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ON THE MATHEMATICAL BASIS OF THE LINEAR SAMPLING METHOD

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Dedicated to the memory of Professor Victor Kupradze on the occasion of the 100^{th} anniversary of his birth

Abstract. The linear sampling method is an algorithm for solving the inverse scattering problem for acoustic and electromagnetic waves. The method is based on showing that a linear integral equation of first kind has a solution that becomes unbounded as a parameter z approaches the boundary of the scatterer D from inside D. However, except for the case of the transmission problem, the case where z is in the exterior of D is unresolved. Since for the inverse scattering problem D is unknown, this step is crucial for the mathematical justification of the linear sampling method. In this paper we give a mathematical justification of the linear sampling method for arbitrary z by using the theory of integral equations of first kind with singular kernels.

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1. Introduction

The inverse scattering problem of interest to us in this paper is that of determining the shape of a two dimensional scattering object from the knowledge of the far field pattern of the scattered wave where the incident field is a time harmonic plane wave. (The restriction to two dimensions is purely for convenience and all our results remain valid in the three-dimensional case). Until recently, the solution of this problem required more information, in particular whether or not the scattering object was penetrable or impenetrable as well as what boundary conditions were satisfied by the scattered field on the boundary of the scatterer D. However, in 1996 a method was introduced, which does not require this extra information and, in addition, is a linear algorithm for determining of the boundary Γ of the scatterer [5]. This method is called the *linear sampling method* and has been the subject of considerable attention since its introduction [1], [9]–[12].

To describe the linear sampling method, we assume the incident field u^i is given by (factoring out a term of the form $e^{-i\omega t}$ where ω is the frequency and t is time)

$$u^i(x) = e^{ikx\cdot d}, \tag{1.1}$$

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where $k > 0$ is the wave number, d is a unit vector and $x \in \mathbb{R}^2$. Then the scattered field $u^s(x) = u^s(x, d)$ has the asymptotic behavior

$$u^s(x) = \frac{e^{ikr}}{\sqrt{r}}\, u_\infty(\widehat{x}, d) + O(r^{-3/2}), \tag{1.2}$$

where u_∞ is the *far field pattern* [6]. The linear sampling method chooses a parameter $z \in \mathbb{R}^2$ and then looks for a solution $g \in L^2(\Omega)$ (where Ω is the unit circle in $\mathbb{R}^2$) of the *far field equation*

$$\int\limits_\Omega u_\infty(\widehat{x}, d) g(d)\, ds(d) = \Phi_\infty(\widehat{x}, z), \tag{1.3}$$

where $\widehat{x} = x/|x|$, and Φ_∞ is the far field pattern of the fundamental solution

$$\Phi(x, z) = H_0^{(1)}(k|x - z|) \tag{1.4}$$

with $H_0^{(1)}$ denoting a Hankel function of first kind of order zero. It can then be shown [4], [5] that for $z \in D$ and almost every k there exists an approximate regularized solution of (1.3) such that

$$\lim_{z \to \Gamma} \left\| g(\cdot, z) \right\|_{L^2(\Omega)} = \infty \tag{1.5}$$

and

$$\lim_{z \to \Gamma} \left\| v_g(\cdot, z) \right\|_{H^1(D)} = \infty, \tag{1.6}$$

where v_g is *Herglotz wave function* with kernel g defined by

$$v_g(x) = \int\limits_\Omega e^{ikx \cdot d} g(d)\, ds(d), \qquad x \in \mathbb{R}^2. \tag{1.7}$$

In particular, Γ is characterized by the norm of the regularized solution of (1.3) becoming unbounded. Note that although the proof of this fact required knowing the boundary conditions of u^s on Γ, the equation (1.3) that one needs to solve to determine Γ is independent of such knowledge, i.e., u_∞ is assumed to be given data (in general noisy).

