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SOLVABILITY AND ASYMPTOTICS OF SOLUTIONS OF CRACK-TYPE BOUNDARY-CONTACT PROBLEMS OF THE COUPLE-STRESS ELASTICITY

O. CHKADUA

Abstract. Spatial boundary value problems of statics of couple-stress elasticity for anisotropic homogeneous media (with contact on a part of the boundary) with an open crack are studied supposing that one medium has a smooth boundary and the other one has an open crack.

Using the method of the potential theory and the theory of pseudodifferential equations on manifolds with boundary, the existence and uniqueness theorems are proved in Besov and Bessel-potential spaces. The smoothness and a complete asymptotics of solutions near the contact boundaries and near crack edge are studied.

Properties of exponents of the first terms of the asymptotic expansion of solutions are established. Classes of isotropic, transversally-isotropic and anisotropic bodies are found, where oscillation vanishes.

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Introduction

The paper is dedicated to the study of solvability and asymptotics of solutions of spatial crack type boundary-contact problems of statics of couple-stress elasticity for anisotropic homogeneous media with contact on a part of the boundary.

A vast number of works are devoted to the justification and axiomatization of elasticity and couple-stress elasticity. The fundamentals of the theory of couple-stress elasticity are included in the works by W. Voight [40], E. Cosserat, F. Cosserat [11], and developed later in the works by E. Aero and E. Kuvshinski [1], G. Grioli [21], R. Mindlin [31], W. Koiter [24], W. Nowacki [34], V. Kupradze, T. Gegelia, M. Basheleishvili, T. Burchuladze [26], T. Burchuladze and T. Gegelia [4] and others.

It is well-known that solutions of elliptic boundary value problems in domains with corners, edges and conical points have singularities regardless the smoothness properties of given data.

Among theoretical investigations the methods suggested and developed by V. Kondrat'ev [25], V, Maz'ya [27], V, Maz'ya and B. Plamenevsky [28]–[30], S. Nazarov and B. Plamenevsky [33], M. Dauge [13], P. Grisvard [22] and others

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attracted attention of many scientists. They used the Mellin transform which allows them to reduce the problem to the investigation of spectral properties of ordinary differential operators depending on the parameter.

The method of the potential theory and the theory of pseudodifferential equations used in this paper makes it possible to obtain more precise asymptotic representations of solutions of the problems posed, which frequently have crucial importance in applications (e.g., in crack extension problems). For the development of this method see [18], [3], [10], [9] and other papers.

The method of the potential theory was successfully applied to the classical problems of elasticity and couple-stress elasticity theory by V. Kupradze and his disciples.

In the present paper we consider the contact of two media, one of which has a smooth boundary, while the other has a boundary containing a closed cuspidal edge (the corresponding dihedral angle is equal to 2π), i.e., an open crack.

Theorems on the existence and uniqueness of solutions of these boundary-contact problems are obtained using the potential theory and the general theory of pseudodifferential equations on a manifold with boundary.

Using the asymptotic expansion of solutions of strongly elliptic pseudodifferential equations obtained in [10] (see also [18], [3]) and also the asymptotic expansion of potential-type functions [9], we obtain a complete asymptotic expansion of solutions of boundary-contact problems near the contact boundaries and near the crack edge. Here it is worth noticing the effective formulae for calculating the exponent of the first terms of asymptotic expansion of solutions of these problems by means of the symbol of the corresponding boundary pseudodifferential equations.

The properties of exponents of the first terms of the asymptotic expansion of solutions are established. Important classes of isotropic, transversally-isotropic and anisotropic bodies are found, where oscillation vanishes.

These results are new even for the problems of elasticity.

1. Formulation of the Problems

Let D_1 be a finite domain, D_2 be a domain that can be both finite or infinite in the Euclidean space $\mathbb{R}^3$ with compact boundaries ∂D_1, ∂D_2 ($\partial D_1 \in C^\infty$), and let there exist a surface S_0 of the class C^∞ of dimension two, which divides the domain D_2 into two subdomains $D_2^{(1)}$ and $D_2^{(2)}$ with C^∞ boundaries $\partial D_2^{(1)}$ and $\partial D_2^{(2)}$ ($D_2^{(1)} \cap D_2^{(2)} = \varnothing$, $\overline{D}_2^{(1)} \cap \overline{D}_2^{(2)} = \overline{S}_0$). Then ∂S_0 is the boundary of the surface S_0 ($\partial S_0 \subset \partial D_2$), representing one-dimensional closed cuspidal edge, where ∂S_0 is the crack edge.

Let the domains D_1 and D_2 have the contact on the two-dimensional manifolds $\overline{S}_0^{(1)}$ and $\overline{S}_0^{(2)}$ of the class C^∞, i.e., $\partial D_1 \cap \partial D_2 = \overline{S}_0^{(1)} \cup \overline{S}_0^{(2)}$, $D_1 \cap D_2 = \varnothing$, $\overline{S}_0^{(1)} \cap \overline{S}_0^{(2)} = \varnothing$, and $S_1 = \partial D_1 \setminus (\overline{S}_0^{(1)} \cup \overline{S}_0^{(2)})$. Then $\partial D_2^{(1)} = S_2^{(1)} \cup \overline{S}_0^{(1)} \cup \overline{S}_0$, $\partial D_2^{(2)} = S_2^{(2)} \cup \overline{S}_0^{(2)} \cup \overline{S}_0$.

Suppose that the domains D_q, $q = 1, 2$, are filled with anisotropic homogeneous elastic materials.

The basic static equations of couple-stress elasticity for anisotropic homogeneous media are written in terms of displacement and rotation components as (see [5], [19])

$$\mathcal{M}^{(q)}(\partial_x)\mathcal{U}^{(q)} + \mathcal{F}^{(q)} = 0 \quad \text{in} \quad D_q, \quad q = 1, 2, \tag{1.1}$$

where $\mathcal{U}^{(q)} = (u^{(q)}, \omega^{(q)})$, $u^{(q)} = (u_1^{(q)}, u_2^{(q)}, u_3^{(q)})$ is the displacement vector, $\omega^{(q)} = (\omega_1^{(q)}, \omega_2^{(q)}, \omega_3^{(q)})$ is the rotation vector, $\mathcal{F}^{(q)} = (\mathcal{F}_1^{(q)}, \dots, \mathcal{F}_6^{(q)})$ is the mass force applied to D_q, and $\mathcal{M}^{(q)}(\partial_x)$ is the matrix differential operator

$$\mathcal{M}^{(q)}(\partial_x) = \begin{pmatrix} \overset{1}{\mathcal{M}}{}^{(q)}(\partial_x) & \overset{2}{\mathcal{M}}{}^{(q)}(\partial_x) \\ \overset{3}{\mathcal{M}}{}^{(q)}(\partial_x) & \overset{4}{\mathcal{M}}{}^{(q)}(\partial_x) \end{pmatrix}_{6\times 6},$$

$$\overset{l}{\mathcal{M}}{}^{(q)}(\partial_x) = \| \overset{l}{\mathcal{M}}{}^{(q)}_{jk}(\partial_x)\|_{3\times 3}, \quad l = \overline{1,4}, \quad q = 1, 2,$$

$$\begin{gathered} \overset{1}{\mathcal{M}}{}^{(q)}_{jk}(\partial_x) = a^{(q)}_{ijlk}\partial_i\partial_l, \quad \overset{2}{\mathcal{M}}{}^{(q)}_{jk}(\partial_x) = b^{(q)}_{ijlk}\partial_i\partial_l - \varepsilon_{lrk}a^{(q)}_{ijlr}\partial_i, \\ \overset{3}{\mathcal{M}}{}^{(q)}_{jk}(\partial_x) = b^{(q)}_{lkij}\partial_i\partial_l + \varepsilon_{irj}a^{(q)}_{irlk}\partial_l, \\ \overset{4}{\mathcal{M}}{}^{(q)}_{jk}(\partial_x) = c^{(q)}_{ijlk}\partial_i\partial_l - b^{(q)}_{lrij}\varepsilon_{lrk}\partial_i + \varepsilon_{irj}b^{(q)}_{irlk}\partial_l - \varepsilon_{ipj}\varepsilon_{lrk}a^{(q)}_{iplr}; \end{gathered} \tag{1.2}$$

ε_{ikj} is the Levi-Civita symbol, $a^{(q)}_{ijlk}$, $b^{(q)}_{ijlk}$, $c^{(q)}_{ijlk}$ are the elastic constants satisfying the conditions

$$a^{(q)}_{ijlk} = a^{(q)}_{lkij}, \quad c^{(q)}_{ijlk} = c^{(q)}_{lkij}, \quad q = 1, 2.$$

In (1.2) and in what follows, under the repeated indices we understand the summation from 1 to 3.

It is assumed that the quadratic forms

$$a^{(q)}_{ijlk}\xi_{ij}\xi_{lk} + 2b^{(q)}_{ijlk}\xi_{ij}\eta_{lk} + c^{(q)}_{ijlk}\eta_{ij}\eta_{lk}, \quad q = 1, 2,$$

with respect to variables ξ_{ij}, η_{ij} are positive-definite, i.e., $\exists M > 0$

$$a^{(q)}_{ijlk}\xi_{ij}\xi_{lk} + 2b^{(q)}_{ijlk}\xi_{ij}\eta_{lk} + c^{(q)}_{ijlk}\eta_{ij}\eta_{lk} \geq M(\xi_{ij}\xi_{ij} + \eta_{lk}\eta_{lk}) \text{ for all } \xi_{ij},\ \eta_{lk},\ q=1,2. \tag{1.3}$$

We introduce the differential stress operator

$$\mathcal{N}^{(q)}(\partial_z, n(z)) = \begin{pmatrix} \overset{1}{\mathcal{N}}{}^{(q)}(\partial_z, n(z)) & \overset{2}{\mathcal{N}}{}^{(q)}(\partial_z, n(z)) \\ \overset{3}{\mathcal{N}}{}^{(q)}(\partial_z, n(z)) & \overset{4}{\mathcal{N}}{}^{(q)}(\partial_z, n(z)) \end{pmatrix}_{6\times 6},$$

$$\overset{l}{\mathcal{N}}{}^{(q)}(\partial_z, n(z)) = \| \overset{l}{\mathcal{N}}{}^{(q)}_{jk}(\partial_z, n(z))\|_{3\times 3}, \quad l = \overline{1,4}, \quad q = 1, 2,$$

$$\overset{1}{\mathcal{N}}{}^{(q)}_{jk}(\partial_z, n(z)) = a^{(q)}_{ijlk}n_i(z)\partial_l, \quad \overset{2}{\mathcal{N}}{}^{(q)}_{jk}(\partial_z, n(z)) = b^{(q)}_{ijlk}n_i(z)\partial_l - a^{(q)}_{ijlk}\varepsilon_{lrk}n_i(z),$$

$$\overset{3}{\mathcal{N}}{}^{(q)}_{jk}(\partial_z, n(z)) = b^{(q)}_{lkij}n_i(z)\partial_l, \quad \overset{4}{\mathcal{N}}{}^{(q)}_{jk}(\partial_z, n(z)) = c^{(q)}_{ijlk}n_i(z)\partial_l - b^{(q)}_{lrij}\varepsilon_{lrk}n_i(z),$$

where $n(z) = (n_1(z), n_2(z), n_3(z))$ is the unit normal of the manifold ∂D_1 at a point $z \in \partial D_1$ (external with respect to D_1) and a point $z \in \partial D_2$ (internal with respect to D_2).

In what follows the stress operators are denoted by

$$\mathcal{N}^{(q)}=\mathcal{N}^{(q)}(\partial_z,n(z)),\quad q=1,2.$$

Let $\mathcal{M}^{(q)}(\xi)$ be the symbol of the differential operator $\mathcal{M}^{(q)}(\partial_x)$ and $\overset{\circ}{\mathcal{M}}{}^{(q)}(\xi)$ be the principal homogeneous symbol of the differential operator $\mathcal{M}^{(q)}(\partial_x)$.

We introduce the following notation for Besov and Bessel potential spaces (see [39]):

$$\mathbb{B}^s_{p,r}=B^s_{p,r}\times B^s_{p,r},\ \ \widetilde{\mathbb{B}}^s_{p,r}=\widetilde{B}^s_{p,r}\times\widetilde{B}^s_{p,r},\ \ \mathbb{H}^s_p=H^s_p\times H^s_p,\ \ \widetilde{\mathbb{H}}^s_p=\widetilde{H}^s_p\times\widetilde{H}^s_p.$$

From the symmetry of the coefficients $a^{(q)}_{ijlk}$, $c^{(q)}_{ijlk}$ and the positive-definiteness of the quadratic forms (1.3) it follows (see [19]) that the operators $\mathcal{M}^{(q)}(\partial_x)$, $q=1,2$, are strongly elliptic, formally self-adjoint differential operators and therefore for any real vector $\xi\in\mathbb{R}^3$ and any complex vector $\eta\in\mathbb{C}^6$ the relations

$$\mathrm{Re}(\overset{\circ}{\mathcal{M}}{}^{(q)}(\xi)\eta,\eta)=(\overset{\circ}{\mathcal{M}}{}^{(q)}(\xi)\eta,\eta)\geq P^{(q)}_0|\xi|^2|\eta|^2$$

are valid, where $P^{(q)}_0=const>0$ depends only on the elastic constants. Thus the matrices $\overset{\circ}{\mathcal{M}}{}^{(q)}(\xi)$ are positive-definite for $\xi\in\mathbb{R}^3\setminus\{0\}$.

Taking into account the property of the Levi-Civita symbol

$$\varepsilon_{ipj}\varepsilon_{lrk}=\det\begin{pmatrix}\delta_{il}&\delta_{ir}&\delta_{ik}\\ \delta_{pl}&\delta_{pr}&\delta_{pk}\\ \delta_{jl}&\delta_{jr}&\delta_{jk}\end{pmatrix}\ \ (\delta_{pl}\ \ \text{is the Kronecker symbol}),$$

it is not difficult to observe that

$$(\mathcal{M}^{(q)}(\xi)\eta,\eta)=(\overset{\circ}{\mathcal{M}}{}^{(q)}(\xi)\eta,\eta),\quad q=1,2.$$

Then we obtain that the matrices $\mathcal{M}^{(q)}(\xi)$, $q=1,2$, are positive-definite for $\xi\in\mathbb{R}^3\setminus\{0\}$.

Since

$$\det\mathcal{M}^{(q)}(\xi)\neq0\quad\text{for}\quad\xi\neq0.$$

Let $\mathcal{U}^{(1)}\in W^1_p(D_1)$, $\mathcal{U}^{(2)}\in W^1_{p,loc}(D_2)$. Then $r_1\mathcal{U}^{(2)}=r_{D^{(1)}_2}\mathcal{U}^{(2)}\in W^1_p(D^{(1)}_2)$ and $r_2\mathcal{U}^{(2)}=r_{D^{(2)}_2}\mathcal{U}^{(2)}\in W^1_{p,loc}(D^{(2)}_2)$, where r_i is the restriction operator on $D^{(i)}_2$, $i=1,2$. From the theorem on traces (see [39]) it follows that the trace of the functions $\mathcal{U}^{(i)}$, $r_i\mathcal{U}^{(2)}$, $i=1,2$, exists on ∂D_1, $\partial D^{(i)}_2$, $i=1,2$, and $\{\mathcal{U}^{(1)}\}^\pm\in\mathbb{B}^{1/p'}_{p,p}(\partial D_1)$, $\{r_i\mathcal{U}^{(2)}\}^\pm\in\mathbb{B}^{1/p'}_{p,p}(\partial D^{(i)}_2)$, $i=1,2$, $p'=p/(p-1)$. Let $\mathcal{U}^{(1)}\in W^1_p(D_1),\mathcal{U}^{(2)}\in W^1_{p,loc}(D_2)$ be such that $\mathcal{M}^{(1)}(\partial_x)\mathcal{U}^{(1)}\in L_p(D_1)$, $\mathcal{M}^{(2)}(\partial_x)\mathcal{U}^{(2)}\in L_{p,comp}(D_2)$. Then $\{\mathcal{N}^{(1)}\mathcal{U}^{(1)}\}^\pm$ and $\{\mathcal{N}^{(2)}(r_i\mathcal{U}^{(2)})\}^\pm$, $i=1,2$, are correctly defined by the equalities

$$\int_{D_1}\big[\overline{\mathcal{V}}^{(1)}\mathcal{M}^{(1)}(\partial_x)\mathcal{U}^{(1)}+E^{(1)}(\mathcal{U}^{(1)},\mathcal{V}^{(1)})\big]dx$$
$$=\pm\langle\{\mathcal{N}^{(1)}\mathcal{U}^{(1)}\}^\pm,\{\mathcal{V}^{(1)}\}^\pm\rangle_{\partial D_1}\ \text{ for all }\ \mathcal{V}^{(1)}\in W^1_{p'}(D_1)\tag{1.4}$$

and

$$\int_{D_2^{(i)}} \big[\overline{\mathcal{V}}_2^{(i)} \mathcal{M}^{(2)}(\partial_x)(r_i\mathcal{U}^{(2)}) + E^{(2)}(r_i\mathcal{U}^{(2)}, \mathcal{V}_2^{(i)})\big]\,dx$$
$$= \mp\langle \{\mathcal{N}^{(2)}(r_i\mathcal{U}^{(2)})\}^{\pm}, \{\mathcal{V}_2^{(i)}\}^{\pm}\rangle_{\partial D_2^{(i)}}, \quad i=1,2, \tag{1.5}$$
$$\text{for all } \mathcal{V}_2^{(1)} \in W^1_{p'}(D_2^{(1)}) \ \ (\mathcal{V}_2^{(2)} \in W^1_{p,comp}(D_2^{(2)})),$$

where

$$E^{(1)}(\mathcal{U}^{(1)}, \mathcal{V}^{(1)}) = a^{(1)}_{ijlk}\xi_{ij}(\mathcal{U}^{(1)})\xi_{lk}(\overline{\mathcal{V}}^{(1)}) + b^{(1)}_{ijlk}\xi_{ij}(\mathcal{U}^{(1)})\eta_{lk}(\overline{\mathcal{V}}^{(1)})$$
$$+ b^{(1)}_{ijlk}\xi_{ij}(\overline{\mathcal{V}}^{(1)})\eta_{lk}(\mathcal{U}^{(1)}) + c^{(1)}_{ijlk}\eta_{ij}(\mathcal{U}^{(1)})\eta_{lk}(\overline{\mathcal{V}}^{(1)});$$

here $\mathcal{U}^{(1)} = (u^{(1)}, \omega^{(1)})$,

$$\xi_{ij}(\mathcal{U}^{(1)}) = \partial_j u_i^{(1)} - \varepsilon_{ijk}\omega_k \ \text{ is the deformation component,}$$
$$\eta_{ij}(\mathcal{U}^{(1)}) = \partial_j\omega_i \ \text{ is the bending torsion component.}$$

The quadratic form $E^{(2)}(r_i\mathcal{U}^{(2)}, \mathcal{V}_2^{(i)})$, $i=1,2$, is defined analogously.

In the case of the infinite domain D_2, for solving equation (1.1) the condition

$$\mathcal{U}^{(2)}(x) = o(1) \quad \text{for } |x| \to \infty \tag{1.6}$$

is assumed to be fulfilled at infinity (see [4]).

We can prove (see [4]) that

$$\partial^\mu \mathcal{U}^{(2)}(x) = O(|x|^{-1-|\mu|}) \quad \text{for } |x| \to \infty$$

is valid for any solution of equation (1.1) satisfying (1.6) the relation.

Let us consider the model problems $\mathbf{M}_1$ and $\mathbf{M}_2$.

We will study the solvability and asymptotics of solutions $\mathcal{U}^{(q)} \in W^1_p(D_q)$, $q = 1,2$, ($\mathcal{U}^{(2)} \in W^1_{p,loc}(D_2)$ with condition (1.6) at infinity) the following boundary-contact problems of couple-stress elasticity:

Problem $\mathbf{M}_1$:

$$\begin{cases} \mathcal{M}^{(q)}(\partial_x)\mathcal{U}^{(q)} = 0 & \text{in } D_q, \ \ q=1,2, \\ \pi_{S_1}\{\mathcal{U}^{(1)}\}^+ = \varphi_1 & \text{on } S_1, \\ \pi_{S_2^{(1)}}\{\mathcal{N}^{(2)}(r_1\mathcal{U}^{(2)})\}^+ = \varphi_2 & \text{on } S_2^{(1)}, \\ \pi_{S_2^{(2)}}\{\mathcal{N}^{(2)}(r_2\mathcal{U}^{(2)})\}^+ = \varphi_3 & \text{on } S_2^{(2)}, \\ \pi_{S_0^{(i)}}\{\mathcal{U}^{(1)}\}^+ - \pi_{S_0^{(i)}}\{r_i\mathcal{U}^{(2)}\}^+ = f_i & \text{on } S_0^{(i)}, \\ \pi_{S_0^{(i)}}\{\mathcal{N}^{(1)}\mathcal{U}^{(1)}\}^+ - \pi_{S_0^{(i)}}\{\mathcal{N}^{(2)}(r_i\mathcal{U}^{(2)})\}^+ = h_i & \text{on } S_0^{(i)}, \ \ i=1,2, \end{cases}$$

where

$$\varphi_1 \in \mathbb{B}^{1/p'}_{p,p}(S_1), \quad \varphi_2 \in \mathbb{B}^{-1/p}_{p,p}(S_2^{(1)}), \quad \varphi_3 \in \mathbb{B}^{-1/p}_{p,p}(S_2^{(2)}),$$
$$f_i \in \mathbb{B}^{1/p'}_{p,p}(S_0^{(i)}), \quad h_i \in \mathbb{B}^{-1/p}_{p,p}(S_0^{(i)}), \ \ i=1,2, \ \ 1<p<\infty, \ \ p' = p/(p-1).$$

If D_2 is a finite domain, then we have the following wedge-type problem:

Wedge-type Problem $\mathbf{M}_2$:

$$\begin{cases} \mathcal{M}^{(q)}(\partial_x)\mathcal{U}^{(q)} = 0 & \text{in } D_q, \ \ q=1,2, \\ \pi_{S_1}\{\mathcal{N}^{(1)}\mathcal{U}^{(1)}\}^+ = \varphi_1 & \text{on } S_1, \\ \pi_{S_2^{(1)}}\{\mathcal{N}^{(2)}(r_1\mathcal{U}^{(2)})\}^+ = \varphi_2 & \text{on } S_2^{(1)}, \\ \pi_{S_2^{(2)}}\{\mathcal{N}^{(2)}(r_2\mathcal{U}^{(2)})\}^+ = \varphi_3 & \text{on } S_2^{(2)}, \\ \pi_{S_0^{(i)}}\{\mathcal{U}^{(1)}\}^+ - \pi_{S_0^{(i)}}\{r_i\mathcal{U}^{(2)}\}^+ = f_i & \text{on } S_0^{(i)}, \\ \pi_{S_0^{(i)}}\{\mathcal{N}^{(1)}\mathcal{U}^{(1)}\}^+ - \pi_{S_0^{(i)}}\{\mathcal{N}^{(2)}(r_i\mathcal{U}^{(2)})\}^+ = h_i & \text{on } S_0^{(i)}, \ \ i=1,2, \end{cases}$$

where

$$\varphi_1 \in \mathbb{B}_{p,p}^{1/p'}(S_1), \quad \varphi_2 \in \mathbb{B}_{p,p}^{1/p'}(S_2^{(1)}), \quad \varphi_3 \in \mathbb{B}_{p,p}^{1/p'}(S_2^{(2)}),$$
$$f_i \in \mathbb{B}_{p,p}^{-1/p}(S_0^{(i)}), \quad h_i \in \mathbb{B}_{p,p}^{1/p'}(S_0^{(i)}), \ \ i=1,2, \ \ 1<p<\infty.$$

2. Fundamental Solutions and Potentials

Consider the fundamental matrix-functions

$$\mathcal{H}^{(q)}(x) = \mathcal{F}^{-1}_{\xi'\to x'}\bigg(\pm\int_{\mathcal{L}_\pm}\big(\mathcal{M}^{(q)}(i\xi',i\tau)\big)^{-1}e^{-i\tau x_3}d\tau\bigg), \quad q=1,2,$$

where the sign "$-$" refers to the case $x_3>0$ and the sign "$+$" to the case $x_3<0$; $x=(x_1,x_2,x_3)$, $x'=(x_1,x_2)$, $\xi'=(\xi_1,\xi_2)$; $\int_{\mathcal{L}_\pm}$ denotes integration over the contour $\mathcal{L}^\pm$, where $\mathcal{L}_+$ ($\mathcal{L}_-$) has the positive orientation and covers all roots of the polynomial $\det\mathcal{M}^{(q)}(i\xi',i\tau)$ with respect to τ in the upper (resp. lower) τ-half-plane. $\mathcal{F}^{-1}$ is the inverse Fourier transform.

