, as a consequence of the hypothesis $\psi(0) = 1$, the boundary condition, for $x = 0$, is trivially satisfied. □

5.2. **L-hyperbolic-type problems.**

Theorem 5.3. *Let* $\hat{\Omega}_x$ *be a* $2nd$ *order differential operator with respect to the variable* x, $D_{nL} := (D_{nL})_t$ *the* nL*-derivative with respect to the* t *variable, and denote by* $\psi(t)$ *and* $\chi(t)$ *two functions such that*

$$D_{nL}\ \psi(t) = \chi(t), \qquad D_{nL}\ \chi(t) = \psi(t), \tag{5.6}$$

$$\psi(0) = 1, \qquad \chi(0) = 0; \tag{5.7}$$

then the abstract L*-hyperbolic-type problem*

$$\begin{cases} \hat{\Omega}_x^2\ S(x,t) = D_{nL}^2\ S(x,t), \quad \text{in the half}-\text{plane} \quad t>0, \\ S(x,0) = q(x), \\ D_{nL}\ S(x,t)|_{t=0} = v(x) \end{cases} \tag{5.8}$$

with analytic initial condition $q(x)$, $v(x)$, *admits an operational solution*

$$S(x,t) = \psi\left(t\hat{\Omega}_x\right) q(x) + \chi\left(t\hat{\Omega}_x\right) w(x), \tag{5.9}$$

where $w(x) := \hat{\Omega}_x^{-1} v(x)$.

Proof. Since $\hat{\Omega}_x$ commutes with $\psi\left(t\hat{\Omega}_x\right)$ and $\chi\left(t\hat{\Omega}_x\right)$, by using conditions (5.6), and the chain rule with respect to the Laguerre derivative (see Remark 5.1 below), we can write

$$D_{nL}\ S(x,t) = \hat{\Omega}_x\ \chi\left(t\hat{\Omega}_x\right) q(x) + \hat{\Omega}_x\ \psi\left(t\hat{\Omega}_x\right) w(x),$$

$$D_{nL}^2\ S(x,t) = \hat{\Omega}_x^2\ \psi\left(t\hat{\Omega}_x\right) q(x) + \hat{\Omega}_x^2\ \chi\left(t\hat{\Omega}_x\right) w(x) = \hat{\Omega}_x^2\ S(x,t).$$

Furthermore, for the initial conditions, by using equations (5.7) and the definition of w we find

$$S(x,0) = \psi(0)q(x) + \chi(0)w(x) = q(x),$$

$$D_{nL}\ S(x,t)|_{t=0} = \hat{\Omega}_x\ \chi(0)q(x) + \hat{\Omega}_x\ \psi(0)w(x) = \hat{\Omega}_x\ w(x) = v(x).$$

Note that conditions (5.6)-(5.7) are satisfied, fixing an arbitrary integral n and assuming

$$\psi(x) := \cosh_{nL}(x), \qquad \chi(x) := \sinh_{nL}(x).$$

□

5.3. **L-elliptic-type problems.**

Theorem 5.4. *Let* $\hat{\Omega}_x$ *be a* $2nd$ *order differential operator with respect to the variable* x, $D_{nL} := (D_{nL})_y$ *the* nL*-derivative with respect to the variable* y, *and denote by* $\varphi(y)$ *and* $\tau(y)$ *two functions such that*

$$D_{nL}\ \varphi(y) = -\tau(y), \qquad D_{nL}\ \tau(y) = \varphi(y), \tag{5.10}$$

$$\varphi(0) = 1, \qquad \tau(0) = 0; \tag{5.11}$$

then the abstract L*-elliptic-type problem*

$$\begin{cases} \hat{\Omega}_x^2 \; S(x,y) + D_{nL}^2 \; S(x,y) = 0, \quad \text{in the half} - \text{plane} \quad y > 0, \\ S(x,0) = q(x), \end{cases} \tag{5.12}$$

with analytic boundary condition $q(x)$, *admits the operational solution*

$$S(x,y) = \varphi\left(y\hat{\Omega}_x\right) q(x). \tag{5.13}$$

Proof. Since $\hat{\Omega}_x$ commutes with $\varphi\left(y\hat{\Omega}_x\right)$, by using conditions (5.10) we can write

$$D_{nL} \; S(x,y) = -\hat{\Omega}_x \; \tau\left(y\hat{\Omega}_x\right) q(x),$$

$$D_{nL}^2 \; S(x,y) = -\hat{\Omega}_x^2 \; \varphi\left(y\hat{\Omega}_x\right) q(x) = -\hat{\Omega}_x^2 \; S(x,y).$$