A major problem with the above approach to solving the inverse scattering problem is that the conclusion (1.5) and (1.6) requires that $z \in D$ and nothing is said about what happens if $z \in \mathbb{R}^2 \setminus \overline{D}$. Since D is unknown, this at first glance seems disastrous. However, numerical experiments indicate that $\|g(\cdot, z)\|_{L^2(\Omega)}$ not only becomes large as z approaches Γ from inside D but continues to become larger as z moves into $\mathbb{R}^2 \setminus \overline{D}$. It is highly desirable to give a mathematical explanation of this observed numerical behaviour of the regularized solution of (1.3) for $z \in \mathbb{R}^2 \setminus \overline{D}$ since this would then put the linear sampling method on a clearer mathematical foundation.

For the case of the acoustic transmission problem this was done in [3] by the use of variational methods. Such an approach could probably also be carried out for other problems in the acoustic scattering theory such as the Dirichlet and mixed boundary value problems, but this has not been done.

Here, in honor of Professor Kupradze's fundamental contributions to the use of integral equation methods in the scattering theory, we propose to carry out the above program for the Dirichlet and mixed boundary value problems using the method of integral equations. For completeness, we also consider the special case of the transmission problem when the total field has no jump across the boundary since this case was excluded in the analysis of [3]. This task is accomplished by factoring the far field operator F in the form $F = \mathcal{F}S\mathcal{H}$ where $\mathcal{H}$ maps functions in $L^2(\Omega)$ onto Herglotz wave functions, S is a solution operator which maps the incident field onto a density function and $\mathcal{F}$ maps the density function onto the far-field pattern.

2. The Linear Sampling Method for the Inverse Obstacle Problem

2.1. **Formulation of the direct and inverse scattering problem.** Let $D \subset \mathbb{R}^2$ be an open, bounded region with Lipschitz boundary Γ such that $\mathbb{R}^2 \setminus \overline{D}$ is connected. We assume that the boundary Γ has a Lipschitz dissection $\Gamma = \Gamma_D \cup \Pi \cup \Gamma_I$, where Γ_D and Γ_I are disjoint, relatively open subsets of Γ, having Π as their common boundary in Γ (see e.g. [8]). Furthermore, boundary conditions of Dirichlet and impedance type with the surface impedance $\lambda \geq 0$ are specified on Γ_D and Γ_I, respectively. Let ν denote the unit outward normal vector defined almost everywhere on $\Gamma_D \cup \Gamma_I$.

We assume that the incident field u^i is given by the plane wave $e^{ikx\cdot d}$ where $x \in \mathbb{R}^2$, $k > 0$ is the wave number and d is a fixed unit vector describing the incident direction. If we denote the scattered field by u^s and define the total field by $u(x) = u^s(x) + e^{ikx\cdot d}$, then *the direct obstacle scattering problem* for the obstacle D is to find a weak solution $u \in H^1_{loc}(\mathbb{R}^2 \setminus \overline{D})$ of the following exterior mixed boundary value problem for the Helmholtz equation

$$\Delta u + k^2 u = 0 \qquad \text{in} \qquad \mathbb{R}^2 \setminus \overline{D}, \tag{2.1}$$

$$u = 0 \qquad \text{on} \qquad \Gamma_D, \tag{2.2}$$

$$\frac{\partial u}{\partial \nu} + ik\lambda u = 0 \qquad \text{on} \qquad \Gamma_I, \tag{2.3}$$

and u^s satisfies the Sommerfeld radiation condition [6]

$$\lim_{r\to\infty} \sqrt{r}\Big(\frac{\partial u^s}{\partial r} - iku^s\Big) = 0, \tag{2.4}$$

where $r = |x|$ and (2.4) is assumed to hold uniformly in $\widehat{x} = x/|x|$. In particular the above formulation covers the Dirichlet boundary if $\Gamma_I = \varnothing$ and the impedance boundary (Neumann boundary) if $\Gamma_D = \varnothing$ and $\lambda > 0$ (if $\Gamma_D = \varnothing$ and $\lambda = 0$).