The simple-layer potentials are of the form

$$\mathbb{V}^{(1)}(g_1)(x) = \int_{\partial D_1}\mathcal{H}^{(1)}(x-y)g_1(y)d_yS, \quad x\notin\partial D_1,$$
$$\mathbb{V}^{(2)}(g_2)(x) = \int_{\partial D_2^{(1)}}\mathcal{H}^{(2)}(x-y)g_2(y)d_yS, \quad x\notin\partial D_2^{(1)},$$
$$\mathbb{V}^{(3)}(g_3)(x) = \int_{\partial D_2^{(2)}}\mathcal{H}^{(2)}(x-y)g_3(y)d_yS, \quad x\notin\partial D_2^{(2)}.$$

For these potentials the theorems below are valid.

Theorem 2.1. *Let $1<p<\infty$, $1\le r\le\infty$. Then the operators $\mathbb{V}^{(i)}$, $i=1,2,3$, admit extensions to the operators which are continuous in the following spaces:*

$$\mathbb{V}^{(1)} \ : \ \mathbb{B}^s_{p,r}(\partial D_1)\to\mathbb{B}^{s+1+1/p}_{p,r}(D_1) \ \big(\mathbb{B}^s_{p,p}(\partial D_1)\to\mathbb{H}^{s+1+1/p}_p(D_1)\big),$$
$$\mathbb{V}^{(2)} \ : \ \mathbb{B}^s_{p,r}(\partial D_2^{(1)})\to\mathbb{B}^{s+1+1/p}_{p,r}(D_2^{(1)}) \ \big(\mathbb{B}^s_{p,p}(\partial D_2^{(1)})\to\mathbb{H}^{s+1+1/p}_p(D_2^{(1)})\big),$$
$$\mathbb{V}^{(3)} \ : \ \mathbb{B}^s_{p,r}(\partial D_2^{(2)})\to\mathbb{B}^{s+1+1/p}_{p,r,loc}(D_2^{(2)}) \ \big(\mathbb{B}^s_{p,p}(\partial D_2^{(2)})\to\mathbb{H}^{s+1+1/p}_{p,loc}(D_2^{(2)})\big).$$

Theorem 2.2. *Let* $1 < p < \infty$, $1 \le r \le \infty$, $\varepsilon > 0$, $g_1 \in \mathbb{B}^{-1+\varepsilon}_{p,r}(\partial D_1)$, $g_2 \in \mathbb{B}^{-1+\varepsilon}_{p,r}(\partial D_2^{(1)})$, $g_3 \in \mathbb{B}^{-1+\varepsilon}_{p,r}(\partial D_2^{(2)})$. *Then*

$$\{\mathbb{V}^{(1)}(g_1)(z)\}^{\pm} = \int_{\partial D_1} \mathcal{H}^{(1)}(z-y)g_1(y)d_yS, \quad z \in \partial D_1,$$

$$\{\mathbb{V}^{(2)}(g_2)(z)\}^{\pm} = \int_{\partial D_2^{(1)}} \mathcal{H}^{(2)}(z-y)g_2(y)d_yS, \quad z \in \partial D_2^{(1)},$$

$$\{\mathbb{V}^{(3)}(g_3)(z)\}^{\pm} = \int_{\partial D_2^{(2)}} \mathcal{H}^{(2)}(z-y)g_3(y)d_yS, \quad z \in \partial D_2^{(2)}.$$

Theorem 2.3. *Let* $1 < p < \infty$, $g_1 \in \mathbb{B}^{-1/p}_{p,p}(\partial D_1)$, $g_2 \in \mathbb{B}^{-1/p}_{p,p}(\partial D_2^{(1)})$, $g_3 \in \mathbb{B}^{-1/p}_{p,p}(\partial D_2^{(2)})$. *Then*

$$\begin{aligned}\big\{\mathcal{N}^{(1)}\mathbb{V}^{(1)}(g_1)(z)\big\}^{\pm} &= \mp\frac{1}{2}\, g_1(z)\\ &\quad + \int_{\partial D_1} \mathcal{N}^{(1)}(\partial_z, n(z))\mathcal{H}^{(1)}(z-y)g_1(y)d_yS,\ z\in\partial D_1,\\ \big\{\mathcal{N}^{(2)}\mathbb{V}^{(2)}(g_2)(z)\big\}^{\pm} &= \pm\frac{1}{2}\, g_2(z)\\ &\quad + \int_{\partial D_2^{(1)}} \mathcal{N}^{(2)}(\partial_z, n(z))\mathcal{H}^{(2)}(z-y)g_2(y)d_yS, \quad z \in \partial D_2^{(1)},\\ \big\{\mathcal{N}^{(2)}\mathbb{V}^{(3)}(g_3)(z)\big\}^{\pm} &= \pm\frac{1}{2}\, g_3(z)\\ &\quad + \int_{\partial D_2^{(2)}} \mathcal{N}^{(2)}(\partial_z, n(z))\mathcal{H}^{(2)}(z-y)g_3(y)d_yS, \quad z \in \partial D_2^{(2)}.\end{aligned}$$

Let us introduce the following notation:

$$\begin{aligned}\mathbb{V}^{(1)}_{-1}(g_1)(z) &= \int_{\partial D_1} \mathcal{H}^{(1)}(z-y)g_1(y)d_yS, \quad z \in \partial D_1,\\ \mathbb{V}^{(2)}_{-1}(g_2)(z) &= \int_{\partial D_2^{(1)}} \mathcal{H}^{(2)}(z-y)g_2(y)d_yS, \quad z \in \partial D_2^{(1)},\\ \mathbb{V}^{(3)}_{-1}(g_3)(z) &= \int_{\partial D_2^{(2)}} \mathcal{H}^{(2)}(z-y)g_3(y)d_yS, \quad z \in \partial D_2^{(2)},\\ \overset{*}{\mathbb{V}}{}^{(1)}_{0}(g_1)(z) &= \int_{\partial D_1} \mathcal{N}^{(1)}(\partial_z, n(z))\mathcal{H}^{(1)}(z-y)g_1(y)d_yS, \quad z \in \partial D_1,\\ \overset{*}{\mathbb{V}}{}^{(2)}_{0}(g_2)(z) &= \int_{\partial D_2^{(1)}} \mathcal{N}^{(2)}(\partial_z, n(z))\mathcal{H}^{(2)}(z-y)g_2(y)d_yS, \quad z \in \partial D_2^{(1)},\\ \overset{*}{\mathbb{V}}{}^{(3)}_{0}(g_3)(z) &= \int_{\partial D_2^{(2)}} \mathcal{N}^{(2)}(\partial_z, n(z))\mathcal{H}^{(2)}(z-y)g_3(y)d_yS, \quad z \in \partial D_2^{(2)}.\end{aligned}$$

Theorem 2.4. *Let* $1 < p < \infty$, $1 \le r \le \infty$. *Then the operators* $\mathbb{V}^{(i)}_{-1}$, $\overset{*}{\mathbb{V}}{}^{(i)}_{0}$, $i = 1, 2, 3$, *admit extensions to the operators which are continuous in the* *following spaces:*

$$\mathbb{V}_{-1}^{(i)} \ : \ \mathbb{H}_p^s(\partial\Omega_i) \to \mathbb{H}_p^{s+1}(\partial\Omega_i)$$
$$\big(\mathbb{B}_{p,r}^s(\partial\Omega_i) \to \mathbb{B}_{p,r}^{s+1}(\partial\Omega_i)\big), \ \ i=1,2,3,$$
$$\overset{*}{\mathbb{V}}{}_0^{(i)} \ : \ \mathbb{H}_p^s(\partial\Omega_i) \to \mathbb{H}_p^{s}(\partial\Omega_i)$$
$$\big(\mathbb{B}_{p,r}^s(\partial\Omega_i) \to \mathbb{B}_{p,r}^{s}(\partial\Omega_i)\big), \ \ i=1,2,3,$$

here $\Omega_1 = D_1$, $\Omega_2 = D_2^{(1)}$, $\Omega_3 = D_2^{(2)}$.

3. Uniqueness, Existence and Smoothness Theorems for Problem $\mathbf{M}_1$

From the ellipticity of the differential operator $\mathcal{M}^{(2)}(\partial_x)$ it follows that any generalized solution of the equation

$$\mathcal{M}^{(2)}(\partial_x)\mathcal{U}^{(2)} = 0 \quad \text{in} \ \ D_2$$

is an analytic function in D_2 (see [17]). Then we see that the equalities

$$\begin{cases} \{r_1\mathcal{U}^{(2)}\}^+ - \{r_2\mathcal{U}^{(2)}\}^+ = 0 & \text{on} \ \ S_0, \\ \{\mathcal{N}^{(2)}(r_1\mathcal{U}^{(2)})\}^+ + \{\mathcal{N}^{(2)}(r_2\mathcal{U}^{(2)})\}^+ = 0 & \text{on} \ \ S_0 \end{cases} \tag{3.1}$$

are valid on S_0.

Let us study the uniqueness of a solution of the boundary-contact problem $\mathbf{M}_1$ in the classes $W_2^1(D_q)$, $q=1,2$ ($W_{2,loc}^1(D_2)$ with condition (1.6) at infinity).

Lemma 3.1. *A solution of the boundary-contact problem* $\mathbf{M}_1$ *is unique in the classes* $W_2^1(D_q)$, $q=1,2$ ($W_{2,loc}^1(D_2)$ *and satisfying condition* (1.6) *at infinity*).

Proof. Let $\mathcal{U}^{(q)}$, $q=1,2$, be a solution of the homogeneous problem $\mathbf{M}_1$.

We write the Green formulae (see (1.4), (1.5)) in the domains D_1, $D_2^{(1)}$, $D_2^{(2)}$ for the vector-functions $\mathcal{U}^{(1)}$, $r_1\mathcal{U}^{(2)}$, $r_2\mathcal{U}^{(2)}$ as

$$\begin{gathered} \int_{D_1} E^{(1)}(\mathcal{U}^{(1)},\mathcal{U}^{(1)})dx = \langle\{\mathcal{N}^{(1)}\mathcal{U}^{(1)}\}^+,\{\mathcal{U}^{(1)}\}^+\rangle_{\partial D_1}, \\ \int_{D_2^{(i)}} E^{(2)}(r_i\mathcal{U}^{(2)},r_i\mathcal{U}^{(2)})dx = -\langle\{\mathcal{N}^{(2)}(r_i\mathcal{U}^{(2)})\}^+,\{r_i\mathcal{U}^{(2)}\}^+\rangle_{\partial D_2^{(i)}}, \ \ i=1,2. \end{gathered} \tag{3.2}$$

Taking into account the boundary and boundary-contact conditions, the Green formulas (3.2) can be rewritten as

$$\begin{gathered} \int_{D_1} E^{(1)}(\mathcal{U}^{(1)},\mathcal{U}^{(1)})dx = \langle\{\mathcal{N}^{(1)}\mathcal{U}^{(1)}\}^+,\{\mathcal{U}^{(1)}\}^+\rangle_{S_0^{(1)}\cup S_0^{(2)}}, \\ \int_{D_2^{(i)}} E^{(2)}(r_i\mathcal{U}^{(2)},r_i\mathcal{U}^{(2)})dx = -\langle\{\mathcal{N}^{(2)}(r_i\mathcal{U}^{(2)})\}^+,\{r_i\mathcal{U}^{(2)}\}^+\rangle_{S_0^{(i)}\cup S_0}, \ i=1,2. \end{gathered} \tag{3.3}$$

Since equalities (3.1) are fulfilled for the function $\mathcal{U}^{(2)}$, we have

$$\langle\{\mathcal{N}^{(2)}(r_1\mathcal{U}^{(2)})\}^+,\{r_1\mathcal{U}^{(2)}\}^+\rangle_{S_0} = -\langle\{\mathcal{N}^{(2)}(r_2\mathcal{U}^{(2)})\}^+,\{r_2\mathcal{U}^{(2)}\}^+\rangle_{S_0}. \tag{3.4}$$

Summing now the Green formulas (3.3) and taking into account (3.4), we obtain

$$\int_{D_1} E^{(1)}(\mathcal{U}^{(1)},\mathcal{U}^{(1)})dx + \int_{D_2^{(1)}} E^{(2)}(r_1\mathcal{U}^{(2)}, r_1\mathcal{U}^{(2)})dx + \int_{D_2^{(2)}} E^{(2)}(r_2\mathcal{U}^{(2)}, r_2\mathcal{U}^{(2)})dx = 0. \tag{3.5}$$

From equality (3.5) and the positive-definiteness of forms (1.3) we have

$$\begin{cases} \partial_j u_i^{(q)} - \varepsilon_{ijk}\omega_k^{(q)} = 0, \\ \partial_j \omega_i^{(q)} = 0, \quad q = 1, 2. \end{cases}$$

Therefore

$$u^{(q)} = [a^{(1)} \times x] + b^{(q)}, \quad \omega^{(q)} = a^{(q)}, \quad q = 1, 2,$$

and

$$\mathcal{U}^{(q)} = \big([a^{(q)} \times x] + b^{(q)}, a^{(q)}\big), \quad q = 1, 2,$$

where $a^{(q)}$ and $b^{(q)}$, $q = 1, 2$, are arbitrary three-dimensional constant vectors.

By virtue of the contact conditions it is clear that

$$a^{(1)} = a^{(2)} \quad \text{and} \quad b^{(1)} = b^{(2)}.$$

Since $\{\mathcal{U}^{(1)}\}^+ = 0$ on S_1, we have

$$\mathcal{U}^{(q)}(x) = 0, \quad x \in D_q, \quad q = 1, 2. \qquad \square$$

Any extension $\Phi^{(1)} \in \mathbb{B}_{p,p}^{1/p'}(\partial D_1)$ of the function φ_1 onto the whole boundary ∂D_1 has the form

$$\Phi^{(1)} = \Phi_0^{(1)} + \varphi_0^{(1)} + \psi_0^{(1)},$$

where $\Phi_0^{(1)}$ is some fixed extension of φ_1 and $\varphi_0^{(1)} \in \widetilde{\mathbb{B}}_{p,p}^{1/p'}(S_0^{(1)})$, $\psi_0^{(1)} \in \widetilde{\mathbb{B}}_{p,p}^{1/p'}(S_0^{(2)})$.

Any extension $\Phi^{(2)} \in \mathbb{B}_{p,p}^{-1/p}(\partial D_2^{(1)})$ of the function φ_2 onto the whole boundary $\partial D_2^{(1)}$ has the form

$$\Phi^{(2)} = \Phi_0^{(2)} + \varphi_0^{(2)} + \psi_0^{(2)},$$

where $\Phi_0^{(2)}$ is some fixed extension of φ_2 and $\varphi_0^{(2)} \in \widetilde{\mathbb{B}}_{p,p}^{-1/p}(S_0^{(1)})$, $\psi_0^{(2)} \in \widetilde{\mathbb{B}}_{p,p}^{-1/p}(S_0)$.

Any extension $\Phi^{(3)} \in \mathbb{B}_{p,p}^{-1/p}(\partial D_2^{(2)})$ of the function φ_3 onto the whole boundary $\partial D_2^{(2)}$ has the form

$$\Phi^{(3)} = \Phi_0^{(3)} + \varphi_0^{(3)} + \psi_0^{(3)},$$

where $\Phi_0^{(3)}$ is some fixed extension of φ_3 and $\varphi_0^{(3)} \in \widetilde{\mathbb{B}}_{p,p}^{-1/p}(S_0^{(2)})$, $\psi_0^{(3)} \in \widetilde{\mathbb{B}}_{p,p}^{-1/p}(S_0)$.

A solution of the boundary-contact problem $\mathbf{M}_1$ will be sought for in the form of the simple-layer potentials

$$\mathcal{U}^{(1)} = \mathbb{V}^{(1)} g_1 \ \text{ in } \ D_1, \qquad \mathcal{U}^{(2)} = \begin{cases} \mathbb{V}^{(2)} g_2 & \text{in } \ D_2^{(1)}, \\ \mathbb{V}^{(3)} g_3 & \text{in } \ D_2^{(2)}. \end{cases}$$

Taking into account the boundary and boundary-contact conditions of problem $\mathbf{M}_1$ and equality (3.1) we obtain a system of equations with respect to $(g_1, g_2, g_3, \varphi_0^{(1)}, \psi_0^{(1)}, \varphi_0^{(2)}, \psi_0^{(2)}, \varphi_0^{(3)}, \psi_0^{(3)})$:

$$\begin{cases} \mathbb{V}_{-1}^{(1)} g_1 - \varphi_0^{(1)} - \psi_0^{(1)} = \Phi_0^{(1)} & \text{on } \partial D_1, \\ (\frac{1}{2}\mathcal{I} + \overset{*}{\mathbb{V}}{}_0^{(2)}) g_2 - \varphi_0^{(2)} - \psi_0^{(2)} = \Phi_0^{(2)} & \text{on } \partial D_2^{(1)}, \\ (\frac{1}{2}\mathcal{I} + \overset{*}{\mathbb{V}}{}_0^{(3)}) g_3 - \varphi_0^{(3)} - \psi_0^{(3)} = \Phi_0^{(3)} & \text{on } \partial D_2^{(2)}, \\ -\pi_{S_0^{(1)}} \mathbb{V}_{-1}^{(2)} g_2 + \varphi_0^{(1)} = f_1 - \pi_{S_0^{(1)}} \Phi_0^{(1)} & \text{on } S_0^{(1)}, \\ \pi_{S_0^{(1)}} (-\frac{1}{2}\mathcal{I} + \overset{*}{\mathbb{V}}{}_0^{(1)}) g_1 - \varphi_0^{(2)} = h_1 + \pi_{S_0^{(1)}} \Phi_0^{(2)} & \text{on } S_0^{(1)}, \\ -\pi_{S_0^{(2)}} \mathbb{V}_{-1}^{(3)} g_3 + \psi_0^{(1)} = f_2 - \pi_{S_0^{(2)}} \Phi_0^{(1)} & \text{on } S_0^{(2)}, \\ \pi_{S_0^{(2)}} (-\frac{1}{2}\mathcal{I} + \overset{*}{\mathbb{V}}{}_0^{(1)}) g_1 - \varphi_0^{(3)} = h_2 + \pi_{S_0^{(2)}} \Phi_0^{(3)} & \text{on } S_0^{(2)}, \\ \pi_{S_0} \mathbb{V}_{-1}^{(2)} g_2 - \pi_{S_0} \mathbb{V}_{-1}^{(3)} g_3 = 0 & \text{on } S_0, \\ \psi_0^{(2)} + \psi_0^{(3)} = -\pi_{S_0} \Phi_0^{(2)} - \pi_{S_0} \Phi_0^{(3)} & \text{on } S_0. \end{cases} \tag{3.6}$$

It is almost obvious that system (3.6) has a solution if and only if the compatibility conditions on ∂S_0

$$\exists \Phi_0^{(2)} \in \mathbb{B}_{p,r}^{s-1}(\partial D_2^{(1)}), \quad \Phi_0^{(3)} \in \mathbb{B}_{p,r}^{s-1}(\partial D_2^{(2)}) : \ \pi_{S_0} \Phi_0^{(2)} + \pi_{S_0} \Phi_0^{(3)} \in \widetilde{\mathbb{B}}_{p,r}^{s-1}(S_0) \tag{3.7}$$

hold for $\varphi_2 \in \mathbb{B}_{p,r}^{s-1}(S_2^{(1)})$, $\varphi_3 \in \mathbb{B}_{p,r}^{s-1}(S_2^{(2)})$, $\pi_{S_0}\Phi_0^{(2)} + \pi_{S_0}\Phi_0^{(3)} \in \mathbb{B}_{p,r}^{s-1}(S_0)$, $1 \le r \le \infty$, $1 < p < \infty$, $1/p - 1/2 < s < 1/p + 1/2$.

Note that these conditions hold automatically when $1/p - 1/2 < s < 1/p$ or $1/p < s < 1/p + 1/2$ (see [39]).

Denote by $\mathcal{A}$ the operator corresponding to system (3.6) and acting in the spaces

$$\mathcal{A} \ : \ \overset{(1)}{\mathcal{H}}{}_p^s \to \overset{(2)}{\mathcal{H}}{}_p^s \quad \big(\overset{(1)}{\mathcal{B}}{}_{p,r}^s \to \overset{(2)}{\mathcal{B}}{}_{p,r}^s \big),$$

where

$$\begin{aligned} \overset{(1)}{\mathcal{H}}{}_p^s = {} & \mathbb{H}_p^{s-1}(\partial D_1) \oplus \mathbb{H}_p^{s-1}(\partial D_2^{(1)}) \oplus \mathbb{H}_p^{s-1}(\partial D_2^{(2)}) \oplus \widetilde{\mathbb{H}}_p^s(S_0^{(1)}) \oplus \widetilde{\mathbb{H}}_p^s(S_0^{(2)}) \\ & \oplus \widetilde{\mathbb{H}}_p^{s-1}(S_0^{(1)}) \oplus \widetilde{\mathbb{H}}_p^{s-1}(S_0) \oplus \widetilde{\mathbb{H}}_p^{s-1}(S_0^{(2)}) \oplus \widetilde{\mathbb{H}}_p^{s-1}(S_0), \\ \overset{(2)}{\mathcal{H}}{}_p^s = {} & \mathbb{H}_p^s(\partial D_1) \oplus \mathbb{H}_p^{s-1}(\partial D_2^{(1)}) \oplus \mathbb{H}_p^{s-1}(\partial D_2^{(2)}) \oplus \mathbb{H}_p^s(S_0^{(1)}) \oplus \mathbb{H}_p^{s-1}(S_0^{(1)}) \\ & \oplus \mathbb{H}_p^s(S_0^{(2)}) \oplus \mathbb{H}_p^{s-1}(S_0^{(2)}) \oplus \mathbb{H}_p^s(S_0) \oplus \mathbb{H}_p^{s-1}(S_0), \\ \overset{(1)}{\mathcal{B}}{}_{p,r}^s = {} & \mathbb{B}_{p,r}^{s-1}(\partial D_1) \oplus \mathbb{B}_{p,r}^{s-1}(\partial D_2^{(1)}) \oplus \mathbb{B}_{p,r}^{s-1}(\partial D_2^{(2)}) \oplus \widetilde{\mathbb{B}}_{p,r}^s(S_0^{(1)}) \oplus \widetilde{\mathbb{B}}_{p,r}^s(S_0^{(2)}) \\ & \oplus \widetilde{\mathbb{B}}_{p,r}^{s-1}(S_0^{(1)}) \oplus \widetilde{\mathbb{B}}_{p,r}^{s-1}(S_0) \oplus \widetilde{\mathbb{B}}_{p,r}^{s-1}(S_0^{(2)}) \oplus \widetilde{\mathbb{B}}_{p,r}^{s-1}(S_0), \\ \overset{(2)}{\mathcal{B}}{}_{p,r}^s = {} & \mathbb{B}_{p,r}^s(\partial D_1) \oplus \mathbb{B}_{p,r}^{s-1}(\partial D_2^{(1)}) \oplus \mathbb{B}_{p,r}^{s-1}(\partial D_2^{(2)}) \oplus \mathbb{B}_{p,r}^s(S_0^{(1)}) \oplus \mathbb{B}_{p,r}^{s-1}(S_0^{(1)}) \\ & \oplus \mathbb{B}_{p,r}^s(S_0^{(2)}) \oplus \mathbb{B}_{p,r}^{s-1}(S_0^{(2)}) \oplus \mathbb{B}_{p,r}^s(S_0) \oplus \mathbb{B}_{p,r}^{s-1}(S_0); \end{aligned}$$

the symbol $\oplus$ denotes a direct sum of the spaces.