Furthermore, for the boundary conditions, by using equations (5.11) we find:

$$S(x,0) = \varphi(0) q(x) = q(x).$$

$\square$

Note that conditions (5.10)–(5.11) are satisfied, fixing an arbitrary integral n and assuming

$$\varphi(x) := \cos_{nL}(x), \qquad \tau(x) := \sin_{nL}(x).$$

Remark 5.1. Note that for the Laguerre derivative, the chain rule

$$\frac{d}{dt} = \frac{d}{dx}\frac{dx}{dt}$$

becomes

$$\frac{d}{dt}t\frac{d}{dt} = \frac{d}{dx}\frac{d}{dt}t\frac{dx}{dt}, \qquad \Leftrightarrow \qquad (D_L)_t = \frac{d}{dx}(D_L)_t \, x$$

and, in general,

$$(D_{nL})_t = \frac{d}{dx}(D_{nL})_t \, x.$$

6. Concluding Remarks

The concepts we have so far developed can be further generalized. Limiting ourselves to the case of second order operators, we note that the function

$$e_{1,m}(x) := \sum_{k=0}^{\infty} \frac{x^k}{k!(k+m)!} \tag{6.1}$$

is such that $e_{1,m}(ax)$ is an eigenfunction of the operator

$$\hat{\Omega}_m := D_L + mD = \frac{d}{dx}x\frac{d}{dx} + m\frac{d}{dx} \tag{6.2}$$

and, indeed,

$$\hat{\Omega}_m \; e_{1,m}(ax) = a \; e_{1,m}(ax). \tag{6.3}$$

It is therefore clear that we can introduce more general forms of L-circular functions as follows:

$$\cos_{1,m}(x) := \frac{e_{1,m}(ix) + e_{1,m}(-ix)}{2}, \tag{6.4}$$

$$\sin_{1,m}(x) := \frac{e_{1,m}(ix) - e_{1,m}(-ix)}{2i}, \tag{6.5}$$

and analogous forms for the hyperbolic case.

It is also evident that the above functions satisfy the differential equation

$$\hat{\Omega}_m^2 \, v(x) = -v(x). \tag{6.6}$$

It is furthermore interesting to note that $e_{1,m}(x)$ can be viewed as the n-th order Tricomi function defined by the generating function

$$\sum_{m=-\infty}^{+\infty} t^m \, e_{1,m}(x) = \exp\left(t + \frac{x}{t}\right) \tag{6.7}$$

and that the formalism we have just envisaged can be exploited in a more general framework, as, e.g., that associated with partial differential equations. Namely, the solution of the evolution problem

$$\begin{cases} \hat{\Omega}_m F(x,t) = \frac{\partial}{\partial t} F(x,t) \\ F(0,t) = g(t) \end{cases} \tag{6.8}$$

can be solved in the form

$$F(x,t) = m! \, e_{1,m}\left(x \frac{\partial}{\partial t}\right) g(t) \tag{6.9}$$

which, once expanded in a series, yields

$$F(x,t) = m! \sum_{k=0}^{\infty} \frac{x^k}{k!(k+m)!} g^{(k)}(t). \tag{6.10}$$

Different solutions, expressed in terms of integral transforms, will be discussed elsewhere.

Before concluding, let us note that more in general we can introduce the functions

$$e_{r,m_1,\ldots,m_r}(x) := \sum_{k=0}^{\infty} \frac{x^k}{k!(k+m_1)!\ldots(k+m_r)!} \tag{6.11}$$

which are essentially the multi-index Bessel functions discussed in ref. [11]. Their role in the definition of more general forms of generalized circular and hyperbolic functions will be discussed elsewhere.

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