In [2] it was proved that the direct scattering problem (2.1)–(2.4) admits a unique weak solution u and the corresponding scattered field u^s has the asymptotic behaviour (1.2) at infinity with the far field pattern $u_\infty(\widehat{x}, d)$. *The inverse obstacle scattering problem* is to determine D from the knowledge of $u_\infty(\widehat{x}, d)$

for $\widehat{x}$ and d on the unit circle Ω and fixed wave number k. The far field pattern u_∞ defines the *far-field operator* $F : L^2(\Omega) \longrightarrow L^2(\Omega)$ by

$$(Fg)(\widehat{x}) := \int_\Omega u_\infty(\widehat{x}, d)g(d)\, ds(d). \tag{2.5}$$

Then the linear sampling method looks for a solution $g = g(\cdot, z) \in L^2(\Omega)$ of the linear far field equation (1.3) which we write in the form [3, 4, 5]

$$(Fg)(\widehat{x}) = \gamma e^{-ik\widehat{x}\cdot z}, \tag{2.6}$$

$$\gamma = \frac{e^{i\pi/4}}{\sqrt{8\pi k}}\,. \tag{2.7}$$

Our main concern in this paper is to study the far field equation (2.6) for various sampling points $z \in \mathbb{R}^2$.

2.2. **The linear sampling method for Dirichlet boundary conditions.** In this section we consider the simple case where only Dirichlet data is given on the Lipschitz boundary Γ, that is $\Gamma_I = \varnothing$ in (2.1)–(2.4), and assume that k^2 is not a Dirichlet eigenvalue for $-\Delta$ in D.

The direct scattering problem is then a special case of the exterior Dirichlet boundary value problem

$$\Delta w + k^2 w = 0 \qquad \text{in} \qquad \mathbb{R}^2 \setminus \overline{D}, \tag{2.8}$$

$$w = f \qquad \text{on} \qquad \Gamma, \tag{2.9}$$

where w satisfies the Sommerfeld radiation condition (2.4). This problem has a unique solution $w \in H^1_{loc}(\mathbb{R}^2 \setminus \overline{D})$ provided $f \in H^{1/2}(\Gamma)$ (see e.g. [8]), and w has the asymptotic behavior (1.2) at infinity.

Let us define the linear operator B which maps the boundary data f onto the far field pattern w_∞ of the radiating solution w of (2.8)–(2.9). Since w depends continuously on the boundary data $f \in H^{1/2}(\Gamma)$, the operator $B : H^{1/2}(\Gamma) \longrightarrow L^2(\Omega)$ is bounded. By superposition, we have the relation

$$(Fg) = -B(\mathcal{H}g), \tag{2.10}$$

where $\mathcal{H}g$ is the trace on the boundary Γ of the Herglotz wave function v_g given by (1.7). Next we define the operator $\mathcal{F} : H^{-1/2}(\Gamma) \longrightarrow L^2(\Omega)$ by

$$(\mathcal{F}\phi)(\widehat{x}) = \int_\Gamma \phi(y)e^{-ik\widehat{x}\cdot y}\, ds(y), \qquad \widehat{x} \in \Omega. \tag{2.11}$$

For given $\phi \in H^{-1/2}(\Gamma)$, the function $(\mathcal{F}\phi)(\widehat{x})$ is the far field pattern of the radiating solution $\gamma^{-1}\mathcal{S}\phi(x)$ where $\mathcal{S}\phi$ is the single layer potential

$$\mathcal{S}\phi(x) := \int_\Gamma \phi(y)\Phi(x, y)ds(y), \qquad x \in \mathbb{R}^2 \setminus \Gamma, \tag{2.12}$$

with $\Phi(x, y)$ defined by (1.4). From the trace theorem we have $\mathcal{S}\phi(x)|_\Gamma = S\phi(x)$, where S is a boundary integral operator defined by

$$(S\phi)(x) := \int_\Gamma \phi(y)\Phi(x, y)ds(y), \qquad x \in \Gamma, \tag{2.13}$$

which under our assumption on k^2 is an isomorphism from $H^{-1/2}(\Gamma)$ onto $H^{1/2}(\Gamma)$ (see Theorem 7.6 in [8]).