Consider the composition of the operators

$$\mathcal{D}\circ\mathcal{A},$$

where $\mathcal{D}$ is the invertible operator of the form

$$\mathcal{D}=\operatorname{diag}\{\mathcal{I},-\mathbb{V}_{-1}^{(2)},-\mathbb{V}_{-1}^{(3)},\mathcal{I},\dots,\mathcal{I}\}_{54\times 54}.$$

Consider now the operator

$$\mathcal{A}_M=\mathcal{T}_M+\mathcal{D}\circ\mathcal{A},\quad M=2,3,\dots,$$

where

$$\mathcal{T}_M=\operatorname{diag}\{0,(-\mathbb{V}_{-1}^{(2)})^M,(-\mathbb{V}_{-1}^{(3)})^M,0,\dots,0\}_{54\times 54}.$$

Since the operator $\mathcal{A}_M$ differs from the operator $\mathcal{D}\circ\mathcal{A}$ in a compact operator, it is sufficient to investigate the operator $\mathcal{A}_M$ acting in the spaces

$$\mathcal{A}_M\ :\ \overset{(1)}{\mathcal{H}}{}_p^s\to\overset{(3)}{\mathcal{H}}{}_p^s\quad\big(\overset{(1)}{\mathcal{B}}{}_{p,r}^s\to\overset{(3)}{\mathcal{B}}{}_{p,r}^s\big),$$

where

$$\begin{aligned}\overset{(3)}{\mathcal{H}}{}_p^s&=\mathbb{H}_p^s(\partial D_1)\oplus\mathbb{H}_p^s(\partial D_2^{(1)})\oplus\mathbb{H}_p^s(\partial D_2^{(2)})\oplus\mathbb{H}_p^s(S_0^{(1)})\oplus\mathbb{H}_p^{s-1}(S_0^{(1)})\\&\oplus\mathbb{H}_p^s(S_0^{(2)})\oplus\mathbb{H}_p^{s-1}(S_0^{(2)})\oplus\mathbb{H}_p^s(S_0)\oplus\mathbb{H}_p^{s-1}(S_0),\\\overset{(3)}{\mathcal{B}}{}_{p,r}^s&=\mathbb{B}_{p,r}^s(\partial D_1)\oplus\mathbb{B}_{p,r}^s(\partial D_2^{(1)})\oplus\mathbb{B}_{p,r}^s(\partial D_2^{(2)})\oplus\mathbb{B}_{p,r}^s(S_0^{(1)})\oplus\mathbb{B}_{p,r}^{s-1}(S_0^{(1)})\\&\oplus\mathbb{B}_{p,r}^s(S_0^{(2)})\oplus\mathbb{B}_{p,r}^{s-1}(S_0^{(2)})\oplus\mathbb{B}_{p,r}^s(S_0)\oplus\mathbb{B}_{p,r}^{s-1}(S_0).\end{aligned}$$

Now consider the system of equations that corresponds to the operator $\mathcal{A}_M$ given by

$$\begin{cases}\mathbb{V}_{-1}^{(1)}\widetilde{g}_1-\widetilde{\varphi}_0^{(1)}-\widetilde{\psi}_0^{(1)}=\Psi_0^{(1)} & \text{on }\ \partial D_1,\\ [(-\mathbb{V}_{-1}^{(2)})^M-\mathbb{V}_{-1}^{(2)}(\frac12\mathcal{I}+\overset{*}{\mathbb{V}}{}_0^{(2)})]\widetilde{g}_2+\mathbb{V}_{-1}^{(2)}\widetilde{\varphi}_0^{(2)}+\mathbb{V}_{-1}^{(2)}\widetilde{\psi}_0^{(2)}=\Psi_0^{(2)} & \text{on }\ \partial D_2^{(1)},\\ (-\mathbb{V}_{-1}^{(3)})^M-\mathbb{V}_{-1}^{(3)}(\frac12\mathcal{I}+\overset{*}{\mathbb{V}}{}_0^{(3)})\widetilde{g}_3+\mathbb{V}_{-1}^{(3)}\widetilde{\varphi}_0^{(3)}+\mathbb{V}_{-1}^{(3)}\widetilde{\psi}_0^{(3)}=\Psi_0^{(3)} & \text{on }\ \partial D_2^{(2)},\\ -\pi_{S_0^{(1)}}\mathbb{V}_{-1}^{(2)}\widetilde{g}_2+\widetilde{\varphi}_0^{(1)}=G_1 & \text{on }\ S_0^{(1)},\\ \pi_{S_0^{(1)}}(-\frac12\mathcal{I}+\overset{*}{\mathbb{V}}{}_0^{(1)})\widetilde{g}_1-\widetilde{\varphi}_0^{(2)}=G_2 & \text{on }\ S_0^{(1)},\\ -\pi_{S_0^{(2)}}\mathbb{V}_{-1}^{(3)}\widetilde{g}_3+\widetilde{\psi}_0^{(1)}=F_1 & \text{on }\ S_0^{(2)},\\ \pi_{S_0^{(2)}}(-\frac12\mathcal{I}+\overset{*}{\mathbb{V}}{}_0^{(1)})\widetilde{g}_1-\widetilde{\varphi}_0^{(3)}=F_2 & \text{on }\ S_0^{(2)},\\ \pi_{S_0}\mathbb{V}_{-1}^{(2)}\widetilde{g}_2-\pi_{S_0}\mathbb{V}_{-1}^{(3)}\widetilde{g}_3=E_1 & \text{on }\ S_0,\\ \widetilde{\psi}_0^{(2)}+\widetilde{\psi}_0^{(3)}=E_2 & \text{on }\ S_0,\end{cases}\tag{3.8}$$

where

$$\Psi_0^{(1)}\in\mathbb{H}_p^s(\partial D_1)\ \big(\mathbb{B}_{p,r}^s(\partial D_1)\big),\quad\Psi_0^{(2)}\in\mathbb{H}_p^s(\partial D_2^{(1)})\ \big(\mathbb{B}_{p,r}^s(\partial D_2^{(1)})\big),$$
$$\Psi_0^{(3)}\in\mathbb{H}_p^s(\partial D_2^{(2)})\ \big(\mathbb{B}_{p,r}^s(\partial D_2^{(2)})\big),$$

$$G_1 \in \mathbb{H}^s_p(S_0^{(1)}) \ \big(\mathbb{B}^s_{p,r}(S_0^{(1)})\big), \quad G_2 \in \mathbb{H}^{s-1}_p(S_0^{(1)}) \ \big(\mathbb{B}^{s-1}_{p,r}(S_0^{(1)})\big),$$
$$F_1 \in \mathbb{H}^s_p(S_0^{(2)}) \ \big(\mathbb{B}^s_{p,r}(S_0^{(2)})\big), \quad F_2 \in \mathbb{H}^{s-1}_p(S_0^{(2)}) \ \big(\mathbb{B}^{s-1}_{p,r}(S_0^{(2)})\big),$$
$$E_1 \in \mathbb{H}^s_p(S_0) \ \big(\mathbb{B}^s_{p,r}(S_0)\big), \quad E_2 \in \mathbb{H}^{s-1}_p(S_0) \ \big(\mathbb{B}^{s-1}_{p,r}(S_0)\big).$$

The ΨDO $-\mathbb{V}^{(1)}_{-1}$ is positive and the operators $-\mathbb{V}^{(i)}_{-1}(\frac{1}{2}\mathcal{I}+\overset{*}{\mathbb{V}}{}^{(i)}_0)$, $i=2,3$, are nonnegative, i.e.,

$$\langle -\mathbb{V}^{(1)}_{-1}\varphi,\varphi\rangle_{\partial D_1} > 0 \quad \text{for all} \ \ \varphi\in\mathbb{H}^{-1/2}_2(\partial D_1), \ \ \varphi\neq 0,$$
$$\Big\langle -\mathbb{V}^{(2)}_{-1}\Big(\frac{1}{2}\mathcal{I}+\overset{*}{\mathbb{V}}{}^{(2)}_0\Big)\psi,\psi\Big\rangle_{\partial D_2^{(1)}} \geq 0 \quad \text{for all} \ \ \psi\in\mathbb{H}^{-1/2}_2(\partial D_2^{(1)})$$

and

$$\Big\langle -\mathbb{V}^{(3)}_{-1}\Big(\frac{1}{2}\mathcal{I}+\overset{*}{\mathbb{V}}{}^{(3)}_0\Big)\psi,\psi\Big\rangle_{\partial D_2^{(2)}} > 0 \quad \text{for all} \ \ \psi\in\mathbb{H}^{-1/2}_2(\partial D_2^{(2)}),$$

the equality being fulfilled only when $\psi=([a\times x]+b,a)$, where a and b are arbitrary three-dimensional constant vectors.

The proof of these inequalities follows from the Green formulae (see [32]). Then the ΨDOs

$$\mathbf{B}^{(i)}_M = (-\mathbb{V}^{(i)}_{-1})^M - \mathbb{V}^{(i)}_{-1}\Big(\frac{1}{2}\mathcal{I}+\overset{*}{\mathbb{V}}{}^{(i)}_0\Big), \quad i=2,3,$$

are positive operators, i.e.,

$$\langle \mathbf{B}^{(2)}_M\varphi,\varphi\rangle_{\partial D_2^{(1)}} > 0 \quad \text{for all} \ \ \psi\in\mathbb{H}^{-1/2}_2(\partial D_2^{(1)}), \ \ \varphi\neq 0,$$
$$\langle \mathbf{B}^{(3)}_M\psi,\psi\rangle_{\partial D_2^{(2)}} > 0 \quad \text{for all} \ \ \psi\in\mathbb{H}^{-1/2}_2(\partial D_2^{(2)}), \ \ \psi\neq 0.$$

Hence the ΨDOs $\mathbb{V}^{(1)}_{-1}$ and $\mathbf{B}^{(i)}_M$, $i=2,3$, are invertible (which is proved as in [32], [7]). The first, second and third equations of system (3.8) imply

$$\widetilde{g}_1 = (\mathbb{V}^{(1)}_{-1})^{-1}\widetilde{\varphi}^{(1)}_0 + (\mathbb{V}^{(1)}_{-1})^{-1}\widetilde{\psi}^{(1)}_0 + (\mathbb{V}^{(1)}_{-1})^{-1}\Psi^{(1)}_0,$$
$$\widetilde{g}_i = -(\mathbf{B}^{(i)}_M)^{-1}\mathbb{V}^{(i)}_{-1}\widetilde{\varphi}^{(i)}_0 - (\mathbf{B}^{(i)}_M)^{-1}\mathbb{V}^{(i)}_{-1}\widetilde{\psi}^{(i)}_0 + (\mathbf{B}^{(i)}_M)^{-1}\Psi^{(i)}_0, \ \ i=2,3.$$

After substituting $\widetilde{g}_1$, $\widetilde{g}_2$, $\widetilde{g}_3$ into the remaining equations of system (3.8), we obtain a system of equations whose corresponding operator has the form

$$\mathcal{P} = \begin{pmatrix} \pi_{S_0^{(1)}}\mathbf{A}(x,D) & 0 & 0 \\ 0 & \pi_{S_0^{(2)}}\mathbf{B}(x,D) & 0 \\ 0 & 0 & \pi_{S_0}\mathbf{C}(x,D) \end{pmatrix}_{36\times 36} + \mathcal{T}_{-\infty},$$

where

$$\mathbf{A}(x,D) = \begin{pmatrix} \pi_{S_0^{(1)}}\mathbb{V}^{(2)}_{-1}(\mathbf{B}^{(2)}_M)^{-1}\mathbb{V}^{(2)}_{-1} & \mathcal{I} \\ -\mathcal{I} & \pi_{S_0^{(1)}}(-\frac{1}{2}\mathcal{I}+\overset{*}{\mathbb{V}}{}^{(1)}_0)(\mathbb{V}^{(1)}_{-1})^{-1} \end{pmatrix},$$

$$\mathbf{B}(x,D) = \begin{pmatrix} \pi_{S_0^{(2)}}\mathbb{V}^{(3)}_{-1}(\mathbf{B}^{(3)}_M)^{-1}\mathbb{V}^{(3)}_{-1} & \mathcal{I} \\ -\mathcal{I} & \pi_{S_0^{(2)}}(-\frac{1}{2}\mathcal{I}+\overset{*}{\mathbb{V}}{}^{(1)}_0)(\mathbb{V}^{(1)}_{-1})^{-1} \end{pmatrix},$$

$$\mathbf{C}(x,D) \;=\; \begin{pmatrix} -\pi_{S_0}\mathbb{V}_{-1}^{(2)}(\mathbf{B}_M^{(2)})^{-1}\mathbb{V}_{-1}^{(2)} & \pi_{S_0}\mathbb{V}_{-1}^{(3)}(\mathbf{B}_M^{(3)})^{-1}\mathbb{V}_{-1}^{(3)} \\ \mathcal{I} & \mathcal{I} \end{pmatrix}$$

and $\mathcal{T}_{-\infty}$ is the operator of order $-\infty$.

Further, after the localization, the operators $\mathbf{A}(x,D)$ and $\mathbf{B}(x,D)$ are reduced by means of lifting to the strongly elliptic ΨDOs of order 1, while the operator $\mathbf{C}(x,D)$ is reduced to positive-definite ΨDO.

Indeed, let $\mathbf{A}(x',D')$ and $\mathbf{B}(x',D')$ be ΨDOs with the symbols $\sigma_{\mathbf{A}}(x',\xi')$ and $\sigma_{\mathbf{B}}(x',\xi')$ ($\xi'=(\xi_1,\xi_2)$) "frozen" at the points and written in terms of some local coordinate system of the manifolds $S_0^{(1)}$ and $S_0^{(2)}$, respectively.

Denote

$$\mathbf{R}(x',D') = \begin{pmatrix} \mathbf{L}_- & 0 \\ 0 & \mathcal{I} \end{pmatrix} \circ \mathbf{A}(x',D') \circ \begin{pmatrix} \mathbf{L}_+ & 0 \\ 0 & \mathcal{I} \end{pmatrix}$$

and

$$\mathbf{Q}(x',D') = \begin{pmatrix} \mathbf{L}_- & 0 \\ 0 & \mathcal{I} \end{pmatrix} \circ \mathbf{B}(x',D') \circ \begin{pmatrix} \mathbf{L}_+ & 0 \\ 0 & \mathcal{I} \end{pmatrix},$$

where $\mathbf{L}_+ = \mathrm{diag}\Lambda_+$, $\mathbf{L}_- = \mathrm{diag}\pi_+\Lambda_-\ell$ are 6×6 matrix operators, $\Lambda_\pm$ is a ΨDO operator with the symbol $\Lambda_\pm(\xi') = \xi_2 \pm i \pm i|\xi_1|$, π_+ denotes the operator of restriction onto $\mathbb{R}^2_+$, and ℓ is an extension operator.

The operators

$$\begin{pmatrix} \mathbf{L}_\pm & 0 \\ 0 & \mathcal{I} \end{pmatrix}$$

are invertible in the respective spaces (see [39]).

The principal homogeneous symbols of the ΨDOs $\mathbf{R}(x',D')$ and $\mathbf{Q}(x',D')$ are written as

$$\sigma_{\mathbf{R}}(x',\xi') = \begin{pmatrix} (\xi_{n-1}-i|\xi''|)\sigma_{\mathbf{N}_2}(x',\xi')(\xi_{n-1}+i|\xi''|) & (\xi_{n-1}-i|\xi''|)\mathcal{I} \\ -(\xi_{n-1}+i|\xi''|)\mathcal{I} & \sigma_{\mathbf{N}_1}(x',\xi') \end{pmatrix}, \quad x'\in \overline{S}_0^{(1)},$$

$$\sigma_{\mathbf{Q}}(x',\xi') = \begin{pmatrix} (\xi_{n-1}-i|\xi''|)\sigma_{\mathbf{N}_3}(x',\xi')(\xi_{n-1}+i|\xi''|) & (\xi_{n-1}-i|\xi''|)\mathcal{I} \\ -(\xi_{n-1}+i|\xi''|)\mathcal{I} & \sigma_{\mathbf{N}_1}(x',\xi') \end{pmatrix}, \quad x'\in \overline{S}_0^{(2)},$$

where $\sigma_{\mathbf{N}_1}(x',\xi')$, $\sigma_{\mathbf{N}_2}(x',\xi')$ and $\sigma_{\mathbf{N}_3}(x',\xi')$ are the principal homogeneous symbols of the ΨDOs

$$\mathbf{N}_1 = \Big(-\frac{1}{2}\mathcal{I} + \overset{*}{\mathbb{V}}{}_0^{(1)}\Big)(\mathbb{V}_{-1}^{(1)})^{-1}, \quad \mathbf{N}_2 = \mathbb{V}_{-1}^{(2)}(\mathbf{B}_M^{(2)})^{-1}\mathbb{V}_{-1}^{(2)}, \quad \mathbf{N}_3 = \mathbb{V}_{-1}^{(3)}(\mathbf{B}_M^{(3)})^{-1}\mathbb{V}_{-1}^{(3)},$$

respectively, written in terms of a given local coordinate system, and $\mathcal{I}$ is the identity matrix.

Let $\lambda_{\mathbf{R}}^{(k)}$, $k=1,\dots,12$, be the eigenvalues of the matrix

$$\big(\sigma_{\mathbf{R}}(x_1,0,+1)\big)^{-1}\sigma_{\mathbf{R}}(x_1,0,-1), \quad x_1\in\partial S_0^{(1)},$$

where

$$\sigma_{\mathbf{R}}(x_1,0,-1) = \begin{pmatrix} \sigma_{\mathbf{N}_2}(x_1,0,-1) & -\mathcal{I} \\ \mathcal{I} & \sigma_{\mathbf{N}_1}(x_1,0,-1) \end{pmatrix},$$

$$\sigma_{\mathbf{R}}(x_1,0,+1)=\begin{pmatrix}\sigma_{\mathbf{N}_2}(x_1,0,+1) & \mathcal{I}\\ -\mathcal{I} & \sigma_{\mathbf{N}_1}(x_1,0,+1)\end{pmatrix},$$

and let $\lambda_{\mathbf{Q}}^{(k)}$, $k=1,\dots,12$, be the eigenvalues of the matrix

$$\big(\sigma_{\mathbf{Q}}(x_1,0,+1)\big)^{-1}\sigma_{\mathbf{Q}}(x_1,0,-1),\quad x_1\in\partial S_0^{(2)},$$

where

$$\sigma_{\mathbf{Q}}(x_1,0,-1)=\begin{pmatrix}\sigma_{\mathbf{N}_3}(x_1,0,-1) & -\mathcal{I}\\ \mathcal{I} & \sigma_{\mathbf{N}_1}(x_1,0,-1)\end{pmatrix},$$

$$\sigma_{\mathbf{Q}}(x_1,0,+1)=\begin{pmatrix}\sigma_{\mathbf{N}_3}(x_1,0,+1) & \mathcal{I}\\ -\mathcal{I} & \sigma_{\mathbf{N}_1}(x_1,0,+1)\end{pmatrix}$$

Introduce the notation

$$\delta_{\mathbf{R}}=\sup_{\substack{1\le j\le 12\\ x_1\in\partial S_0^{(1)}}}\Big|\frac{1}{2\pi}\arg\lambda_{\mathbf{R}}^{(j)}(x_1)\Big|,\quad \delta_{\mathbf{Q}}=\sup_{\substack{1\le j\le 12\\ x_1\in\partial S_0^{(2)}}}\Big|\frac{1}{2\pi}\arg\lambda_{\mathbf{Q}}^{(j)}(x_1)\Big|,$$

$$\delta=\max(\delta_{\mathbf{R}},\delta_{\mathbf{Q}}).$$

Using the general theory of pseudodifferential operators (ΨDOs) (see [35], [36], the following propositions are valid.

Lemma 3.2. *Let $1<p<\infty$, $1\le r\le\infty$, $1/p-1/2+\delta<s<1/p+1/2-\delta$. Then the operators*

$$\begin{aligned}\mathbf{R}(x',D') \;:\;& \widetilde{\mathbb{H}}_p^s(\mathbb{R}_+^2)\oplus\widetilde{\mathbb{H}}_p^s(\mathbb{R}_+^2)\to\mathbb{H}_p^{s-1}(\mathbb{R}_+^2)\oplus\mathbb{H}_p^{s-1}(\mathbb{R}_+^2)\\ &\big(\widetilde{\mathbb{B}}_{p,r}^s(\mathbb{R}_+^2)\oplus\widetilde{\mathbb{B}}_{p,r}^s(\mathbb{R}_+^2)\to\mathbb{B}_{p,r}^{s-1}(\mathbb{R}_+^2)\oplus\mathbb{B}_{p,r}^{s-1}(\mathbb{R}_+^2)\big),\\ \mathbf{Q}(x',D') \;:\;& \widetilde{\mathbb{H}}_p^s(\mathbb{R}_+^2)\oplus\widetilde{\mathbb{H}}_p^s(\mathbb{R}_+^2)\to\mathbb{H}_p^{s-1}(\mathbb{R}_+^2)\oplus\mathbb{H}_p^{s-1}(\mathbb{R}_+^2)\\ &\big(\widetilde{\mathbb{B}}_{p,r}^s(\mathbb{R}_+^2)\oplus\widetilde{\mathbb{B}}_{p,r}^s(\mathbb{R}_+^2)\to\mathbb{B}_{p,r}^{s-1}(\mathbb{R}_+^2)\oplus\mathbb{B}_{p,r}^{s-1}(\mathbb{R}_+^2)\big)\end{aligned}$$

are Fredholm.

Note that the ΨDOs $\mathbf{R}(x',D')$ and $\mathbf{Q}(x',D')$ are Fredholm in the anisotropic Bessel potential spaces with weight

$$\widetilde{\mathbb{H}}_p^{(\mu,s),k}(\mathbb{R}_+^2)\oplus\widetilde{\mathbb{H}}_p^{(\mu,s),k}(\mathbb{R}_+^2)\to\mathbb{H}_p^{(\mu,s-1),k}(\mathbb{R}_+^2)\oplus\mathbb{H}_p^{(\mu,s-1),k}(\mathbb{R}_+^2)$$

for all $\mu\in\mathbb{R}$ and $k=0,1,\dots$ (see [10]).

Since the operators $\pi_{S_0^{(1)}}\mathbf{N}_1$, $\pi_{S_0^{(1)}}\mathbf{N}_2$, $\pi_{S_0^{(2)}}\mathbf{N}_1$ and $\pi_{S_0^{(2)}}\mathbf{N}_2$ are positive-definite, we obtain a strong Gårding inequality for the operators $\mathbf{A}(x,D)$ and $\mathbf{B}(x,D)$ (see [6, Lemma 3.3]). Hence, using the results obtained in [2], [23], we have

Lemma 3.3. *Let $1<p<\infty$, $1\le r\le\infty$, $1/p-1/2+\delta<s<1/p+1/2-\delta$. Then the operators*

$$\begin{aligned}\pi_{S_0^{(1)}}\mathbf{A}(x,D) \;:\;& \widetilde{\mathbb{H}}_p^{s-1}(S_0^{(1)})\oplus\widetilde{\mathbb{H}}_p^s(S_0^{(1)})\to\mathbb{H}_p^s(S_0^{(1)})\oplus\mathbb{H}_p^{s-1}(S_0^{(1)})\\ &\big(\widetilde{\mathbb{B}}_{p,r}^{s-1}(S_0^{(1)})\oplus\widetilde{\mathbb{B}}_{p,r}^s(S_0^{(1)})\to\mathbb{B}_{p,r}^s(S_0^{(1)})\oplus\mathbb{B}_{p,r}^{s-1}(S_0^{(1)})\big)\\ \pi_{S_0^{(2)}}\mathbf{A}(x,D) \;:\;& \widetilde{\mathbb{H}}_p^{s-1}(S_0^{(2)})\oplus\widetilde{\mathbb{H}}_p^s(S_0^{(2)})\to\mathbb{H}_p^s(S_0^{(2)})\oplus\mathbb{H}_p^{s-1}(S_0^{(2)})\end{aligned}$$

$$\big(\widetilde{\mathbb{B}}^{s-1}_{p,r}(S_0^{(2)}) \oplus \widetilde{\mathbb{B}}^{s}_{p,r}(S_0^{(2)}) \to \mathbb{B}^{s}_{p,r}(S_0^{(2)}) \oplus \mathbb{B}^{s-1}_{p,r}(S_0^{(2)})\big)$$

are invertible.

Let us consider the operator $\mathbf{C}(x,D)$. The corresponding system

$$\pi_{S_0}\mathbf{C}(x,D)\begin{pmatrix} \varphi_0^{(2)} \\ \varphi_0^{(3)} \end{pmatrix} = \begin{pmatrix} \widetilde{E}_1 \\ E_2 \end{pmatrix}$$

is reduced to pseudodifferential equation on the open manifold S_0

$$\pi_{S_0}\mathbf{N}\widetilde{\psi}_0^{(3)} = \widetilde{E}_1, \qquad \widetilde{\psi}_0^{(2)} = -\widetilde{\psi}_0^{(3)} + E_2,$$

where $\mathbf{N} = \mathbf{N}_2 + \mathbf{N}_3$.

The ΨDO $\pi_{S_0}\mathbf{N}$ is positive-definite and the following proposition holds for it.

Lemma 3.4. *Let $1 < p < \infty$, $1 \le r \le \infty$, $1/p - 1/2 + \delta < s < 1/p + 1/2 - \delta$. Then the ΨDOs*

$$\begin{aligned} \pi_{S_0}\mathbf{N} \quad &: \quad \widetilde{\mathbb{H}}^{s-1}_p(S_0) \to \mathbb{H}^s_p(S_0) \\ &\quad \big(\widetilde{\mathbb{B}}^{s-1}_{p,r}(S_0) \to \mathbb{B}^s_{p,r}(S_0)\big) \end{aligned}$$

and

$$\begin{aligned} \pi_{S_0}\mathbf{C}(x,D) \quad &: \quad \widetilde{\mathbb{H}}^{s-1}_p(S_0) \oplus \widetilde{\mathbb{H}}^{s-1}_p(S_0) \to \mathbb{H}^s_p(S_0) \oplus \mathbb{H}^s_p(S_0) \\ &\quad \big(\widetilde{\mathbb{B}}^{s-1}_{p,r}(S_0) \oplus \widetilde{\mathbb{B}}^{s-1}_{p,r}(S_0) \to \mathbb{B}^s_{p,r}(S_0) \oplus \mathbb{B}^s_{p,r}(S_0)\big) \end{aligned}$$

are invertible.

Note that the ΨDO $\pi_{S_0}\mathbf{N}$ is invertible in anisotropic Bessel potential spaces with weight $\widetilde{\mathbb{H}}^{(\mu,s-1),k}_p \to \mathbb{H}^{(\mu,s),k}_p(S_0)$ (see [10]).

Lemmas 3.3 and 3.4 imply the validity of the following proposition.

Lemma 3.5. *Let $1 < p < \infty$, $1 \le r \le \infty$, $1/p - 1/2 + \delta < s < 1/p + 1/2 - \delta$. Then the operator*

$$\mathcal{P} \;:\; \begin{matrix} \widetilde{\mathbb{H}}^{s-1}_p(S_0^{(1)}) \oplus \widetilde{\mathbb{H}}^{s}_p(S_0^{(1)}) \\ \oplus \\ \widetilde{\mathbb{H}}^{s-1}_p(S_0^{(2)}) \oplus \widetilde{\mathbb{H}}^{s}_p(S_0^{(2)}) \\ \oplus \\ \widetilde{\mathbb{H}}^{s-1}_p(S_0) \oplus \widetilde{\mathbb{H}}^{s-1}_p(S_0) \end{matrix} \;\to\; \begin{matrix} \mathbb{H}^{s}_p(S_0^{(1)}) \oplus \mathbb{H}^{s-1}_p(S_0^{(1)}) \\ \oplus \\ \mathbb{H}^{s}_p(S_0^{(2)}) \oplus \mathbb{H}^{s-1}_p(S_0^{(2)}) \\ \oplus \\ \mathbb{H}^{s}_p(S_0) \oplus \mathbb{H}^{s}_p(S_0) \end{matrix}$$

$$\left(\begin{matrix} \widetilde{\mathbb{B}}^{s-1}_{p,r}(S_0^{(1)}) \oplus \widetilde{\mathbb{B}}^{s}_{p,r}(S_0^{(1)}) \\ \oplus \\ \widetilde{\mathbb{B}}^{s-1}_{p,r}(S_0^{(2)}) \oplus \widetilde{\mathbb{B}}^{s}_{p,r}(S_0^{(2)}) \\ \oplus \\ \widetilde{\mathbb{B}}^{s-1}_{p,r}(S_0) \oplus \widetilde{\mathbb{B}}^{s-1}_{p,r}(S_0) \end{matrix} \;\to\; \begin{matrix} \mathbb{B}^{s}_{p,r}(S_0^{(1)}) \oplus \widetilde{\mathbb{B}}^{s-1}_{p,r}(S_0^{(1)}) \\ \oplus \\ \mathbb{B}^{s}_{p,r}(S_0^{(2)}) \oplus \widetilde{\mathbb{B}}^{s-1}_{p,r}(S_0^{(2)}) \\ \oplus \\ \mathbb{B}^{s}_{p,r}(S_0) \oplus \mathbb{B}^{s}_{p,r}(S_0) \end{matrix} \right)$$

is invertible.

Theorem 3.6. *Let* $1 < p < \infty$, $1 \le r \le \infty$, $1/p - 1/2 + \delta < s < 1/p + 1/2 - \delta$, $M = 2, 3, \ldots$. *Then the operator*

$$\mathcal{A}_M \;:\; \overset{(1)}{\mathcal{H}}{}^s_p \to \overset{(3)}{\mathcal{H}}{}^s_p \quad \big(\overset{(1)}{\mathcal{B}}{}^s_{p,r} \to \overset{(3)}{\mathcal{B}}{}^s_{p,r} \big)$$

is invertible.

Lemma 2.1 and Theorem 3.6 imply that the following proposition is valid.

Theorem 3.7. *Let* $1 < p < \infty$, $1 \le r \le \infty$, $1/p - 1/2 + \delta < s < 1/p + 1/2 - \delta$. *Then the operator*

$$\mathcal{A} \;:\; \overset{(1)}{\mathcal{H}}{}^s_p \to \overset{(2)}{\mathcal{H}}{}^s_p \quad \big(\overset{(1)}{\mathcal{B}}{}^s_{p,r} \to \overset{(2)}{\mathcal{B}}{}^s_{p,r} \big)$$

is invertible.

If we take $s = 1/p'$ in the condition for the operator $\mathcal{A}$ to be invertible (see Theorem 3.7), we conclude that p must satisfy the equality

$$\frac{4}{3 - 2\delta} < p < \frac{4}{1 + 2\delta}.$$

Theorem 3.7 and the above reasoning imply the validity of the solution existence and uniqueness for problem $\mathbf{M}_1$.

Theorem 3.8. *Let* $4/(3 - 2\delta) < p < 4/(1 + 2\delta)$ *and the compatibility condition* (3.7) *be fulfilled for* $s = 1 - 1/p$. *Then the boundary-contact problem* $\mathbf{M}_1$ *has a unique solution in the classes* $W^1_p(D_q)$, $q = 1, 2$, ($W^1_{p,loc}(D_2)$ *with condition* (1.6) *at infinity*), *which is given by the formulae*

$$\mathcal{U}^{(1)} = \mathbb{V}^{(1)} g_1 \quad in \;\; D_1, \qquad \mathcal{U}^{(2)} = \begin{cases} \mathbb{V}^{(2)} g_2 & in \;\; D_2^{(1)}, \\ \mathbb{V}^{(3)} g_3 & in \;\; D_2^{(2)}, \\ \pi_{S_0} \mathbb{V}^{(2)}_{-1} g_2 = \pi_{S_0} \mathbb{V}^{(3)}_{-1} g_3 & on \;\; S_0, \end{cases}$$

where g_q, $q = 1, 2, 3$, *are obtained from system* (3.6).

Theorems 2.1, 3.7 and the embedding theorems (see [39]) imply

Theorem 3.9. *Let* $4/(3 - 2\delta) < p < 4/(1 + 2\delta)$, $1 \le t \le \infty$, $1 < r < \infty$, $1/r - 1/2 + \delta < s < 1/r + 1/2 - \delta$, *the compatibility condition* (3.7) *with* t *instead of* p *be fulfilled*, $\mathcal{U}^{(q)} \in W^1_p(D_q)$, $q = 1, 2$, ($\mathcal{U}^{(2)} \in W^1_{p,loc}(D_2)$ *with condition* (1.6) *at infinity*) *be a solution of the boundary-contact problem* $\mathbf{M}_1$. *Then:*

if $\varphi_1 \in \mathbb{B}^s_{r,r}(S_1)$, $\varphi_2 \in \mathbb{B}^{s-1}_{r,r}(S_2^{(1)})$, $\varphi_3 \in \mathbb{B}^{s-1}_{r,r}(S_2^{(2)})$, $f_i \in \mathbb{B}^s_{r,r}(S_0^{(i)})$, $h_i \in \mathbb{B}^{s-1}_{r,r}(S_0^{(i)})$, $i = 1, 2$, *we have* $\mathcal{U}^{(q)} \in \mathbb{H}^{s+1/r}_r(D_q)$, $q = 1, 2$, ($\mathcal{U}^{(2)} \in \mathbb{H}^{s+1/r}_{r,loc}(D_2)$);

if $\varphi_1 \in \mathbb{B}^s_{r,t}(S_1)$, $\varphi_2 \in \mathbb{B}^{s-1}_{r,t}(S_2^{(1)})$, $\varphi_3 \in \mathbb{B}^{s-1}_{r,t}(S_2^{(2)})$, $f_i \in \mathbb{B}^s_{r,t}(S_0^{(i)})$, $h_i \in \mathbb{B}^{s-1}_{r,t}(S_0^{(i)})$, $i = 1, 2$, *we have* $\mathcal{U}^{(q)} \in \mathbb{B}^{s+1/r}_{r,t}(D_q)$, $q = 1, 2$, ($\mathcal{U}^{(2)} \in \mathbb{B}^{s+1/r}_{r,t,loc}(D_2)$).

4. Asymptotics of Solutions

Now we will write the asymptotics of solutions of the boundary-contact problem $\mathbf{M}_1$. The boundary and contact data are assumed to be sufficiently smooth, i.e.,

$$\varphi_1 \in \mathbb{H}_r^{(\infty,s+2M+1),\infty}(S_1), \quad \varphi_2 \in \mathbb{H}_r^{(\infty,s+2M),\infty}(S_2^{(1)}), \quad \varphi_3 \in \mathbb{H}_r^{(\infty,s+2M),\infty}(S_2^{(2)}),$$
$$f_i \in \mathbb{H}_r^{(\infty,s+2M+1),\infty}(S_0^{(i)}), \quad h_i \in \mathbb{H}_r^{(\infty,s+2M),\infty}(S_0^{(i)}), \quad i=1,2;$$

here the numbers r and s satisfy the conditions of Theorem 3.9.

Let $m_1,\dots,m_{2\ell}$ be algebraic multiplicities of the eigenvalues $\lambda_1,\dots,\lambda_{2\ell}$, $\sum_{j=1}^{2\ell} m_j = 12$, where $\lambda_j = \lambda_{\mathbf{R}}^{(j)}$, $j=1,\dots,2\ell$.

We introduce the notation

$$b_{\mathbf{R}}(x_1) = \big(\sigma_{\mathbf{R}}(x_1,0,+1)\big)^{-1}\sigma_{\mathbf{R}}(x_1,0,-1).$$

Let

$$b_{0\mathbf{R}}(x_1) = \mathcal{K}^{-1}(x_1) b_{\mathbf{R}}(x_1)\mathcal{K}(x_1), \quad x_1 \in \partial S_0^{(1)},$$

be a quasi-diagonal form, where $\mathcal{K}$ is some nondegenerate matrix function $\det\mathcal{K}(x_1)\neq 0$, and $\mathcal{K}\in C^\infty$ (see [20]).

The asymptotics of solutions to a strongly elliptic pseudodifferential equation (see [10, Theorem 2.1]) implies the asymptotics of solutions of the pseudodifferential equation

$$\mathbf{R}(x',D')\chi = F, \quad F \in \mathbb{H}_{r,comp}^{(\infty,s+M),\infty}(\mathbb{R}_+^2)\times \mathbb{H}_{r,comp}^{(\infty,s+M),\infty}(\mathbb{R}_+^2)$$

written in terms of some local coordinate system of the manifold $S_0^{(1)}$.

Thus we obtain an asymptotic expansion of the solution $\chi = (\chi_1,\chi_2)^\top$

$$\chi(x_1,x_{2,+}) = \mathcal{K}(x_1) x_{2,+}^{1/2+\Delta(x_1)} \mathbb{B}_{a_{pr}}^0\Big(-\frac{1}{2\pi i}\log x_{2,+}\Big)\mathcal{K}^{-1}(x_1)c_0(x_1)$$
$$+\sum_{k=1}^{M}\mathcal{K}(x_1)x_{2,+}^{1/2+\Delta(x_1)+k}\mathbb{B}_k(x_1,\log x_{2,+}) + \chi_{M+1}(x_1,x_{2,+}), \tag{4.1}$$

for all sufficiently small $x_{2,+}>0$, $\chi_{M+1}\in\widetilde{\mathbb{H}}_{r,comp}^{(\infty,s+M+1),\infty}(\mathbb{R}_+^2)\times\widetilde{\mathbb{H}}_{r,comp}^{(\infty,s+M+1),\infty}(\mathbb{R}_+^2)$; $\mathbb{B}_{a_{pr}}^0(t)$ is defined in [10]; the 12×12 matrix function $\mathbb{B}_k(x_1,t)$ is a polynomial of order $\nu_k = k(2m_0-1)+m_0-1$, $m_0=\max\{m_1,\dots,m_{2\ell}\}$ with respect to the variable t with 12-dimensional vector coefficients which depend on the variable x_1 and

$$\Delta(x_1) = (\Delta_1(x_1),\Delta_2(x_1));$$

here

$$\Delta_j(x_1) = (\underbrace{\delta_1^{(j)}(x_1),\dots,\delta_1^{(j)}(x_1)}_{m_1\text{-times}},\dots,\underbrace{\delta_\ell^{(j)}(x_1),\dots,\delta_\ell^{(j)}(x_1)}_{m_\ell\text{-times}}), \quad j=1,2,$$
$$\delta_k^{(1)}(x_1) = \frac{1}{2\pi}\arg\lambda_k(x_1) - \frac{i}{2\pi}|\lambda_k(x_1)|,$$
$$\delta_k^{(2)}(x_1) = -\frac{1}{2\pi}\arg\lambda_k(x_1) - \frac{i}{2\pi}|\lambda_k(x_1)|, \quad k=1,\dots,\ell.$$

Without loss of generality suppose that the matrix $\mathbb{B}^0_{a_{pr}}(t)$ has the form

$$\mathbb{B}^0_{a_{pr}}(t) = \operatorname{diag}\{B^0_{a_{pr}}(t), B^0_{a_{pr}}(t)\};$$

here $B^0_{a_{pr}}(t)$ is the upper triangular block-diagonal 6×6-matrix-function defined in [9].

Hence for the functions χ_1 and χ_2 we can write an asymptotic expansion. Indeed, let

$$\mathcal{K}(x_1) = \begin{pmatrix} \mathcal{K}_{11}(x_1) & \mathcal{K}_{12}(x_1) \\ \mathcal{K}_{21}(x_1) & \mathcal{K}_{22}(x_1) \end{pmatrix}_{12\times 12}$$

and

$$\mathcal{K}^{-1}(x_1)c_0(x_1) = (c_0^{(1)}(x_1), c_0^{(2)}(x_1))^\top, \tag{4.2}$$

where $\mathcal{K}_{ij}(x_1)$, $i,j=1,2$, are 6×6-matrices, $c_0^{(i)}$, $i=1,2$, are six-dimensional vector functions. Then

$$\chi_i(x_1,x_{2,+}) = \sum_{j=1}^{2} \mathcal{K}_{ij}(x_1) x_{2,+}^{1/2+\Delta_j(x_1)} B^0_{a_{pr}}\Big(-\frac{1}{2\pi i}\log x_{2,+}\Big) c_0^{(i)}(x_1)$$

$$+\sum_{j=1}^{2}\sum_{k=1}^{M} \mathcal{K}_{ij}(x_1) x_{2,+}^{1/2+\Delta_j(x_1)+k} B^{(i)}_{kj}(x_1,\log x_{2,+}) + \chi^{(i)}_{M+1}(x_1,x_{2,+}), \quad i=1,2, \tag{4.3}$$

where $\chi^{(i)}_{M+1} \in \widetilde{\mathbb{H}}^{(\infty,s+M+1)}_{r,comp}(\mathbb{R}^2_+)$ and $B^{(i)}_{kj}(x_1,t)$ is a polynomial of order $\nu_k = k(2m_0-1)+m_0-1$ with respect to the variable t with six-dimensional vector coefficients which depend on the variable x_1.

We can also obtain an analogous asymptotic expansion of the solution $\widetilde{\chi} = (\widetilde{\chi}_1, \widetilde{\chi}_2)$ of the strongly elliptic equation

$$\mathbf{Q}(x',D')\widetilde{\chi} = \widetilde{F}, \quad \widetilde{F} \in \mathbb{H}^{(\infty,s+M),\infty}_{r,comp}(\mathbb{R}^2_+) \times \mathbb{H}^{(\infty,s+M),\infty}_{r,comp}(\mathbb{R}^2_+)$$

in terms of some local coordinate system on the manifold $S_0^{(2)}$. Indeed, we have

$$\widetilde{\chi}_i(x_1,x_{2,+}) = \sum_{j=1}^{2} \mathcal{K}_{ij}(x_1) x_{2,+}^{1/2+\Delta_j(x_1)} B^0_{a_{pr}}\Big(-\frac{1}{2\pi i}\log x_{2,+}\Big) b_0^{(i)}(x_1)$$

$$+\sum_{j=1}^{2}\sum_{k=1}^{M} \mathcal{K}_{ij}(x_1) x_{2,+}^{1/2+\widetilde{\Delta}_j(x_1)+k} B^{(i)}_{kj}(x_1,\log x_{2,+}) + \widetilde{\chi}^{(i)}_{M+1}(x_1,x_{2,+}), \quad i=1,2, \tag{4.4}$$

where $\widetilde{\chi}^{(i)}_{M+1} \in \widetilde{\mathbb{H}}^{(\infty,s+M+1),\infty}_{r,comp}(\mathbb{R}^2_+)$, $\widetilde{\Delta}_j$, $j=1,2$, are defined as Δ_j, $j=1,2$, by means of the eigenvalues $\lambda^{(k)}_{\mathbf{Q}}$, $k=1,\dots,12$, of the matrix $b_{\mathbf{Q}}$.

Let us consider the pseudodifferential equation

$$\pi_{S_0}\mathbf{N}\widetilde{\psi}_0^{(3)} = \widetilde{E}_1 \quad\text{and}\quad \widetilde{\psi}_0^{(2)} = -\widetilde{\psi}_0^{(3)} + E_2,$$

where

$$\mathbf{N} = \mathbb{V}^{(2)}_{-1}(\mathbf{B}^{(2)}_M)^{-1}\mathbb{V}^{(2)}_{-1} + \mathbb{V}^{(3)}_{-1}(\mathbf{B}^{(3)}_M)^{-1}\mathbb{V}^{(3)}_{-1}.$$

The following equalities hold for the principal homogeneous symbols of the operators $\mathbb{V}_{-1}^{(i)}$ and $\overset{*}{\mathbb{V}}{}_0^{(i)}$, $i=2,3$:

$$\begin{aligned} &\sigma_{\mathbb{V}_{-1}^{(2)}}(x',\xi') = \sigma_{\mathbb{V}_{-1}^{(3)}}(x',\xi') \quad \text{for } \ x'\in \overline{S}_0,\\ &\sigma_{\overset{*}{\mathbb{V}}{}_0^{(2)}}(x',\xi') = -\sigma_{\overset{*}{\mathbb{V}}{}_0^{(3)}}(x',\xi') \quad \text{for } \ x'\in \overline{S}_0. \end{aligned} \tag{4.5}$$

In view of equality (4.5) we can write the symbol $\sigma_{\mathbf{N}}(x',\xi')$ of the ΨDO operator $\mathbf{N}$ as follows:

$$\begin{aligned} \sigma_{\mathbf{N}}(x',\xi') = \Big[\Big(\frac12\mathcal{I} + \sigma_{\overset{*}{\mathbb{V}}{}_0^{(2)}}(x',\xi')\Big)\big(\sigma_{-\mathbb{V}_{-1}^{(2)}}(x',\xi')\big)^{-1}\Big]^{-1}&\\ + \Big[\Big(\frac12\mathcal{I} - \sigma_{\overset{*}{\mathbb{V}}{}_0^{(2)}}(x',\xi')\Big)\big(\sigma_{-\mathbb{V}_{-1}^{(2)}}(x',\xi')\big)^{-1}\Big]^{-1}&. \end{aligned}$$

Since the symbol $\sigma_{\overset{*}{\mathbb{V}}{}_0^{(2)}}(x',\xi')$ is an odd matrix function with respect to ξ', while the symbol $\sigma_{-\mathbb{V}_{-1}^{(2)}}(x',\xi')$ is an even matrix function, one can easily ascertain that the symbol $\sigma_{\mathbf{N}}(x',\xi')$ is even with respect to the variable ξ', i.e.,

$$\sigma_{\mathbf{N}}(x',-\xi') = \sigma_{\mathbf{N}}(x',\xi'), \quad x'\in\overline{S}_0,$$

and all eigenvalues of the matrix

$$\big(\sigma_{\mathbf{N}}(x_1,0,+1)\big)^{-1}\sigma_{\mathbf{N}}(x_1,0,-1) = \mathcal{I}, \quad x_1\in\partial S_0,$$

are trivial $\lambda_{\mathbf{N}}^{(j)}=1$, $j=1,\dots,6$.