Hence the following relation holds:

$$(\mathcal{F}\phi) = \gamma^{-1} B(S\phi). \tag{2.14}$$

Combining (2.10) and (2.14), we obtain the following factorization of the far field operator F:

$$(Fg) = -\gamma \mathcal{F} S^{-1}(\mathcal{H}g), \tag{2.15}$$

and hence the far field equation (2.6) can be written as

$$\mathcal{F} S^{-1}(\mathcal{H}g) = -e^{-ik\widehat{x}\cdot z}. \tag{2.16}$$

Theorem 2.1. *The operator $\mathcal{F} : H^{-1/2}(\Gamma) \longrightarrow L^2(\Omega)$ is injective and has dense range provided k^2 is not a Dirichlet eigenvalue for the negative Laplacian in D.*

Proof. Let $\mathcal{F}\phi = 0$ and $\phi \in H^{-1/2}(\Gamma)$. Then the radiating solution $\mathcal{S}\phi(x)$ has zero far field pattern, whence $\mathcal{S}\phi(x) \equiv 0$ for $x \in \mathbb{R}^2 \setminus \overline{D}$ (see Theorem 2.13 in [6]). The trace theorem yields $S\phi = 0$ almost everywhere on the boundary Γ, and from the injectivity of the boundary operator $\phi \equiv 0$ in $H^{-1/2}(\Gamma)$. Hence $\mathcal{F}$ is injective.

The dual (or transpose) operator $\mathcal{F}^\top : L^2(\Omega) \longrightarrow H^{1/2}(\Gamma)$ of $\mathcal{F}$ is given by

$$(\mathcal{F}^\top g)(y) = \int_\Omega g(\widehat{x})e^{-ik\widehat{x}\cdot y}\, ds(\widehat{x}), \qquad y \in \Gamma. \tag{2.17}$$

Let $\mathcal{F}^\top g = 0$ in $H^{1/2}(\Gamma)$. Then

$$v_g(y) := \int_\Omega g(\widehat{x})e^{-ik\widehat{x}\cdot y}\, ds(\widehat{x}) \tag{2.18}$$

defines a Herglotz wave function which solves the homogeneous Dirichlet problem in the interior of Γ. Since k^2 is not an eigenvalue v_g vanishes in D, and since v_g is analytic in $\mathbb{R}^2$ it follows that $v_g \equiv 0$ everywhere. Theorem 3.15 of [6] yields $g = 0$ on Ω, which means that $\mathcal{F}^\top$ is injective.

Now the range $\mathcal{F}$ can be characterized as follows (see, e.g., [8], p. 23)

$$\operatorname{kern} \mathcal{F}^\top = (\operatorname{range} \mathcal{F})^a, \tag{2.19}$$

where the annihilator $(\operatorname{range} \mathcal{F})^a$ is a closed subset of $L^2(\Omega)$ defined by

$$(\operatorname{range} \mathcal{F})^a = \left\{ g \in L^2(\Omega) : \langle g, \psi \rangle = 0 \quad \text{for all} \quad \psi \in \operatorname{range} \mathcal{F} \right\} \tag{2.20}$$

with $\langle g, \psi \rangle$ being the duality pairing given by $\int_\Gamma g\psi \, ds$. Therefore from (2.19) and the injectivity of $\mathcal{F}^\top$ we have

$$\left\{ g \in L^2(\Omega) : \langle g, \psi \rangle = 0 \quad \text{for all} \quad \psi \in \operatorname{range} \mathcal{F} \right\} = \{0\}\,, \tag{2.21}$$

whence the range of $\mathcal{F}$ is dense in $L^2(\Omega)$. □

We remark that from (2.14) the range of $\mathcal{F}$ coincides with the range of B since the operator $S : H^{-1/2}(\Gamma) \longrightarrow H^{1/2}(\Gamma)$ is bijective.