Applying the result on strongly elliptic pseudodifferential equations (see [10, Theorem 2.1]), we obtain, in terms of some local coordinate system, the following result on asymptotic expansion of the functions $\psi_0^{(i)}$, $i=2,3$:

$$\psi_0^{(i)}(x_1,x_{2,+}) = (-1)^{i+1}c_0(x_1)x_{2,+}^{-1/2} + \sum_{k=1}^{M} x_{2,+}^{-1/2+k} d_k^{(i)}(x_1) + \psi_{M+1}^{(i)}(x_1,x_{2,+}), \tag{4.6}$$

where $c_0, d_k^{(i)}\in C_0^\infty(\mathbb{R})$, and the remainder $\psi_{M+1}^{(i)}\in \widetilde{\mathbb{H}}^{(\infty,s+M+1),\infty}_{r,comp}(\mathbb{R}^2_+)$, $i=2,3$, $M\in\mathbb{N}$. As we can see from (4.6), due to the properties of the symbol $\sigma_{\mathbf{N}}(x',\xi')$ (see [12]) there are no logarithms in the entire asymptotic expansion.

Let $g=(g_1,g_2,g_3,\varphi_0^{(1)},\psi_0^{(1)}\varphi_0^{(2)},\psi_0^{(2)},\varphi_0^{(3)},\psi_0^{(3)})$ be a solution of system (3.6), i.e.,

$$\mathcal{A}g = \Phi,$$

where

$$\begin{aligned} \Phi = \big(\Phi_0^{(1)},\Phi_0^{(2)},\Phi_0^{(3)}, f_1 - \pi_{S_0^{(1)}}\Phi_0^{(1)}, h_1 + \pi_{S_0^{(1)}}\Phi_0^{(2)}, f_2 - \pi_{S_0^{(2)}}\Phi_0^{(1)},&\\ h_2 + \pi_{S_0^{(2)}}\Phi_0^{(3)}, 0, -(\pi_{S_0}\Phi_0^{(2)} + \pi_{S_0}\Phi_0^{(3)})\big)&. \end{aligned}$$

Then

$$\mathcal{D}\circ\mathcal{A}g = \Psi; \tag{4.7}$$

here

$$\Psi = \big(\Phi_0^{(1)}, -\mathbb{V}_{-1}^{(2)}\Phi_0^{(2)}, -\mathbb{V}_{-1}^{(3)}\Phi_0^{(3)}, f_1 - \pi_{S_0^{(1)}}\Phi_0^{(1)}, h_1 + \pi_{S_0^{(1)}}\Phi_0^{(2)}, f_2 - \pi_{S_0^{(2)}}\Phi_0^{(1)},$$

$$h_2 + \pi_{S_0^{(2)}}\Phi_0^{(3)}, 0, -\big(\pi_{S_0}\Phi_0^{(2)} + \pi_{S_0}\Phi_0^{(3)}\big)\big).$$

Now adding the expression

$$\mathcal{T}_{2M+1}g = \operatorname{diag}\big\{0, -(\mathbb{V}_{-1}^{(2)})^{2M+1}, -(\mathbb{V}_{-1}^{(3)})^{2M+1}, 0, \dots, 0\big\}g$$

to both parts of system (4.7), we obtain the equality

$$\mathcal{A}_{2M+1}g = \widetilde{\Psi}, \tag{4.8}$$

where

$$\widetilde{\Psi} = \big(\Phi_0^{(1)}, -\mathbb{V}_{-1}^{(2)}\Phi_0^{(2)} - (\mathbb{V}_{-1}^{(2)})^{2M+1}g_2, -\mathbb{V}_{-1}^{(2)}\Phi_0^{(3)} - (\mathbb{V}_{-1}^{(3)})^{2M+1}g_3, f_1 - \pi_{S_0^{(1)}}\Phi_0^{(1)},$$
$$h_1 + \pi_{S_0^{(1)}}\Phi_0^{(2)}, f_2 - \pi_{S_0^{(2)}}\Phi_0^{(1)}, h_2 + \pi_{S_0^{(2)}}\Phi_0^{(3)}, 0, -\big(\pi_{S_0}\Phi_0^{(2)} + \pi_{S_0}\Phi_0^{(3)}\big)\big).$$

Thus $(-\mathbf{L}_+^{-1}\varphi_0^{(2)}, \varphi_0^{(1)})$ satisfies, in some local coordinate system of the manifold $S_0^{(1)}$, the pseudodifferential equation

$$\mathbf{R}(x', D')\begin{pmatrix}\chi_1\\ \chi_2\end{pmatrix} = F,$$

and $(-\mathbf{L}_+^{-1}\psi_0^{(3)}, \varphi_0^{(1)})$ satisfies, in some local coordinate system of the manifold $S_0^{(2)}$, the pseudodifferential equation

$$\mathbf{Q}(x', D')\begin{pmatrix}\widetilde{\chi}_1\\ \widetilde{\chi}_2\end{pmatrix} = \widetilde{F},$$

where

$$F = (\mathbf{L}_- F_1, F_2),$$
$$F_1 = f_1 - \pi_{S_0^{(1)}}\Phi_0^{(1)} - \pi_{S_0^{(1)}}\mathbb{V}_{-1}^{(2)}(\mathbf{B}_{2M+1}^{(2)})^{-1}\mathbb{V}_{-1}^{(2)}\Phi_0^{(2)}$$
$$-\pi_{S_0^{(1)}}\mathbb{V}_{-1}^{(2)}(\mathbf{B}_{2M+1}^{(2)})^{-1}\mathbb{V}_{-1}^{(2)}\psi_0^{(2)} - \pi_{S_0^{(1)}}\mathbb{V}_{-1}^{(2)}(\mathbf{B}_{2M+1}^{(2)})^{-1}(\mathbb{V}_{-1}^{(2)})^{2M+1}g_2,$$
$$F_2 = h_1 + \pi_{S_0^{(1)}}\Phi_0^{(2)} - \pi_{S_0^{(1)}}\Big(-\frac{1}{2}\mathcal{I} + \overset{*}{\mathbb{V}}{}_0^{(1)}\Big)(\mathbb{V}_{-1}^{(1)})^{-1}\Phi_0^{(1)}$$
$$-\pi_{S_0^{(1)}}\Big(-\frac{1}{2}\mathcal{I} + \overset{*}{\mathbb{V}}{}_0^{(1)}\Big)(\mathbb{V}_{-1}^{(1)})^{-1}\psi_0^{(1)}$$

and

$$\widetilde{F} = (\mathbf{L}_-\widetilde{F}_1, \widetilde{F}_2),$$
$$\widetilde{F}_1 = f_2 - \pi_{S_0^{(2)}}\Phi_0^{(1)} + \pi_{S_0^{(2)}}\mathbb{V}_{-1}^{(3)}(\mathbf{B}_{2M+1}^{(3)})^{-1}\mathbb{V}_{-1}^{(3)}\Phi_0^{(3)}$$
$$-\pi_{S_0^{(2)}}\mathbb{V}_{-1}^{(3)}(\mathbf{B}_{2M+1}^{(3)})^{-1}(\mathbb{V}_{-1}^{(3)})^{2M+1}\psi_0^{(3)} - \pi_{S_0^{(2)}}\mathbb{V}_{-1}^{(3)}(\mathbf{B}_{2M+1}^{(3)})^{-1}(\mathbb{V}_{-1}^{(3)})^{2M+1}g_3,$$
$$\widetilde{F}_2 = h_2 + \pi_{S_0^{(2)}}\Phi_0^{(3)} - \pi_{S_0^{(2)}}\Big(-\frac{1}{2}\mathcal{I} + \overset{*}{\mathbb{V}}{}_0^{(1)}\Big)(\mathbb{V}_{-1}^{(1)})^{-1}\Phi_0^{(1)}$$
$$-\pi_{S_0^{(2)}}\Big(-\frac{1}{2}\mathcal{I} + \overset{*}{\mathbb{V}}{}_0^{(1)}\Big)(\mathbb{V}_{-1}^{(1)})^{-1}\varphi_0^{(1)};$$

here

$$F_i \in \mathbb{H}_{r,comp}^{(\infty,s+2M),\infty}(\mathbb{R}_+^2), \quad \widetilde{F}_i \in \mathbb{H}_{r,comp}^{(\infty,s+2M),\infty}(\mathbb{R}_+^2), \quad i = 1, 2.$$

Further, we have that $(\psi_0^{(2)}, \psi_0^{(3)})$ is a solution of the system

$$\pi_{S_0} C(x,D) \begin{pmatrix} \psi_0^{(2)} \\ \psi_0^{(3)} \end{pmatrix} = \begin{pmatrix} 0 \\ -(\pi_{S_0}\Phi_0^{(2)} + \pi_{S_0}\Phi_0^{(3)}) \end{pmatrix}.$$

This system can be reduced to a pseudodifferential equation with the positive-definite operator

$$\pi_{S_0}\mathbf{N}\psi_0^{(3)} = E_1, \quad \psi_0^{(2)} = -\psi_0^{(3)} - (\pi_{S_0}\Phi_0^{(2)} + \pi_{S_0}\Phi_0^{(3)}),$$

where

$$\mathbf{N} = \mathbb{V}_{-1}^{(2)}(\mathbf{B}_{2M+1}^{(2)})^{-1}\mathbb{V}_{-1}^{(2)} + \mathbb{V}_{-1}^{(3)}(\mathbf{B}_{2M+1}^{(3)})^{-1}\mathbb{V}_{-1}^{(3)}$$

and

$$\begin{aligned} E_1 = &-\pi_{S_0}\mathbb{V}_{-1}^{(2)}(\mathbf{B}_{2M+1}^{(2)})^{-1}\mathbb{V}_{-1}^{(2)}\Phi_0^{(2)} - \pi_{S_0}\mathbb{V}_{-1}^{(2)}(\mathbf{B}_{2M+1}^{(2)})^{-1}(\mathbb{V}_{-1}^{(2)})^{2M+1}g_2 \\ &+ \pi_{S_0}\mathbb{V}_{-1}^{(2)}(\mathbf{B}_{2M+1}^{(2)})^{-1}\mathbb{V}_{-1}^{(2)}\varphi_0^{(2)} - \pi_{S_0}\mathbb{V}_{-1}^{(2)}(\mathbf{B}_{2M+1}^{(2)})^{-1}\mathbb{V}_{-1}^{(2)}(\pi_{S_0}\Phi_0^{(2)} + \pi_{S_0}\Phi_0^{(3)}); \end{aligned}$$

here

$$E_1 \in \mathbb{H}_{r,comp}^{(\infty,s+2M),\infty}(\mathbb{R}_+^2).$$

Hence we can obtain asymptotic expansions (4.3), (4.4) and (4.6) for the functions $-\mathbf{L}_+^{-1}\varphi_0^{(2)}$, $\varphi_0^{(2)}$, $-\mathbf{L}_+^{-1}\psi_0^{(3)}$, $\varphi_0^{(1)}$ and $\psi_0^{(3)}$, $\psi_0^{(2)}$, respectively.

We now define g_1, g_2 and g_3 by the first three equations of system (4.8)

$$g_1 = (\mathbb{V}_{-1}^{(1)})^{-1}\varphi_0^{(1)} + (\mathbb{V}_{-1}^{(1)})^{-1}\psi_0^{(1)} + (\mathbb{V}_{-1}^{(1)})^{-1}\Phi_0^{(1)}, \tag{4.9}$$

$$\begin{aligned} g_i = &-(\mathbf{B}_{2M+1}^{(i)})^{-1}\mathbb{V}_{-1}^{(i)}\varphi_0^{(i)} - (\mathbf{B}_{2M+1}^{(i)})^{-1}\mathbb{V}_{-1}^{(i)}\psi_0^{(i)} \\ &+ (\mathbf{B}_{2M+1}^{(i)})^{-1}\Phi_0^{(i)} + G_i, \quad i = 2,3, \end{aligned} \tag{4.10}$$

where

$$G_i = (\mathbf{B}_{2M+1}^{(i)})^{-1}(-\mathbb{V}_{-1}^{(i)})^{2M+1}g_i, \quad i = 2,3,$$
$$G_2 \in \mathbb{H}_r^{(\infty,s+2M),\infty}(\partial D_2^{(1)}), \quad G_3 \in \mathbb{H}_r^{(\infty,s+2M),\infty}(\partial D_2^{(2)}).$$

Using expansions (4.9) and (4.10), we obtain the following representation, i.e., the solutions of the boundary-contact problem $\mathbf{M}_1$ are expressed by the potential-type functions

$$\mathcal{U}^{(1)} = \mathbb{V}^{(1)}(\mathbb{V}_{-1}^{(1)})^{-1}\varphi_0^{(1)} + \mathbb{V}^{(1)}(\mathbb{V}_{-1}^{(1)})^{-1}\psi_0^{(1)} + R_1, \tag{4.11}$$

$$r_1\mathcal{U}^{(2)} = -\mathbb{V}^{(2)}(\mathbf{B}_{2M+1}^{(2)})^{-1}\mathbb{V}_{-1}^{(2)}\varphi_0^{(2)} - \mathbb{V}^{(2)}(\mathbf{B}_{2M+1}^{(2)})^{-1}\mathbb{V}_{-1}^{(2)}\psi_0^{(2)} + R_2, \tag{4.12}$$

$$r_2\mathcal{U}^{(2)} = -\mathbb{V}^{(3)}(\mathbf{B}_{2M+1}^{(3)})^{-1}\mathbb{V}_{-1}^{(3)}\varphi_0^{(3)} - \mathbb{V}^{(3)}(\mathbf{B}_{2M+1}^{(3)})^{-1}\mathbb{V}_{-1}^{(3)}\psi_0^{(3)} + R_3, \tag{4.13}$$

where

$$R_1 \in C^{M+1}(\overline{D}_1), \quad R_2 \in C^{M+1}(\overline{D}_2^{(1)}), \quad R_3 \in C^{M+1}(\overline{D}_2^{(2)}),$$
$$\operatorname{supp}\varphi_0^{(1)} \subset \overline{S}_0^{(1)}, \quad \operatorname{supp}\psi_0^{(1)} \subset \overline{S}_0^{(2)}, \quad \operatorname{supp}\varphi_0^{(2)} \subset \overline{S}_0^{(1)},$$
$$\operatorname{supp}\psi_0^{(2)} \subset \overline{S}_0, \quad \operatorname{supp}\varphi_0^{(3)} \subset \overline{S}_0^{(2)}, \quad \operatorname{supp}\psi_0^{(3)} \subset \overline{S}_0.$$

Thus, taking into account (4.11), (4.12), (4.13), using the asymptotic expansions of the functions $-\mathbf{L}_+^{-1}\varphi_0^{(2)}, \varphi_0^{(1)}$, $-\mathbf{L}_+^{-1}\psi_0^{(3)}, \varphi_0^{(1)}$ and $\psi_0^{(3)}, \psi_0^{(2)}$ (see (4.3), (4.4), (4.6)), the asymptotic expansion of potential-type functions (see [9, Theorems 2.2 and 2.3]) we derive the following asymptotic expansions of the solutions of the considered boundary-contact problem $\mathbf{M}_1$ in terms of some local coordinate systems of curves $\partial S_0^{(1)}$, $\partial S_0^{(2)}$, ∂S_0:

a) the asymptotic expansion near the contact boundary $\partial S_0^{(1)}$:

$$\begin{aligned}
\mathcal{U}^{(q)}(x_1,x_2,x_3) &= (u^{(q)},\omega^{(q)})(x_1,x_2,x_3)\\
&= \sum_{\theta=\pm1}\sum_{j=1}^{2}\sum_{s=1}^{\ell_0}\mathrm{Re}\bigg\{\sum_{m=0}^{n_s-1}x_3^m\Big[d_{sjm}^{(q)}(x_1,\theta)(z_{s,\theta}^{(q)})^{1/2+\Delta_j(x_1)-m}\\
&\qquad\times B_{a_{pr}}^0\Big(-\frac{1}{2\pi i}\log z_{s,\theta}^{(q)}\Big)\Big]c_{jm}^{(q)}(x_1)\\
&\quad+\sum_{\substack{k,l=0\\ k+l+p+m\neq0}}^{M+2}\sum_{p+m=0}^{M+2-l}x_2^l x_3^m d_{slmpj}^{(q)}(x_1,\vartheta)(z_{s,\vartheta}^{(q)})^{1/2+\Delta_j(x_1)+k+p}B_{skmpj}^{(q)}(x_1,\log z_{s,\vartheta}^{(q)})\bigg\}\\
&\quad+\mathcal{U}_{M+1}^{(q)}(x_1,x_2,x_3)\ \text{ for }\ M>\frac{2}{r}-\min\{[s-1],0\},\ \ q=1,2,
\end{aligned}\tag{4.14}$$

with the coefficients $d_{sjm}^{(q)}(\cdot,\pm1)$, $c_{jm}^{(q)}$, $d_{slmpj}^{(q)}(\cdot,\pm1)\in C_0^\infty(\mathbb{R})$ and the remainder $\mathcal{U}_{M+1}^{(q)}\in C_0^{M+1}(\overline{\mathbb{R}^3_\pm})$, $q=1,2$, where the signs "+" and "-" refer to the cases $q=1$ and $q=2$, respectively. Here

$$\begin{gathered}
z_{s,+1}^{(q)}=(-1)^q[x_2+x_3\tau_{s,+1}^{(q)}],\quad z_{s,-1}^{(q)}=(-1)^{q+1}[x_2-x_3\tau_{s,-1}^{(q)}],\\
-\pi<\mathrm{Arg}z_{s,\pm1}<\pi,\quad \tau_{s,\pm1}\in C_0^\infty(\mathbb{R}),
\end{gathered}$$

$\{\tau_{s,\pm1}^{(q)}\}_{s=1}^{\ell_0}$ are all different roots of the polynomial $\det\overset{\circ}{\mathcal{M}}{}^{(q)}((\mathcal{J}_\varkappa^\top(x_1,0))^{-1}\cdot(0,\pm1,\tau))$ of multiplicity n_s, $s=1,\dots,\ell_0$, in the complex lower half-plane (n_s and ℓ_0 depend on q). $B_{skmpj}^{(q)}(x_1,t)$ is a polynomial of order $\nu_{kmp}=\nu_k+p+m$ ($\nu_k=k(2m_0-1)+m_0-1$, $m_0=\max\{m_1,\dots,m_\ell\}$, $\sum\limits_{j=1}^{\ell}m_j=6$) with respect to the variable t with vector coefficients which depend on the variable x_1.

We write the following relation between the leading (first) coefficients of the asymptotic expansions (4.14) and (4.3) (see [9, Theorem 2.3]):

$$\begin{aligned}
d_{sjm}^{(1)}(x_1,+1)&=\frac{1}{2\pi}\mathcal{G}_\varkappa(x_1,0)\overset{1}{\mathbb{V}}{}_{-1,m}^{(s)}(x_1,0,+1)\sigma^{-1}_{\mathbb{V}_{-1}^{(1)}}(x_1,0,+1)\mathcal{K}_{2j}(x_1),\\
d_{sjm}^{(1)}(x_1,-1)&=-\frac{1}{2\pi}\mathcal{G}_\varkappa(x_1,0)\overset{1}{\mathbb{V}}{}_{-1,m}^{(s)}(x_1,0,-1)\sigma^{-1}_{\mathbb{V}_{-1}^{(1)}}(x_1,0,-1)\\
&\qquad\qquad\times\mathcal{K}_{2j}(x_1)e^{i\pi(-1/2-\Delta_j(x_1))},\\
d_{sjm}^{(2)}(x_1,+1)&=\frac{(-1)^{m+1}}{2\pi}\mathcal{G}_\varkappa(x_1,0)\overset{2}{\mathbb{V}}{}_{-1,m}^{(s)}(x_1,0,+1)\sigma^{-1}_{\frac12\mathcal{I}+\overset{*}{\mathbb{V}}{}_0^{(2)}}(x_1,0,+1)\mathcal{K}_{1j}(x_1),\\
d_{sjm}^{(2)}(x_1,-1)&=\frac{(-1)^{m+1}}{2\pi}\mathcal{G}_\varkappa(x_1,0)\overset{2}{\mathbb{V}}{}_{-1,m}^{(s)}(x_1,0,-1)\sigma^{-1}_{\frac12\mathcal{I}+\overset{*}{\mathbb{V}}{}_0^{(2)}}(x_1,0,-1)
\end{aligned}$$

$$\times \mathcal{K}_{1j}(x_1) e^{i\pi(-1/2-\Delta_j(x_1))},$$
$$j=1,2, \quad s=1,\dots,\ell_0, \quad m=0,\dots,n_s-1;$$

here $\mathcal{G}_\varkappa$ is the square root from the Gram determinant of the diffeomorphism $\varkappa$ and

$$\overset{q}{\mathbb{V}}{}^{(s)}_{-1,m}(x_1,0,\pm1) = \frac{i^{m+1}}{m!(n_s-1-m)!} \frac{d^{n_s-1-m}}{d\tau^{n_s-1-m}} (\tau - \tau^{(q)}_{s,\pm1})^{n_s}$$
$$\times \Big[\overset{\circ}{\mathcal{M}}{}^{(q)}((\mathcal{J}_\varkappa^\top(x_1,0))^{-1}(0,\pm1,\tau))\Big]^{-1}\Big|_{\tau=\tau^{(q)}_{s,\pm1}}, \quad q=1,2.$$

The coefficients $c^{(q)}_{jm}(x_1)$ in (4.14) are defined as follows:

$$\begin{aligned} c^{(1)}_{jm}(x_1) &= a_{jm}(x_1) B^-_{a_{pr}}\Big(\frac{1}{2}+\Delta_j(x_1)\Big) c^{(2)}_0(x_1), \\ c^{(2)}_{jm}(x_1) &= a_{jm}(x_1) B^-_{a_{pr}}\Big(\frac{1}{2}+\Delta_j(x_1)\Big) c^{(1)}_0(x_1), \end{aligned} \qquad j=1,2, \quad m=0,\dots,n_s-1,$$

where

$$B^-_{a_{pr}}(t) = \operatorname{diag}\{B^{m_1}_-(t),\dots,B^{m_l}_-)t)\},$$
$$B^{m_r}_-(t) = B^{m_r}\Big(-\frac{1}{2\pi i}\partial t\Big)\Big(\Gamma(t+1)e^{\frac{i\pi(t+1}{2}}\Big),$$
$$B^{m_r}(t) = \|b^{m_r}_{kp}(t)\|_{m_r\times m_r},$$
$$b^{m_r}_{kp}(t) = \begin{cases} \Big(\frac{1}{2\pi i}\Big)^{p-k} \frac{(-1)^{p+k}}{(p-k)!} \frac{d^{p-k}}{dt^{p-k}}\Big(\Gamma(t+1)e^{\frac{i\pi(t+1)}{2}}\Big), & k\le p, \\ 0, & k>p, \end{cases}$$
$$p=0,\dots,m_r-1, \quad r=1,\dots,\ell.$$

Further,

$$a_{jm}(x_1) = \operatorname{diag}\{a^{m_1}(\lambda^{(j)}_1),\dots,a^{m_\ell}(\lambda^{(j)}_\ell)\}, \quad j=1,2,$$
$$\lambda^{(1)}_r(x_1) = -\frac{3}{2} - \frac{1}{2\pi}\arg\lambda_r(x_1) + \frac{i}{2\pi}\log|\lambda_r(x_1)| + m,$$
$$\lambda^{(2)}_r(x_1) = -\frac{3}{2} + \frac{1}{2\pi}\arg\lambda_r(x_1) + \frac{i}{2\pi}\log|\lambda_r(x_1)| + m,$$
$$m=0,1,\dots,n_s-1;$$
$$a^{m_r}(\lambda^{(j)}_r) = \|a^{m_r}_{kp}(\lambda^{(j)}_r\|_{m_r\times m_r},$$

where

$$a^{m_r}_{kp}(\lambda^{(j)}_r) = \begin{cases} -i\sum\limits_{l=k}^{p} \frac{(-1)^{p+k}(2\pi i)^{l-p} b^{m_r}_{kl}(\mu^{(j)}_r)}{(\lambda^{(j)}_r+1)^{p-l+1}}, & m=0, \quad k\le p, \\ (-1)^{p+k} b^{m_r}_{kp}(\lambda^{(j)}_r), & m=1,2,\dots,n_s-1, \quad k\le p, \\ 0, & k>p; \end{cases}$$

here $\lambda^{(j)}_r = -1+m+\mu^{(j)}_r$, $0<\operatorname{Re}\mu^{(j)}_r<1$, $j=1,2$, $r=1,\dots,\ell$, and $c^{(1)}_0(x_1)$, $c^{(2)}_0(x_1)$ are defined using the first coefficients of the asymptotic expansion of the functions $-\mathbf{L}_+^{-1}\varphi^{(2)}_0$ and $\varphi^{(1)}_0$, respectively (see (4.3)).

b) the asymptotics of solutions near the contact boundary $\partial S_0^{(2)}$:

$$\mathcal{U}^{(q)}(x_1,x_2,x_3)=(u^{(q)},\omega^{(q)})(x_1,x_2,x_3)$$
$$=\sum_{\vartheta=\pm1}\sum_{j=1}^{2}\sum_{s=1}^{\ell_0}\mathrm{Re}\Big\{\sum_{m=0}^{n_s-1}x_3^m\Big[d_{sjm}^{(q)}(x_1,\theta)(z_{z,\theta}^{(q)})^{1/2+\widetilde{\Delta}_j(x_1)-m}$$
$$\times B_{a_{pr}}^0\Big(-\frac{1}{2\pi i}\log z_{s,\theta}^{(q)}\Big)\Big]c_{jm}^{(q)}(x_1)$$
$$+\sum_{\substack{k,l=0\\ k+l+p+m\neq0}}^{M+2}\sum_{p+m=0}^{M+2-l}x_2^lx_3^m d_{slmpj}^{(q)}(x_1,\vartheta)(z_{s,\vartheta}^{(q)})^{1/2+\Delta_j(x_1)+k+p}B_{skmpj}^{(q)}(x_1,\log z_{s,\vartheta}^{(q)})\Big\}$$
$$+\mathcal{U}_{M+1}^{(q)}(x_1,x_2,x_3)\ \text{ for }\ M>\frac{2}{r}-\min\{[s-1],0\},\ \ q=1,2; \tag{4.15}$$

here $z_{s,+1}^{(q)}=(-1)^q[x_2+x_3\tau_{s,+1}^{(q)}]$, $z_{s,-1}^{(q)}=(-1)^{q+1}[x_2-x_3\tau_{s,-1}^{(q)}]$. The coefficients $d_{sjm}^{(q)}(\cdot,\pm1)$ are calculated similarly as in **a)** and the coefficients $b_{jm}^{(q)}$ are defined as in **a)** by using the first coefficients $b_0^{(q)}$ $(q=1,2)$ of the asymptotic expansion (4.4).

c) the asymptotics of solutions near the cuspidal edge ∂S_0:

$$(r_i\mathcal{U}^{(2)})(x_1,x_2,x_3)=r_i(u^{(2)},\omega^{(2)})(x_1,x_2,x_3)$$
$$=\sum_{\vartheta=\pm1}\sum_{s=1}^{\ell_0}\mathrm{Re}\Big\{\sum_{j=0}^{n_s-1}x_3^jz_{s,\theta}^{1/2-j}d_{sj}^{(i)}(x_1,\theta)+\sum_{\substack{k,l=0\\ l+k+j+p\neq1}}^{M+1}\sum_{j+p=1}^{M+2-l}x_2^lx_3^jz_{s,\vartheta}^{-1/2+p+k}d_{slkjp}^{(i)}(x_1)\Big\}$$
$$+\mathcal{U}_{M+1}^{(i)}(x_1,x_2,x_3)\ \text{ for }\ M>\frac{2}{r}-\min\{[s],0\},\ \ i=1,2, \tag{4.16}$$

with the coefficients $d_{sj}^{(i)}(\cdot,\pm1),d_{slkjp}^{(i)}\in C_0^\infty(\mathbb{R})$ and the remainder $\mathcal{U}_{M+1}^{(i)}\in C_0^{M+1}(\overline{\mathbb{R}_\pm^3})$, $i=1,2$; here

$z_{s,+1}=-x_2-x_3\tau_{s,+1}^{(2)}$, $z_{s,-1}=x_2-x_3\tau_{s,-1}^{(2)}$, $-\pi<\mathrm{Arg}z_{s,\pm1}<\pi$, $\tau_{s,\pm1}\in C_0^\infty(\mathbb{R})$,

$\{\tau_{s,\pm1}^{(2)}\}_{s=1}^{\ell_0}$ are all different roots of the polynomial $\det\overset{\circ}{\mathcal{M}}{}^{(2)}((\mathcal{J}_\varkappa^\top(x_1,0))^{-1}\cdot(0,\pm1,\tau))$ of multiplicity n_s, $s=1,\dots,\ell_0$, in the complex lower half-plane. The coefficients $d_{sj}^{(i)}(x_1,\pm1)$ have the form (see [9, Theorem 2.3]):

$$d_{sj}^{(1)}(x_1,+1)=\mathcal{G}_\varkappa(x_1,0)\overset{2}{\mathbb{V}}{}_{-1,j}^{(s)}(x_1,0,+1)\sigma_{\frac12\mathcal{I}+\overset{*}{\mathbb{V}}{}_0^{(2)}}^{-1}(x_1,0,+1)c^{(j)}(x_1),$$
$$d_{sj}^{(1)}(x_1,-1)=-i\mathcal{G}_\varkappa(x_1,0)\overset{2}{\mathbb{V}}{}_{-1,j}^{(s)}(x_1,0,-1)\sigma_{\frac12\mathcal{I}+\overset{*}{\mathbb{V}}{}_0^{(2)}}^{-1}(x_1,0,-1)c^{(j)}(x_1)$$
$$d_{sj}^{(2)}(x_1,+1)=(-1)^{j+1}\mathcal{G}_\varkappa(x_1,0)\overset{2}{\mathbb{V}}{}_{-1,j}^{(s)}(x_1,0,+1)\sigma_{\frac12\mathcal{I}+\overset{*}{\mathbb{V}}{}_0^{(3)}}^{-1}(x_1,0,+1)c^{(j)}(x_1),$$
$$d_{sj}^{(2)}(x_1,-1)=(-1)^{j+1}i\mathcal{G}_\varkappa(x_1,0)\overset{2}{\mathbb{V}}{}_{-1,j}^{(s)}(x_1,0,-1)\sigma_{\frac12\mathcal{I}+\overset{*}{\mathbb{V}}{}_0^{(3)}}^{-1}(x_1,0,-1)c^{(j)}(x_1),$$

$$s=1,\dots,l_0,\quad j=0,\dots,n_s-1,$$

where $\mathcal{G}_\varkappa$ is the square root from the Gram determinant of the diffeomorphism $\varkappa$,

$$\overset{2}{\mathbb{V}}{}^{(s)}_{-1,j}(x_1,0,\pm1)=\frac{i^{j+1}}{j!(n_s-1-j)!}\,\frac{d^{n_s-1-j}}{d\tau^{n_s-1-j}}\,(\tau-\tau^{(2)}_{s,\pm1})^{n_s}$$
$$\times\Big(\overset{\circ}{\mathcal{M}}{}^{(2)}((\mathcal{J}_\varkappa^\top(x_1,0))^{-1}\cdot(0,\pm1,\tau))\Big)^{-1}\Big|_{\tau=\tau^{(2)}_{s,\pm1}},$$
$$c^{(j)}(x_1)=\frac{i^j}{2\sqrt{\pi}}\Gamma\Big(j-\frac12\Big)c_0(x_1)$$

and $c_0(x_1)$ is the first coefficient of the asymptotic expansion in (4.6).

5. Investigation of the Properties of Exponents of the First Terms of an Asymptotic Expansion of Solutions of Problem $\mathbf{M}_1$ in the Neighbouhoods of Contact Boundaries

We consider the properties of exponents of the first terms of an asymptotic expansion of solutions of the boundary-contact problem $\mathbf{M}_1$ in the neighbourhood of the contact boundary $\partial S_0^{(1)}$. Analogous properties will be valid in the neighbourhood of the contact boundary $\partial S_0^{(2)}$, too.

For the sake of brevity, by γ_m and δ_m $(m=1,\dots,6)$ we denote the real and the imaginary part of the first term exponent of the asymptotic expansions (4.14), and (4.15), i.e.,

$$\gamma_m(x_1)=\frac12-\Big|\frac{1}{2\pi}\arg\lambda_m(x_1)\Big|,\quad \delta_m(x_1)=-\frac{1}{2\pi}\log|\lambda_m(x_1)|,$$
$$m=1,\dots,6,\quad x_1\in\partial S_0^{(1)},$$

where $\lambda_m(x_1)=\lambda^{(m)}_{\mathbf{R}}(x_1)$ $m=1,\dots,12$, are the eigenvalues of the matrix

$$b_{\mathbf{R}}(x_1)=(\sigma_{\mathbf{R}}(x_1,0,+1))^{-1}\sigma_{\mathbf{R}}(x_1,0,-1).$$

Theorem 5.1. *The real parts γ_m, $m=1,\dots,6$, of the exponents of the first terms of the asymptotic expansion of solutions of the boundary-contact problem $\mathbf{M}_1$ near the contact boundary $\partial S_0^{(1)}$ depend on the elastic constants and also on the geometry of the contact boundaries and may take any values from the interval $]0,1/2[$, i.e.,*

(a) *if the elastic constants satisfy the limit conditions*

$$\frac{a^{(2)}_{ijlk}}{a^{(1)}_{ijlk}}\to0,\quad \frac{b^{(2)}_{ijlk}}{b^{(1)}_{ijlk}}\to0,\quad \frac{c^{(2)}_{ijlk}}{c^{(1)}_{ijlk}}\to0,$$

then $\gamma_m\to1/2$ $(m=1,\dots,6)$;

(b) *if for*

$$\frac{a^{(2)}_{ijlk}}{a^{(1)}_{ijlk}}\to\infty,\quad \frac{b^{(2)}_{ijlk}}{b^{(1)}_{ijlk}}\to\infty,\quad \frac{c^{(2)}_{ijlk}}{c^{(1)}_{ijlk}}\to\infty$$

the limiting relations

1. $M_2^+\dfrac{|\zeta_1|}{|\zeta_2|}+M_1^+\dfrac{|\zeta_2|}{|\zeta_1|}\to 0$; 2. $\dfrac{|\mathrm{Im}\langle\zeta_1,\zeta_2\rangle|}{|\zeta_1|\,|\zeta_2|}\to c\ (c>0)$,

are valid, then $\gamma_m\to 0$ *and* $\delta_m\to 0$ $(m=1,\dots,6)$,

where $\zeta=(\zeta_1,\zeta_2)$ *is the eigenvector of the matrix* $b_{\mathbf{R}}$, *while* M_1^+ *and* M_2^+ *are maximal eigenvalues of the matrices* $\sigma_{\mathbf{N}_1}^+=\sigma_{\mathbf{N}_1}(x_1,0,+1)$ *and* $\sigma_{\mathbf{N}_2}^+=\sigma_{\mathbf{N}_2}(x_1,0,+1)$.

Here the limits of the coefficients $a_{ijlk}^{(q)}$, $b_{ijlk}^{(q)}$, $c_{ijlk}^{(q)}$ *are understood in the uniform sense with respect to the indices* i, j, l, k.

Proof. Multiply the coefficients $a_{ijlk}^{(1)}$, $b_{ijlk}^{(1)}$, $c_{ijlk}^{(1)}$ by α $(\alpha>0)$, and the coefficients $a_{ijlk}^{(2)}$, $b_{ijlk}^{(2)}$, $c_{ijlk}^{(2)}$ by β $(\beta>0)$; i.e., we consider the differential equations with the elastic constants $\alpha a_{ijlk}^{(1)}$, $\alpha b_{ijlk}^{(1)}$, $\alpha c_{ijlk}^{(1)}$ and $\beta a_{ijlk}^{(2)}$, $\beta b_{ijlk}^{(2)}$, $\beta c_{ijlk}^{(2)}$.

(a) Taking into consideration the estimate obtained in [6] we have

$$\Big|\frac{1}{2\pi}\arg\lambda_m(x_1)\Big|\le\frac{1}{\pi}\mathrm{arctg}\Big(\frac{\sqrt{\beta}}{\sqrt{\alpha}}\,\frac{1}{\sqrt{m_1^+m_2^+}}\Big),$$

where αm_1^+, $\beta^{-1}m_2^+$ are minimal eigenvalues of the matrices $\sigma_{\mathbf{N}_1}^+$ and $\sigma_{\mathbf{N}_2}^+$, respectively.

Consequently, as $\beta/\alpha\to 0$ we get $\gamma_m\to\frac{1}{2}$ $(m=1,\dots,6)$.

(b) Since

$$\frac{1}{2\pi}\arg\lambda_m(x_1)=\frac{1}{2\pi}\mathrm{arctg}\frac{2\mathrm{Im}\langle\zeta_1,\zeta_2\rangle}{\langle\sigma_{\mathbf{N}_2}^+\zeta_1,\zeta_1\rangle+\langle\sigma_{\mathbf{N}_1}^+\zeta_2,\zeta_2\rangle}+\frac{1}{2\pi}\mathrm{arctg}\frac{2\mathrm{Im}\langle\zeta_1,\zeta_2\rangle}{\langle\sigma_{\mathbf{N}_2}^-\zeta_1,\zeta_1\rangle+\langle\sigma_{\mathbf{N}_1}^-\zeta_2,\zeta_2\rangle},$$

we obtain the estimate

$$\Big|\frac{1}{2\pi}\arg\lambda_m(x_1)\Big|\ge\frac{1}{\pi}\mathrm{arctg}\frac{\frac{2|\mathrm{Im}\langle\zeta_1,\zeta_2\rangle|}{|\zeta_1|\,|\zeta_2|}}{\beta^{-1}M_2^+\frac{|\zeta_1|}{|\zeta_2|}+\alpha M_1^+\frac{|\zeta_2|}{|\zeta_1|}}\,. \tag{5.1}$$

Further, since

$$|\lambda_m(x_1)|^2=\frac{(\langle\sigma_{\mathbf{N}_2}^-\zeta_1,\zeta_1\rangle+\langle\sigma_{\mathbf{N}_1}^-\zeta_2,\zeta_2\rangle)^2+(2\mathrm{Im}\langle\zeta_1,\zeta_2\rangle)^2}{(\langle\sigma_{\mathbf{N}_2}^+\zeta_1,\zeta_1\rangle+\langle\sigma_{\mathbf{N}_1}^+\zeta_2,\zeta_2\rangle)^2+(2\mathrm{Im}\langle\zeta_1,\zeta_2\rangle)^2}\,,$$

we get

$$\frac{1}{b}\le|\lambda_m(x_1)|^2\le b, \tag{5.2}$$

where

$$b=\frac{(\beta^{-1}M_2^+|\zeta_1|^2+\alpha M_1^+|\zeta_2|^2)^2+(2\mathrm{Im}\langle\zeta_1,\zeta_2\rangle)^2}{(\beta^{-1}m_2^+|\zeta_1|^2+\alpha m_1^+|\zeta_2|^2)^2+(2\mathrm{Im}\langle\zeta_1,\zeta_2\rangle)^2}\,;$$

here αM_1^+ and $\beta^{-1}M_2^+$ are maximal eigenvalues of the matrices $\sigma_{\mathbf{N}_1}^+$ and $\sigma_{\mathbf{N}_2}^+$ respectively, while the vectors ζ_1 and ζ_2 depend in general on α and β.

Thus, using the inequalities (5.1) and (5.2) and taking into consideration the limiting relations 1 and 2 from subsection (b), for $\beta/\alpha \to \infty$ we get $\gamma_m \to 0$ and $\delta_m \to 0$ $(m = 1, \dots, 6)$. □

Remark 5.2. In the centrally symmetric isotropic case the exponents of the first terms of the asymptotic expansion of solutions of the boundary-contact problem $\mathbf{M}_1$ $\gamma_j + i\delta_j$ $(j = 1, \dots, 6)$ are calculated explicitly.

Indeed, the differential operator of the couple-stress elasticity theory for a homogeneous isotropic centrally symmetric medium takes the form (see [26]):

$$\begin{aligned}
\mathcal{M}^{(q)}(\partial_x) &= \begin{pmatrix} \overset{1}{\mathcal{M}}{}^{(q)}(\partial_x) & \overset{2}{\mathcal{M}}{}^{(q)}(\partial_x) \\ \overset{3}{\mathcal{M}}{}^{(q)}(\partial_x) & \overset{4}{\mathcal{M}}{}^{(q)}(\partial_x) \end{pmatrix}_{6\times 6}, \\
\overset{k}{\mathcal{M}}{}^{(q)}(\partial_x) &= \big\| \overset{k}{\mathcal{M}}{}^{(q)}_{ij}(\partial_x) \big\|_{3\times 3}, \quad k = \overline{1,4}, \quad q = 1, 2, \\
\overset{1}{\mathcal{M}}{}^{(q)}_{ij}(\partial_x) &= (\mu_q + \alpha_q)\delta_{ij}\Delta + (\lambda_q + \mu_q - \alpha_q)\partial_i\partial_j, \\
\overset{2}{\mathcal{M}}{}^{(q)}_{ij}(\partial_x) &= \overset{3}{\mathcal{M}}{}^{(q)}_{ij}(\partial_x) = -2\alpha_q \sum_{k=1}^{3} \varepsilon_{ijk}\partial_k, \\
\overset{4}{\mathcal{M}}{}^{(q)}_{ij}(\partial_x) &= \delta_{ij}\big[(\nu_q + \beta_q)\Delta - 4\alpha_q\big] + (\varepsilon_q + \nu_q - \beta_q)\partial_i\partial_j,
\end{aligned}$$

where δ_{ij} and ε_{ijk} are respectively the Kronecker and the Levi-Civita symbol. The coefficients λ_q, μ_q, α_q, ν_q, β_q, ε_q, $q = 1, 2$, are the elastic constants satisfying the conditions

$$\mu_q > 0, \quad 3\lambda_q + 2\mu_q > 0, \quad \alpha_q > 0, \quad \nu_q > 0, \quad 3\varepsilon_q + 2\nu_q > 0, \quad \beta_q > 0.$$

The stress operator of couple-stress elasticity is written as

$$\begin{aligned}
\mathcal{N}^{(q)}(\partial_z, n(z)) &= \begin{pmatrix} \overset{1}{\mathcal{N}}{}^{(q)}(\partial_z, n(z)) & \overset{2}{\mathcal{N}}{}^{(q)}(\partial_z, n(z)) \\ \overset{3}{\mathcal{N}}{}^{(q)}(\partial_z, n(z)) & \overset{4}{\mathcal{N}}{}^{(q)}(\partial_z, n(z)) \end{pmatrix}_{6\times 6}, \\
\overset{k}{\mathcal{N}}{}^{(q)}(\partial_z, n(z)) &= \big\| \overset{k}{\mathcal{N}}{}^{(q)}_{ij}(\partial_z, n(z)) \big\|_{3\times 3}, \quad k = \overline{1,4}, \quad q = 1, 2, \\
\overset{1}{\mathcal{N}}{}^{(q)}_{ij}(\partial_z, n(z)) &= \lambda_q n_i(z)\partial_j + (\mu_q - \alpha_q)n_j(z)\partial_i + (\mu_q + \alpha_q)\delta_{ij}\frac{\partial}{\partial n(z)}, \\
\overset{2}{\mathcal{N}}{}^{(q)}_{ij}(\partial_z, n(z)) &= -2\alpha_q \sum_{k=1}^{3} \varepsilon_{ijk} n_k(z), \quad \overset{3}{\mathcal{N}}{}^{(q)}_{ij}(\partial_z, n(z)) = 0, \\
\overset{4}{\mathcal{N}}{}^{(q)}_{ij}(\partial_z, n(z)) &= \varepsilon_q n_i(z)\partial_j + (\nu_q - \beta_q)n_i(z)\partial_i + (\nu_q + \beta_q)\delta_{ij}\frac{\partial}{\partial n(z)}.
\end{aligned}$$