Now let us consider a far field equation in the form (2.16) and assume first that $z \in D$. Since $-e^{-ik\widehat{x}\cdot z}$ is in the range of $\mathcal{F}$ ($\gamma e^{-ik\widehat{x}\cdot z}$ is the far field pattern of the fundamental solution $\Phi(x, z)$), it follows that for any $z \in D$ there exists a unique $\phi_z \in H^{-1/2}(\Gamma)$ such that

$$(\mathcal{F}\phi_z)(\widehat{x}) = -e^{-ik\widehat{x}\cdot z}. \tag{2.22}$$

Hence, to solve the far field equation we have to find $g(\cdot, z) \in L^2(\Omega)$ such that $S^{-1}\mathcal{H}g(\cdot, z) = \phi_z$ or $\mathcal{H}g(\cdot, z) = S\phi_z$. Thus the corresponding Herglotz wave function $v_g(\cdot, z)$ solves the Dirichlet problem in D with boundary data $S\phi_z \in H^{1/2}(\Gamma)$. Unfortunately, this cannot always be done. However from the result of Colton and Sleeman [7] (justified for Lipschitz domains in [2]) we can approximate the solution $\mathcal{S}\phi_z \in H^1(D)$ of the Helmholtz equation by a Herglotz wave function $v_g(\cdot, z)$ and therefore by the continuity of the trace operator we can approximate $S\phi_z \in H^{1/2}(\Gamma)$ by $\mathcal{H}g(\cdot, z) := v_g(\cdot, z)|_\Gamma$. Hence, since S^{-1} is continuous, for every $\epsilon > 0$ we can find $g(\cdot, z) \in L^2(\Omega)$ such that

$$\left\| S^{-1}\mathcal{H}g(\cdot, z) - \phi_z \right\|_{H^{-1/2}(\Gamma)} < \epsilon, \tag{2.23}$$

which yields

$$\left\| \mathcal{F}S^{-1}\mathcal{H}g(\cdot, z) + e^{-ik\widehat{x}\cdot z} \right\|_{L^2(\Omega)} < c\epsilon \tag{2.24}$$

for some positive constant c or, in other words, $g(\cdot, z) \in L^2(\Omega)$ satisfies the far field inequality

$$\left\| Fg(\cdot, z) - \gamma e^{-ik\widehat{x}\cdot z} \right\|_{L^2(\Omega)} < c\epsilon. \tag{2.25}$$

Furthermore, since $\gamma e^{-ik\widehat{x}\cdot z} = \Phi_\infty(x, z)$, we have $S\phi_z \equiv -\Phi(x, z)|_{x\in\Gamma}$. Thus for this $g(\cdot, z)$ it follows from the boundedness of the trace operator and the fact that, for $z \in \Gamma$, $\Phi(\cdot, z) \notin H^1(D)$ that

$$\lim_{z\to\Gamma} \left\| v_g(\cdot, z) \right\|_{H^1(D)} = \infty \tag{2.26}$$

and hence

$$\lim_{z\to\Gamma} \left\| g(\cdot, z) \right\|_{L^2(\Omega)} = \infty. \tag{2.27}$$

Now let us consider $z \in \mathbb{R}^2 \setminus \overline{D}$. In this case $-e^{-ik\widehat{x}\cdot z}$ does not belong to the range of $\mathcal{F}$. But, from Theorem 2.1, by using Tikhonov regularization we can construct a regularized solution of (2.22). In particular, if $\phi_z^\alpha \in H^{-1/2}(\Gamma)$ is the regularized solution of (2.22) corresponding to the regularization parameter α (chosen by a regular regularization strategy, e.g., the Morozov discrepancy principle), we have

$$\left\| (\mathcal{F}\phi_z^\alpha)(\widehat{x}) + e^{-ik\widehat{x}\cdot z} \right\|_{L^2(\Omega)} < \delta, \qquad \delta > 0, \tag{2.28}$$

and

$$\lim_{\alpha\to 0}\left\|\phi_z^\alpha\right\|_{H^{-1/2}(\Gamma)}=\infty. \tag{2.29}$$