The matrices $\sigma^{\pm}_{\mathbf{N}_q} = \sigma_{\mathbf{N}_q}(x_1, 0, \pm 1)$, $q = 1, 2$, have the following expressions

$$\sigma^{\pm}_{\mathbf{N}_1} = \begin{pmatrix} \overset{1}{\sigma}{}^{\pm}_{\mathbf{N}_1} & 0 \\ 0 & \overset{2}{\sigma}{}^{\pm}_{\mathbf{N}_1} \end{pmatrix}_{6\times 6}, \quad \sigma^{\pm}_{\mathbf{N}_2} = \begin{pmatrix} \overset{1}{\sigma}{}^{\pm}_{\mathbf{N}_2} & 0 \\ 0 & \overset{2}{\sigma}{}^{\pm}_{\mathbf{N}_2} \end{pmatrix}_{6\times 6},$$

where

$$\overset{1}{\sigma}\,{}^{\pm}_{\mathbf{N}_1} = \begin{pmatrix} \mu_1+\alpha_1 & 0 & 0 \\ 0 & a_1 & \mp ib_1 \\ 0 & \pm ib_2 & a_1 \end{pmatrix}_{3\times 3}, \quad \overset{2}{\sigma}\,{}^{\pm}_{\mathbf{N}_1} = \begin{pmatrix} \nu_1+\beta_1 & 0 & 0 \\ 0 & c_1 & \mp id_1 \\ 0 & \pm id_1 & c_1 \end{pmatrix}_{3\times 3},$$

$$\overset{1}{\sigma}\,{}^{\pm}_{\mathbf{N}_2} = \begin{pmatrix} \dfrac{1}{\mu_2+\alpha_2} & 0 & 0 \\ 0 & \dfrac{a_2}{a_2^2-b_2^2} & \mp i\dfrac{b_2}{a_2^2-b_2^2} \\ 0 & \pm i\dfrac{b_2}{a_2^2-b_2^2} & \dfrac{a_2}{a_2^2-b_2^2} \end{pmatrix}_{3\times 3},$$

$$\overset{2}{\sigma}\,{}^{\pm}_{\mathbf{N}_2} = \begin{pmatrix} \dfrac{1}{\nu_2+\beta_2} & 0 & 0 \\ 0 & \dfrac{c_2}{c_2^2-d_2^2} & \mp i\dfrac{d_2}{c_2^2-d_2^2} \\ 0 & \pm i\dfrac{d_2}{c_2^2-d_2^2} & \dfrac{c_2}{c_2^2-d_2^2} \end{pmatrix}_{3\times 3},$$

here

$$a_q = \frac{2(\lambda_q+2\mu_q)(\mu_q+\alpha_q)}{\lambda_q+\alpha_q+3\mu_q}, \quad b_q = \frac{2(\mu_q+\alpha_q)^2}{\lambda_q+\alpha_q+3\mu_q}, \quad q=1,2,$$

and

$$c_q = \frac{2(\varepsilon_q+2\nu_q)(\nu_q+\beta_q)}{\varepsilon_q+\beta_q+3\nu_q}, \quad d_q = \frac{2(\nu_q+\beta_q)^2}{\varepsilon_q+\beta_q+3\nu_q}, \quad q=1,2.$$

Hence in the considered case the eigenvalues λ_j $(j=1,\dots,12)$ of the matrix

$$b_{\mathbf{R}}(x_1) = \big(\sigma_{\mathbf{R}}(x_1,0,+1)\big)^{-1}\sigma_{\mathbf{R}}(x_1,0,-1)$$

are calculated in the manner as follows:

$$\lambda_{1,2} = \frac{(\mu_1+\alpha_1)-(\mu_2+\alpha_2)\pm 2i\sqrt{(\mu_1+\alpha_1)(\mu_2+\alpha_2)}}{\mu_1+\alpha_1+\mu_2+\alpha_2},$$

$$\lambda_{3,4} = \begin{cases} \frac{B\pm\sqrt{B^2-AC}}{A} & \text{for } B^2-AC\geq 0, \\ \frac{B\pm i\sqrt{AC-B^2}}{A} & \text{for } B^2-AC<0, \end{cases}$$

$$\lambda_{5,6} = \begin{cases} \frac{B\pm\sqrt{B^2-AC}}{C} & \text{for } B^2-AC\geq 0, \\ \frac{B\pm i\sqrt{AC-B^2}}{C} & \text{for } B^2-AC<0, \end{cases}$$

$$\lambda_{7,8} = \frac{(\nu_1+\beta_1)-(\nu_2+\beta_2)\pm 2i\sqrt{(\nu_1+\beta_1)(\nu_2+\beta_2)}}{\nu_1+\beta_1+\nu_2+\beta_2},$$

$$\lambda_{9,10} = \begin{cases} \frac{\widetilde{B}\pm\sqrt{\widetilde{B}^2-\widetilde{A}\widetilde{C}}}{\widetilde{A}} & \text{for } \widetilde{B}^2-\widetilde{A}\widetilde{C}\geq 0, \\ \frac{\widetilde{B}\pm i\sqrt{\widetilde{A}\widetilde{C}-\widetilde{B}^2}}{\widetilde{A}} & \text{for } \widetilde{B}^2-\widetilde{A}\widetilde{C}<0, \end{cases}$$

$$\lambda_{11,12} = \begin{cases} \frac{\widetilde{B}\pm\sqrt{\widetilde{B}^2-\widetilde{A}\widetilde{C}}}{\widetilde{C}} & \text{for} \quad \widetilde{B}^2 - \widetilde{A}\widetilde{C} \geq 0, \\ \frac{\widetilde{B}\pm i\sqrt{\widetilde{A}\widetilde{C}-\widetilde{B}^2}}{\widetilde{C}} & \text{for} \quad \widetilde{B}^2 - \widetilde{A}\widetilde{C} < 0, \end{cases}$$

where

$$A = a_2a_1 + b_1b_2 + a_2^2 - b_2^2 + a_2b_1 + b_2a_1, \quad B = a_2a_1 - b_1b_2 - a_2^2 + b_2^2,$$
$$C = a_2a_1 + b_1b_2 + a_2^2 - b_2^2 - a_2b_1 - b_2a_1,$$
$$\widetilde{A} = c_2c_1 + d_1d_2 + c_2^2 - d_2^2 + c_2d_1 + d_2c_1, \quad \widetilde{B} = c_2c_1 - d_1d_2 - c_2^2 + d_2^2,$$
$$\widetilde{C} = c_2c_1 + d_1d_2 + c_2^2 - d_2^2 - c_2d_1 - d_2c_1.$$

Hence we obtain

$$\gamma_1 = \frac{1}{2} - \frac{1}{\pi}\,\mathrm{arctg}\frac{\sqrt{\mu_2+\alpha_2}}{\sqrt{\mu_1+\alpha_1}}\,, \quad \delta_1 = 0,$$

$$\gamma_{2,3} = \begin{cases} \frac{1}{2}\,, & B^2 - AC \geq 0, \\ \frac{1}{2} - \frac{1}{2\pi}\,\mathrm{arctg}\frac{\sqrt{AC-B^2}}{B}\,, & B^2 - AC < 0, \;\; B > 0, \\ \frac{1}{2\pi}\,\mathrm{arctg}\frac{\sqrt{AC-B^2}}{|B|} & B^2 - AC < 0, \;\; B < 0, \end{cases}$$

$$\delta_{2,3} = \begin{cases} -\frac{1}{2\pi}\,\log\frac{B\pm\sqrt{B^2-AC}}{A}\,, & B^2 - AC \geq 0, \\ \mp\frac{1}{2\pi}\,\log\left(\frac{A}{C}\right), & B^2 - AC < 0, \end{cases}$$

$$\gamma_4 = \frac{1}{2} - \frac{1}{\pi}\,\mathrm{arctg}\frac{\sqrt{\nu_2+\beta_2}}{\sqrt{\nu_1+\beta_1}}\,, \quad \delta_4 = 0,$$

$$\gamma_{5,6} = \begin{cases} \frac{1}{2}\,, & \widetilde{B}^2 - \widetilde{A}\widetilde{C} \geq 0, \\ \frac{1}{2} - \frac{1}{2\pi}\,\mathrm{arctg}\frac{\sqrt{\widetilde{A}\widetilde{C}-\widetilde{B}^2}}{\widetilde{B}}\,, & \widetilde{B}^2 - \widetilde{A}\widetilde{C} < 0, \;\; \widetilde{B} > 0, \\ \frac{1}{2\pi}\,\mathrm{arctg}\frac{\sqrt{\widetilde{A}\widetilde{C}-\widetilde{B}^2}}{|\widetilde{B}|}\,, & \widetilde{B}^2 - \widetilde{A}\widetilde{C} < 0, \;\; \widetilde{B} < 0, \end{cases}$$

$$\delta_{5,6} = \begin{cases} -\frac{1}{2\pi}\,\log\frac{\widetilde{B}\pm\sqrt{\widetilde{B}^2-\widetilde{A}\widetilde{C}}}{\widetilde{A}}\,, & \widetilde{B}^2 - \widetilde{A}\widetilde{C} \geq 0, \\ \mp\frac{1}{2\pi}\,\log\left(\frac{\widetilde{A}}{\widetilde{C}}\right), & \widetilde{B}^2 - \widetilde{A}\widetilde{C} < 0. \end{cases}$$

Remark 5.3. If $B^2 - AC \neq 0$ and $\widetilde{B}^2 - \widetilde{A}\widetilde{C} \neq 0$, then $B^0_{a_{pr}}(t) = \mathcal{I}$, i.e., the first terms of the asymptotic expansion of solutions of the boundary-contact problem $\mathbf{M}_1$ contain no logarithms near the contact boundary $\partial S_0^{(i)}$, $i = 1, 2$.

Remark 5.4. Note that in the isotropic case we obtain more exact limit relations (for the elasticity case see [6]):

a) if $B > 0$, $B^2 - AC \geq 0$ and $\mu_2 + \alpha_2 \to 0$ or $\mu_1 \to \infty$, then $\gamma_1 \to 1/2$, $\gamma_2 = \gamma_3 = 1/2$;

b) if $B > 0$, $B^2 - AC < 0$ and $\mu_1 \to \infty$, then $\gamma_m \to 1/2$, $m = 1, 2, 3$;

c) if $B > 0$, $B^2 - AC < 0$ and $\mu_2 + \alpha_2 \to 0$, then $\gamma_1 \to 1/2$, $\gamma_{2,3} \to 1/4$;

d) if $B < 0$ and $\mu_1 + \alpha_1 \to 0$ or $\mu_2 \to \infty$, then $\gamma_m \to 0$, $m = 1, 2, 3$;

e) if $|\mu_1-\mu_2|\to 0$, $|\alpha_1-\alpha_2|\to 0$, then $\gamma_m\to 1/4$, $m=1,2,3$,
and

ã) if $\widetilde{B}>0$, $\widetilde{B}^2-\widetilde{A}\widetilde{C}\geq 0$ and $\nu_2+\beta_2\to 0$ or $\nu_1\to\infty$, then $\gamma_4\to 1/2$, $\gamma_{5,6}=1/2$;

b̃) if $\widetilde{B}>0$, $\widetilde{B}^2-\widetilde{A}\widetilde{C}<0$ and $\nu_1\to\infty$, then $\gamma_m\to 1/2$, $m=4,5,6$;

c̃) if $\widetilde{B}>0$, $\widetilde{B}^2-\widetilde{A}\widetilde{C}<0$ and $\nu_2+\beta_2\to 0$, then $\gamma_4\to 1/2$, $\gamma_{5,6}\to 1/4$;

d̃) if $\widetilde{B}<0$, $\nu_1+\beta_1\to 0$ or $\nu_2\to\infty$, then $\gamma_m\to 0$, $m=4,5,6$;

ẽ) if $|\nu_1-\nu_2|\to 0$, $|\beta_1-\beta_2|\to 0$, then $\gamma_m\to 1/4$, $m=4,5,6$.

It is not difficult to see that if the conditions

$$B^2-AC>0 \quad\text{and}\quad \widetilde{B}^2-\widetilde{A}\widetilde{C}>0$$

hold, then the exponent of the first term of the asymptotics of solutions of the boundary-contact problem $\mathbf{M}_1$ has the form

$$\gamma=\frac{1}{2}-\frac{1}{\pi}\operatorname{arctg}\max\left\{\frac{\sqrt{\mu_2+\alpha_2}}{\sqrt{\mu_1+\alpha_1}},\frac{\sqrt{\nu_2+\beta_2}}{\sqrt{\nu_1+\beta_1}}\right\}.$$

Since $\gamma<\frac{1}{2}$, $\gamma_{2,3}=\gamma_{5,6}=\frac{1}{2}$, the oscillation of solutions vanishes in some neighbourhood of the contact boundary and therefore solutions describe the real physical process.

Note that this class has been found only in the spatial case since in the plane case it is known that the oscillation does not vanish.

In the general case we have found a class of anisotropic bodies when the oscillation in the asymptotic expansion vanishes and the real parts γ_j $j=1,\dots,6$, of exponents of the first terms of the asymptotic expansion are calculated by simpler formulas.

Let $\alpha_j>0$ and $\beta_j>0$, $j=1,\dots,6$, be the eigenvalues of the matrices $\sigma^+_{\mathbf{N}_1}$ and $\sigma^+_{\mathbf{N}_2}$, respectively.

Theorem 5.5. *If the conditions*

1) rank $\begin{pmatrix}\sigma^+_{\mathbf{N}_1}-\alpha_j\mathcal{I}\\ \sigma^+_{\mathbf{N}_2}-\beta_j\mathcal{I}\end{pmatrix}<6$, $j=1,\dots,6$;

2) $\langle\zeta_1^{(j)},\overline{\zeta}_1^{(j)}\rangle\neq 0$, $j=1,\dots,6$,

are satisfied, where $\zeta_1^{(j)}$ $(j=1,\dots,6)$ *is the common eigenvector of the matrices* $\sigma^+_{\mathbf{N}_1}$ *and* $\sigma^+_{\mathbf{N}_2}$*, which correspond to the eigenvalues* α_j *and* β_j*, then the oscillation participanting in the asymptotic expansion of solutions of the boundary-contact problem* $\mathbf{M}_1$ *vanishes, i.e.,* $\delta_j=0$, $j=1,\dots,6$, *and the real parts of the exponents of the first terms of the asymptotic expansion are calculated by a simpler formula*

$$\gamma_j=\frac{1}{2}-\frac{1}{\pi}\operatorname{arctg}\frac{1}{\sqrt{\alpha_j\beta_j}},\quad j=1,\dots,6.$$

Proof. From the condition 1) we obtain the existence of the common eigenvectors $\zeta_1^{(j)}$, $j=1,\dots,6$, for the matrices $\sigma^+_{\mathbf{N}_1}$ and $\sigma^+_{\mathbf{N}_2}$, i.e.,

$$\sigma^+_{\mathbf{N}_1}\zeta_1^{(j)}=\alpha_j\zeta_1^{(j)},\quad \sigma^+_{\mathbf{N}_2}\zeta_1^{(j)}=\beta_j\zeta_1^{(j)},\quad j=1,\dots,6. \tag{5.3}$$

Since $\overline{\sigma}^+_{\mathbf{N}_1} = \sigma^-_{\mathbf{N}_1}$ and $\overline{\sigma}^+_{\mathbf{N}_2} = \sigma^-_{\mathbf{N}_2}$, we have

$$\sigma^-_{\mathbf{N}_1}\overline{\zeta}_1^{(j)} = \alpha_j\overline{\zeta}_1^{(j)}, \quad \sigma^-_{\mathbf{N}_2}\overline{\zeta}_1^{(j)} = \beta_j\overline{\zeta}_1^{(j)}, \quad j = 1, \dots, 6. \tag{5.4}$$

Further, we look for an eigenvector $\zeta^{(j)}$ of the matrix

$$\big(\sigma_{\mathbf{R}}(x_1, 0, +1)\big)^{-1}\sigma_{\mathbf{R}}(x_1, 0, -1)$$

in the form $\zeta^{(j)} = (\zeta_1^{(j)}, \widetilde{\gamma}_j\zeta_1^{(j)})$. We obtain

$$\big(\sigma_{\mathbf{R}}(x_1, 0, +1)\big)^{-1}\sigma_{\mathbf{R}}(x_1, 0, -1)\zeta^{(j)} = \lambda_j\zeta^{(j)}.$$

Taking into consideration the expression of the matrices $\sigma_{\mathbf{R}}(x_1, 0, \pm 1)$, we get

$$\begin{cases} \sigma^-_{\mathbf{N}_2}\zeta_1^{(j)} - \widetilde{\gamma}_j\zeta_1^{(j)} = \lambda_j\sigma^+_{\mathbf{N}_2}\zeta_1^{(j)} + \lambda_j\widetilde{\gamma}_j\zeta_1^{(j)}, \\ \zeta_1^{(j)} + \widetilde{\gamma}_j\sigma^-_{\mathbf{N}_1}\zeta_1^{(j)} = -\lambda_j\zeta_1^{(j)} + \lambda_j\widetilde{\gamma}_j\sigma^+_{\mathbf{N}_1}\zeta_1^{(j)}. \end{cases} \tag{5.5}$$

Now, by the scalar multiplication of both equations (5.5) by the vector $\overline{\zeta}^{(j)}$ we have

$$\begin{cases} \langle\sigma^-_{\mathbf{N}_2}\zeta_1^{(j)}, \overline{\zeta}_1^{(j)}\rangle - \widetilde{\gamma}_j\langle\zeta_1^{(j)}, \overline{\zeta}_1^{(j)}\rangle = \lambda_j\langle\sigma^+_{\mathbf{N}_2}\zeta_1^{(j)}, \overline{\zeta}_1^{(j)}\rangle + \lambda_j\widetilde{\gamma}_j\langle\zeta_1^{(j)}, \overline{\zeta}_1^{(j)}\rangle, \\ \langle\zeta_1^{(j)}, \overline{\zeta}_1^{(j)}\rangle + \widetilde{\gamma}_j\langle\sigma^-_{\mathbf{N}_1}\zeta_1^{(j)}, \overline{\zeta}_1^{(j)}\rangle = -\lambda_j\langle\zeta_1^{(j)}, \overline{\zeta}_1^{(j)}\rangle + \lambda_j\widetilde{\gamma}_j\langle\sigma^+_{\mathbf{N}_1}\zeta_1^{(j)}, \overline{\zeta}_1^{(j)}\rangle. \end{cases} \tag{5.6}$$

Substituting (5.3) and (5.4) into (5.6), we get

$$\begin{cases} \beta_j\langle\zeta_1^{(j)}, \overline{\zeta}_1^{(j)}\rangle - \widetilde{\gamma}_j\langle\zeta_1^{(j)}, \overline{\zeta}_1^{(j)}\rangle = \lambda_j\beta_j\langle\zeta_1^{(j)}, \overline{\zeta}_1^{(j)}\rangle + \lambda_j\widetilde{\gamma}_j\langle\zeta_1^{(j)}, \overline{\zeta}_1^{(j)}\rangle, \\ \langle\zeta_1^{(j)}, \overline{\zeta}_1^{(j)}\rangle + \widetilde{\gamma}_j\alpha_j\langle\zeta_1^{(j)}, \overline{\zeta}_1^{(j)}\rangle = -\lambda_j\langle\zeta_1^{(j)}, \overline{\zeta}_1^{(j)}\rangle + \lambda_j\widetilde{\gamma}_j\alpha_j\langle\zeta_1^{(j)}, \overline{\zeta}_1^{(j)}\rangle. \end{cases}$$

Taking into account the condition 2), i.e., $\langle\zeta_1^{(j)}, \overline{\zeta}_1^{(j)}\rangle \neq 0$, we obtain the following system of equations with respect to λ_j and $\widetilde{\gamma}_j$ $(j = 1, \dots, 6)$:

$$\begin{cases} \beta_j - \widetilde{\gamma}_j = \lambda_j\beta_j + \lambda_j\widetilde{\gamma}_j, \\ 1 + \widetilde{\gamma}_j\alpha_j = -\lambda_j + \lambda_j\widetilde{\gamma}_j\alpha_j. \end{cases}$$

Further,

$$\lambda_j = \frac{\sqrt{\alpha_j\beta_j} \mp i}{\sqrt{\alpha_j\beta_j} \pm i}, \quad \widetilde{\gamma}_j = \pm i\sqrt{\frac{\beta_j}{\alpha_j}}, \quad j = 1, \dots, 6.$$

Clearly,

$$|\lambda_j| = 1, \quad j = 1, \dots, 6.$$

Hence

$$\delta_j = 0, \quad j = 1, \dots, 6.$$

Calculating γ_j $(j = 1, \dots, 6)$, we get

$$\gamma_j = \frac{1}{2} - \frac{1}{\pi}\operatorname{arctg}\frac{1}{\sqrt{\alpha_j\beta_j}}, \quad j = 1, \dots, 6.$$

□

If under the conditions of Theorem 5.5

$$\alpha_i\beta_i \neq \alpha_j\beta_j \quad \text{for all} \quad i \neq j, \quad i,j = 1,\dots,6,$$

then $B^0_{a_{pr}}(t) = \mathcal{I}$, i.e., the first terms of the asymptotic expansions (4.14), (4.15) contain no logarithms.

Remark 5.6. Let us assume that in the neighborhood of the boundary the contact surfaces $S_0^{(1)}$ and $S_0^{(2)}$ are parallel to the isotropic plane. We consider the case, where the coefficients $b^{(q)}_{ijlk} = 0$ and, instead of $a^{(q)}_{ijlk}$ and $c^{(q)}_{ijlk}$, we have the elastic constants of transversally-isotropic elastic bodies, i.e., instead of $a^{(q)}_{ijlk}$ we have

$$a^{(q)}_{11}, \quad a^{(q)}_{33}, \quad a^{(q)}_{13}, \quad a^{(q)}_{55}, \quad a^{(q)}_{66}, \quad q = 1,2,$$

and, instead of $c^{(q)}_{ijlk}$, we have

$$c^{(q)}_{11}, \quad c^{(q)}_{33}, \quad c^{(q)}_{13}, \quad c^{(q)}_{55}, \quad c^{(q)}_{66}, \quad q = 1,2,$$

which satisfy the following conditions:

$$\begin{cases} a^{(q)}_{11} - a^{(q)}_{66} > 0, \ \ a^{(q)}_{55} > 0, \ \ a^{(q)}_{66} > 0, \\ a^{(q)}_{33} > \dfrac{(a^{(q)}_{13})^2}{a^{(q)}_{11} - a^{(q)}_{66}}, \quad q = 1,2, \end{cases} \qquad \begin{cases} c^{(q)}_{11} - c^{(q)}_{66} > 0, \ \ c^{(q)}_{55} > 0, \ \ c^{(q)}_{66} > 0, \\ c^{(q)}_{33} > \dfrac{(c^{(q)}_{13})^2}{c^{(q)}_{11} - c^{(q)}_{66}}, \quad q = 1,2. \end{cases}$$

It is not difficult to see that in this case the eigenvalues α_j and β_j, $j = 1,\dots,6$, of the matrices $\sigma^+_{\mathbf{N}_1}$ and $\sigma^+_{\mathbf{N}_2}$ are calculated explicitly.

Let $a^{(q)}_{11} \neq a^{(q)}_{33}$ and $c^{(q)}_{11} \neq c^{(q)}_{33}$ $(q = 1,2)$; then the conditions 1)–2) of Theorem 5.5 have the form

$$\frac{C_1 - D_1}{D_2 - C_2} = \frac{B_1}{B_2}, \qquad \frac{\widetilde{C}_1 - \widetilde{D}_1}{\widetilde{D}_2 - \widetilde{C}_2} = \frac{\widetilde{B}_1}{\widetilde{B}_2},$$

where

$$B_q = \frac{a^{(q)}_{55}\left(\sqrt{a^{(q)}_{11}}\sqrt{a^{(q)}_{33}} - a^{(q)}_{13}\right)}{a^{(q)}_{55} + \sqrt{a^{(q)}_{11}}\sqrt{a^{(q)}_{33}}}, \quad C_q = \frac{1}{2}\,\frac{a^{(q)}_{11}a^{(q)}_{55}\left(\sqrt{a^{(q)}_{2}} + \sqrt{a^{(q)}_{3}}\right)}{a^{(q)}_{55} + \sqrt{a^{(q)}_{11}}\sqrt{a^{(q)}_{33}}},$$

$$D_q = \frac{1}{2}\,\frac{\sqrt{a^{(q)}_{33}}\sqrt{a^{(q)}_{11}}a^{(q)}_{55}\left(\sqrt{a^{(q)}_{2}} + \sqrt{a^{(q)}_{3}}\right)}{a^{(q)}_{55} + \sqrt{a^{(q)}_{11}}\sqrt{a^{(q)}_{33}}}, \quad q = 1,2,$$

$a^{(q)}_2$ and $a^{(q)}_3$ are the roots of the equation (see [26])

$$a^{(q)}_{11}a^{(q)}_{55}a^2 + \left[(a^{(q)}_{13} + a^{(q)}_{55})^2 - a^{(q)}_{11}a^{(q)}_{33} - (a^{(q)}_{55})^2\right]a + a^{(q)}_{33}a^{(q)}_{55} = 0, \quad q = 1,2,$$

and

$$\widetilde{B}_q = \frac{c^{(q)}_{55}\left(\sqrt{c^{(q)}_{11}}\sqrt{c^{(q)}_{33}} - c^{(q)}_{13}\right)}{c^{(q)}_{55} + \sqrt{c^{(q)}_{11}}\sqrt{c^{(q)}_{33}}}, \quad \widetilde{C}_q = \frac{1}{2}\,\frac{c^{(q)}_{11}c^{(q)}_{55}\left(\sqrt{c^{(q)}_{2}} + \sqrt{c^{(q)}_{3}}\right)}{c^{(q)}_{55} + \sqrt{c^{(q)}_{11}}\sqrt{c^{(q)}_{33}}},$$

$$\widetilde{D}_q = \frac{1}{2}\,\frac{\sqrt{c_{33}^{(q)}}\sqrt{c_{11}^{(q)}}c_{55}^{(q)}\left(\sqrt{c_2^{(q)}}+\sqrt{c_3^{(q)}}\right)}{c_{55}^{(q)}+\sqrt{c_{11}^{(q)}}\sqrt{c_{33}^{(q)}}}, \quad q=1,2,$$

where $c_2^{(q)}$ and $c_3^{(q)}$ are the roots of the equation

$$c_{11}^{(q)}c_{55}^{(q)}c^2+\big[(c_{13}^{(q)}+c_{55}^{(q)})^2-c_{11}^{(q)}c_{33}^{(q)}-(c_{55}^{(q)})^2\big]c+c_{33}^{(q)}c_{55}^{(q)}=0, \quad q=1,2.$$

If we assume that $a_{13}^{(q)}=-a_{55}^{(q)}$ and $c_{13}^{(q)}=-c_{55}^{(q)}$, $q=1,2$, then the conditions 1)–2) of Theorem 5.5 can be rewritten in a simpler form:

$$\frac{\sqrt{a_{11}^{(1)}}-\sqrt{a_{33}^{(1)}}}{\sqrt{a_{33}^{(2)}}-\sqrt{a_{11}^{(2)}}}=\frac{\sqrt{a_{55}^{(1)}}}{\sqrt{a_{55}^{(2)}}} \quad \text{and} \quad \frac{\sqrt{c_{11}^{(1)}}-\sqrt{c_{33}^{(1)}}}{\sqrt{c_{33}^{(2)}}-\sqrt{c_{11}^{(2)}}}=\frac{\sqrt{c_{55}^{(1)}}}{\sqrt{c_{55}^{(2)}}}.$$

Now let us consider the case, where the domains D_q, $q=1,2$, are filled with the same material. The following theorem is fulfilled (for analogous results in the case of elasticity theory see [7], [8]; see also [14]).

Theorem 5.7. *If the domains D_q, $q=1,2$, are filled with the same material, then the asymptotic expansion of solutions of the boundary-contact problem $\mathbf{M}_1$ near the contact boundary $\partial S_0^{(1)}$ takes the form*

$$\mathcal{U}^{(q)}(x_1,x_2,x_3)=(u^{(q)},\omega^{(q)})(x_1,x_2,x_3)$$
$$=\sum_{\vartheta=\pm1}\sum_{j=1}^{2}\sum_{s=1}^{l_0}\mathrm{Re}\Big\{\sum_{m=0}^{n_s-1}x_3^m\Big[d_{sjm}^{(q)}(x_1,\theta)(z_{s,theta}^{(q)})^{\frac14+\Delta_j(x_1)-m}\Big]c_{jm}^{(q)}(x_1)$$
$$+\sum_{\substack{k,l=0\\k+l+p+m\neq0}}^{M+2}\sum_{p+m=0}^{M+2-l}x_2^l x_3^m d_{slmpj}^{(q)}(x_1,\vartheta)(z_{s,\vartheta}^{(q)})^{\frac14+\Delta_j(x_1)+k+p}B_{skmpj}^{(q)}(x_1,\log z_{s,\vartheta}^{(q)})\Big\}$$
$$+\mathcal{U}_{M+1}^{(q)}(x_1,x_2,x_3) \quad \textit{for} \quad M>\frac{2}{r}-\min\{[s-1],1\}, \;\; q=1,2, \qquad (5.7)$$

with the coefficients $d_{sjm}^{(q)}(\cdot,\pm1)$, $c_{jm}^{(q)}$, $d_{slmpj}^{(q)}(\cdot,\pm1)\in C_0^\infty(\mathbb{R})$ and the remainder $\mathcal{U}_{M+1}^{(q)}\in C_0^{M+1}(\overline{\mathbb{R}}_\pm^3)$, $q=1,2$,

$$z_{s,+1}^{(q)}=(-1)^q(x_2+x_3\tau_{s,+1}^{(q)}), \;\; z_{s,-1}^{(q)}=(-1)^{q+1}(x_2-x_3\tau_{s,-1}^{(q)}), \;\; -\pi<\mathrm{Arg}\tau_{s,\pm1}^{(q)}<\pi.$$

In this case the parameters Δ_j are calculated by

$$\Delta_j(x_1)=\big(\delta_1^{(j)}(x_1),\dots,\delta_6^{(j)}(x_1)\big), \quad j=1,2,$$

where

$$\delta_k^{(1)}(x_1)=-\frac{i}{2\pi}\log|\lambda_k(x_1)|, \quad \delta_k^{(2)}(x_1)=\frac12-\frac{i}{2\pi}\log|\lambda_k(x_1)|, \;\; k=1,\dots,6.$$

Proof. When the domains D_q, $q=1,2$, are filled with the same material, we can show, like in [7], that the eigenvalues $\lambda_k(x_1)$, $k=1,\dots,12$, of the matrix

$$\big(\sigma_{\mathbf{R}}(x_1,0,+1)\big)^{-1}\sigma_{\mathbf{R}}(x_1,0,-1), \quad x_2\in\partial S_0^{(1)},$$

are calculated by means of the eigenvalues $\beta_k(x_1)$, $k=1,2,3$, of the matrix $\sigma_{\mathbb{V}_0^*}$, i.e.,

$$\lambda_k(x_1)=\begin{cases} i\sqrt{\dfrac{1-2\beta_k(x_1)}{1+2\beta_k(x_1)}}\,, & \text{if} \quad k=1,\dots,6, \\ -i\sqrt{\dfrac{1-2\beta_{k-6}(x_1)}{1+2\beta_{k-6}(x_1)}}\,, & \text{if} \quad k=7,\dots,12, \end{cases}$$

where $-1/2<\beta_k<1/2$, $k=1,\dots,6$.

Hence the exponent of the first term of the asymptotic expansion of solutions has the form

$$\frac{1}{2}\mp\frac{1}{4}-\frac{i}{2\pi}\,\log|\lambda_k(x_1)|, \quad k=1,\dots,6. \qquad \square$$

Note that in this case the asymptotic expansion has step equal to one half (see [8]).

The same asymptotic expansion can also be obtained near the contact boundary $\partial S_0^{(2)}$.

6. Solvability and Asymptotics of Solutions of the Boundary-Contact Wedge-Type Problem $\mathbf{M}_2$

We will formulate theorems of the uniqueness and existence of solutions of problem $\mathbf{M}_2$.

Assume that the following compatibility conditions are fulfilled on the curves $\partial S_0^{(1)}$, $\partial S_0^{(2)}$ ∂S_0:

$$\begin{gathered}\exists\Phi_0^{(i)},\ \ i=1,2,3,\ \ \Phi_0^{(i)}\in\mathbb{B}_{p,r}^{s-1}(\partial D_1),\ \ \Phi_0^{(2)}\in\mathbb{B}_{p,r}^{s-1}(\partial D_2^{(1)}),\ \ \Phi_0^{(3)}\in\mathbb{B}_{p,r}^{s-1}(\partial D_2^{(2)}),\\ \begin{aligned} &h_1-\pi_{S_0^{(1)}}\Phi_0^{(1)}+\pi_{S_0^{(1)}}\Phi_0^{(2)}\in\widetilde{\mathbb{B}}_{p,r}^{s-1}(S_0^{(1)}),\\ &h_2-\pi_{S_0^{(2)}}\Phi_0^{(1)}+\pi_{S_0^{(2)}}\Phi_0^{(3)}\in\widetilde{\mathbb{B}}_{p,r}^{s-1}(S_0^{(2)})\\ &\pi_{S_0}\Phi_0^{(2)}+\pi_{S_0}\Phi_0^{(3)}\in\widetilde{\mathbb{B}}_{p,r}^{s-1}(S_0)\end{aligned}\end{gathered} \tag{6.1}$$

hold for $\varphi_1\in\mathbb{B}_{p,r}^{s-1}(\partial D_1)$, $\varphi_2\in\mathbb{B}_{p,r}^{s-1}(\partial D_2^{(1)})$, $\varphi_3\in\mathbb{B}_{p,r}^{s-1}(\partial D_2^{(2)})$, $1<p<\infty$, $1\le r\le\infty$, $1/p-1/2<s<1/p+1/2$.

Here $\Phi_0^{(i)}$, $i=1,2,3$, are some fixed extensions of the functions φ_i, $i=1,2,3$, on ∂D_1, $\partial D_2^{(1)}$, $\partial D_2^{(2)}$, respectively.

Theorem 6.1. *Let $4/3<p<4$ and the compatibility conditions* (6.1) *be fulfilled on $\partial S_0^{(1)}$, $\partial S_0^{(2)}$ ∂S_0 for $s=1-1/p$. Then the boundary-contact wedge-type problem $\mathbf{M}_2$ has solutions of the classes $W_p^1(D_q)$, $q=1,2$, if and only if the condition*

$$\begin{gathered}\int_{S_1}\varphi_1([a\times z]+b,a)\,ds-\int_{S_2^{(1)}}\varphi_2([a\times z]+b,a)\,ds-\int_{S_2^{(2)}}\varphi_3([a\times z]+b,a)\,ds\\ +\int_{S_0^{(1)}}h_1([a\times z]+b,a)\,ds+\int_{S_0^{(2)}}h_2([a\times z]+b,a)\,ds=0\end{gathered}$$

is fulfilled, where a and b are arbitrary three-dimensional constant vectors.

Solutions of the boundary-contact problem $\mathbf{M}_2$ are given by the potential-type functions

$$\mathcal{U}^{(1)} = \mathbb{V}^{(1)}(\mathbb{V}^{(1)}_{-1})^{-1}(\mathbf{B}^{(1)}_{2M+1})^{-1}\varphi^{(1)}_0 + \mathbb{V}^{(1)}(\mathbb{V}^{(1)}_{-1})^{-1}(\mathbf{B}^{(1)}_{2M+1})^{-1}\psi^{(1)}_0 + R_1, \quad (6.2)$$

$$r_1\mathcal{U}^{(2)} = \mathbb{V}^{(2)}(\mathbb{V}^{(2)}_{-1})^{-1}(\mathbf{B}^{(2)}_{2M+1})^{-1}\varphi^{(2)}_0 + \mathbb{V}^{(2)}_{-1}(\mathbb{V}^{(2)}_{-1})^{-1}(\mathbf{B}^{(2)}_{2M+1})^{-1}\psi^{(2)}_0 + R_2, \quad (6.3)$$

$$r_2\mathcal{U}^{(2)} = \mathbb{V}^{(3)}(\mathbb{V}^{(3)}_{-1})^{-1}(\mathbf{B}^{(3)}_{2M+1})^{-1}\varphi^{(3)}_0 + \mathbb{V}^{(3)}_{-1}(\mathbb{V}^{(3)}_{-1})^{-1}(\mathbf{B}^{(3)}_{2M+1})^{-1}\psi^{(3)}_0 + R_3, \quad (6.4)$$

with $R_1 \in C^{M+1}(\overline{D}_1)$, $R_2 \in C^{M+1}(\overline{D}{}^{(1)}_2)$, $R_3 \in C^{M+1}(\overline{D}{}^{(2)}_2)$, while the functions $(\varphi^{(1)}_0, \varphi^{(2)}_0)$, $(\psi^{(1)}_0, \varphi^{(3)}_0)$ and $(\psi^{(2)}_0, \psi^{(3)}_0)$ are solutions of the systems

$$\begin{cases} \pi_{S^{(1)}_0}\mathbf{N}_1\varphi^{(1)}_0 = \Psi_1, \\ \varphi^{(2)}_0 = \varphi^{(1)}_0 + G_1, \end{cases} \qquad \begin{cases} \pi_{S^{(2)}_0}\mathbf{N}_2\psi^{(1)}_0 = \Psi_2, \\ \varphi^{(3)}_0 = \psi^{(1)}_0 + G_2, \end{cases} \qquad \begin{cases} \pi_{S_0}\mathbf{N}_3\psi^{(2)}_0 = \Psi_3, \\ \psi^{(3)}_0 = -\psi^{(2)}_0 + G_3, \end{cases}$$

respectively. Here

$$\mathbf{N}_1 = (\mathbf{B}^{(1)}_{2M+1})^{-1} - (\mathbf{B}^{(2)}_{2M+1})^{-1},$$
$$\mathbf{N}_2 = (\mathbf{B}^{(1)}_{2M+1})^{-1} - (\mathbf{B}^{(3)}_{2M+1})^{-1},$$
$$\mathbf{N}_3 = -(\mathbf{B}^{(2)}_{2M+1})^{-1} - (\mathbf{B}^{(3)}_{2M+1})^{-1},$$

where

$$\mathbf{B}^{(1)}_{2M+1} = -(\mathbb{V}^{(1)}_{-1})^{2M+1} + \Big(-\frac{1}{2}\mathcal{I} + \overset{*}{\mathbb{V}}{}^{(1)}_0\Big)(\mathbb{V}^{(1)}_{-1})^{-1},$$
$$\mathbf{B}^{(i)}_{2M+1} = (\mathbb{V}^{(i)}_{-1})^{2M+1} + \Big(\frac{1}{2}\mathcal{I} + \overset{*}{\mathbb{V}}{}^{(i)}_0\Big)(\mathbb{V}^{(i)}_{-1})^{-1}, \quad i = 2, 3.$$

The operators $\mathbf{N}_j$, $j = 1, 2, 3$, are positive-definite ΨDOs.

Let

$$\Delta_{\mathbf{N}_j}(x_1) = \big(\underbrace{\delta^{(1)}_j(x_1), \dots, \delta^{(1)}_j(x_1)}_{m_1\text{-times}}, \dots, \underbrace{\delta^{(\ell)}_j(x_1), \dots, \delta^{(\ell)}_j(x_1)}_{m_\ell\text{-times}}\big),$$

$$\delta^{(k)}_j(x_1) = -\frac{i}{2\pi}\log|\lambda^{(k)}_{\mathbf{N}_j}(x_1)|, \quad j = 1, 2, 3, \quad k = 1, \dots, \ell$$

(generally speaking, m_k and ℓ depend on j),

where $\lambda^{(k)}_{\mathbf{N}_j}$, $j = 1, 2$, $k = 1, \dots, \ell$, are the eigenvalues of the matrix

$$b_{\mathbf{N}_j}(x_1) = \big(\sigma_{\mathbf{N}_j}(x_1, 0, +1)\big)^{-1}\sigma_{\mathbf{N}_j}(x_1, 0, -1)$$

of multiplicity m_k, $k = 1, \dots, \ell$, $\sum\limits_{k=1}^{6} m_k = 6$; here m_k, $k = 1, \dots, \ell$, and ℓ depend on j, and the eigenvalues of the matrix $b_{\mathbf{N}_3} = I$ are trivial, $\lambda^{(k)}_{\mathbf{N}_3} = 1$, $k = 1, \dots, 6$.

Note that the boundary data of the problem $\mathbf{M}_2$ are assumed to be sufficiently smooth, i.e., $\varphi_1 \in \mathbb{H}^{(\infty,S+2M),\infty}_r(S_1)$, $\varphi_2 \in \mathbb{H}^{(\infty,S+2M),\infty}_r(S^{(1)}_2)$, $\varphi_3 \in \mathbb{H}^{(\infty,S+2M),\infty}_r(S^{(2)}_2)$, $f_i \in \mathbb{H}^{(\infty,S+2M+1),\infty}_r(S^{(i)}_0)$, $h_i \in \mathbb{H}^{(\infty,S+2M),\infty}_r(S^{(i)}_0)$, $i = 1, 2$, $\frac{1}{r} - \frac{1}{2} < s < \frac{1}{r} + \frac{1}{2}$.

Recalling that solutions of the problem $\mathbf{M}_2$ are represented by potential-type functions (see (6.2), (6.3), (6.4)) and using the asymptotic expansion of such functions (see [9, Theorems 2.2 and 2.3]), we obtain the following asymptotic expansions of problem $\mathbf{M}_2$ in terms of some local coordinate systems of curves $\partial S_0^{(1)}$, $\partial S_0^{(2)}$, ∂S_0:

a) Asymptotic expansion near the contact boundaries $\partial S_0^{(j)}$, $j=1,2$:

$$\begin{aligned}&\mathcal{U}^{(q)}(x_1,x_2,x_3)=(u^{(q)},\omega^{(q)})(x_1,x_2,x_3)\\&=\sum_{\vartheta=\pm1}\sum_{s=1}^{\ell_0}\mathrm{Re}\bigg\{\sum_{j=0}^{n_s-1}x_3^j\Big[d_{sj}^{(q)}(x_1,\theta)(z_{s,\theta}^{(q)})^{\frac{1}{2}-j+i\Delta(x_1)}\Big]c^{(j)}(x_1)\\&+\sum_{\substack{l,k=0\\k+l+p+j\neq1}}^{M+1}\sum_{p+j=1}^{M+2-l}x_2^lx_3^jd_{sljp}^{(q)}(x_1,\vartheta)z_{s,\vartheta}^{-\frac{1}{2}+i\Delta(x_1)+p+k}B_{skjp}^{(q)}(x_1,z_{s,\vartheta}^{(q)})\bigg\}\\&\mathcal{U}_{M+1}^{(q)}(x_1,x_2,x_3)\quad\text{for}\quad M>\frac{2}{r}-\min\{[s],0\},\quad q=1,2,\end{aligned}\tag{6.5}$$

with $\mathcal{U}_{M+1}^{(q)}\in C_0^{M+1}(\overline{\mathbb{R}}_\pm^3)$, $q=1,2$. Here $\Delta=\Delta_{\mathbf{N}_j}$, $j=1,2$, and

$z_{s,+1}^{(q)}=(-1)^q\big[x_2+x_3\tau_{s,+1}^{(q)}\big]$, $z_{s,-1}^{(q)}=(-1)^q\big[x_2-x_3\tau_{s,-1}^{(q)}\big]$, $-\pi<\mathrm{Arg}z_{s,\pm1}<\pi$.

The first coefficients $d_{sj}^{(q)}(\cdot,\pm1)$, $c^{(j)}$ are calculated as in(4.14) (see [9]).

b) Asymptotic expansions near the cuspidal edge ∂S_0 (see (4.16)):

$$\begin{aligned}&r_i\mathcal{U}^{(2)}(x_1,x_2,x_3)=r_i(u^{(2)},\omega^{(2)})(x_1,x_2,x_3)\\&=\sum_{\vartheta=\pm1}\sum_{s=1}^{\ell_0}\mathrm{Re}\bigg\{\sum_{j=0}^{n_s-1}x_3^jz_{s,\theta}^{\frac{1}{2}-j}d_{sj}^{(i)}(x_1,\theta)+\sum_{\substack{l,k=0\\l+k+j+p\neq1}}^{M+1}\sum_{j+p=1}^{M+2-l}x_2^lx_3^jz_{s,\vartheta}^{-\frac{1}{2}+p+k}d_{slkjp}^{(i)}(x_1)\bigg\}\\&+\mathcal{U}_{M+1}^{(i)}(x_1,x_2,x_3)\quad\text{for}\quad M>\frac{2}{r}-\min\{[s],0\},\quad i=1,2,\end{aligned}\tag{6.6}$$

with $\mathcal{U}_{M+1}^{(i)}\in C_0^{M+1}(\overline{\mathbb{R}}_\pm^3)$, $i=1,2$. Here

$$z_{s,+1}=-x_2-x_3\tau_{s,+1}^{(2)},\quad z_{s,-1}=-x_2-x_3\tau_{s,-1}^{(2)},\quad -\pi<\mathrm{Arg}z_{s,\pm1}<\pi.$$

The coefficients $d_{sj}^{(i)}(x_1,\pm1)$ are calculated as in (4.16).

Remark 6.2. Note that if we consider the boundary-contact problem $\mathbf{M}_2$ for the nonhomogeneous equations

$$\mathcal{M}^{(q)}(\partial_x)\mathcal{U}^{(q)}+\mathcal{F}^{(q)}=0\quad\text{in}\quad D_q,\quad q=1,2,$$

and use the boundary and boundary-contact data

$$\varphi_1=0,\quad\varphi_2=0,\quad\varphi_3=0,\quad f_i=h_i=0,\quad i=1,2,$$

then the compatibility conditions (6.1) are fulfilled automatically on $\partial S_0^{(1)}$, $\partial S_0^{(2)}$, ∂S_0.

Remark 6.3. It is easy to see that if the matrices $b_{\mathbf{N}_j}$, $j = 1, 2$, are unitary ($b^*_{\mathbf{N}_j} = b^{-1}_{\mathbf{N}_j}$), then the oscillation in the asymptotic expansion (42) vanishes, i.e., $\Delta_j = 0$, $j = 1, 2$.

In the case of transversally isotropic bodies we obtain the following necessary and sufficient conditions under which the oscillation vanishes:

$$\frac{B_1}{B_2} = \frac{4C_1D_1 - B_1^2}{4C_2D_2 - B_2^2}, \quad \frac{\widetilde{B}_1}{\widetilde{B}_2} = \frac{4\widetilde{C}_1\widetilde{D}_1 - \widetilde{B}_1^2}{4\widetilde{C}_2\widetilde{D}_2 - \widetilde{B}_2^2}. \tag{6.7}$$

For the centrally symmetric isotropic case condition (6.7) tales the form

$$\begin{aligned} &\lambda_1 + \mu_1 - \alpha_1 = \lambda_2 + \mu_2 - \alpha_2, \\ &\varepsilon_1 + \nu_1 - \beta_1 = \varepsilon_2 + \nu_2 - \beta_2. \end{aligned} \tag{6.8}$$

Condition (6.8) written in terms of the Poisson constants was found in [16], when investigating the asymptotic properties of solutions of boundary positive-definite pseudodifferential equations of crack-type problems of elasticity (see also [38]).

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Author's address:

A. Razmadze Mathematical Institute
Georgian Academy of Sciences
1, M. Aleksidze St., Tbilisi 0193
Georgia
E-mail: chkadua@rmi.acnet.ge