The above considerations for ϕ_z in the case of $z\in D$ are valid for ϕ_z^α as well. In particular for every $\epsilon'>0$ we can find a function $g_\alpha(\cdot,z)\in L^2(\Omega)$ such that

$$\left\|S^{-1}\mathcal{H}g_\alpha(\cdot,z)-\phi_z^\alpha\right\|_{H^{-1/2}(\Gamma)}<\epsilon'. \tag{2.30}$$

Combining (2.28) and (2.30) we have that $g_\alpha(\cdot,z)\in L^2(\Omega)$ satisfies

$$\left\|\mathcal{F}S^{-1}\mathcal{H}g_\alpha(\cdot,z)+e^{-ik\widehat{x}\cdot z}\right\|_{L^2(\Omega)}<\epsilon+\delta \tag{2.31}$$

or, in other words, the far field inequality

$$\left\|Fg_\alpha(\cdot,z)-\gamma e^{-ik\widehat{x}\cdot z}\right\|_{L^2(\Omega)}<\epsilon+\delta \tag{2.32}$$

for $\epsilon>0$ and $\delta>0$ arbitrary small. In addition, (2.29) and (2.30) yield

$$\lim_{\alpha\to 0}\left\|\mathcal{H}g_\alpha(\cdot,z)\right\|_{H^{1/2}(\Gamma)}=\infty, \tag{2.33}$$

and hence

$$\lim_{\alpha\to 0}\left\|g_\alpha(\cdot,z)\right\|_{L^2(\Gamma)}=\infty \tag{2.34}$$

and

$$\lim_{\alpha\to 0}\left\|v_{g_\alpha}(\cdot,z)\right\|_{H^1(D)}=\infty, \tag{2.35}$$

where v_{g_α} is the Herglotz wave function with kernel g_α.

We summarize these results in the following theorem, noting that for $z\in\mathbb{R}^2\setminus\overline{D}$ we have that $\alpha\to 0$ as $\delta\to 0$.

Theorem 2.2. *Assume that Γ is Lipschitz and k^2 is not a Dirichlet eigenvalue for the negative Laplacian in the interior of Γ. Then if F is the far field operator corresponding to the scattering problem for Dirichlet boundary conditions, i.e. (2.1)–(2.4) with $\Gamma_I=\varnothing$, we have that*

1) if $z\in D$, then for every $\epsilon>0$ there exists a solution $g^\epsilon(\cdot,z)\in L^2(\Omega)$ of the inequality

$$\left\|Fg^\epsilon(\cdot,z)-\gamma e^{-ik\widehat{x}\cdot z}\right\|_{L^2(\Omega)}<\epsilon$$

such that

$$\lim_{z\to\Gamma}\left\|g^\epsilon(\cdot,z)\right\|_{L^2(\Omega)}=\infty \quad \text{and} \quad \lim_{z\to\Gamma}\left\|v_{g^\epsilon}(\cdot,z)\right\|_{H^1(D)}=\infty,$$

where v_{g^ϵ} is the Herglotz wave function with kernel g^ϵ, and

2) if $z\in\mathbb{R}^2\setminus\overline{D}$, then for every $\epsilon>0$ and $\delta>0$ there exists a solution $g^{\epsilon,\delta}(\cdot,z)\in L^2(\Omega)$ of the inequality

$$\left\|Fg^{\epsilon,\delta}(\cdot,z)-\gamma e^{-ik\widehat{x}\cdot z}\right\|_{L^2(\Omega)}<\epsilon+\delta$$

such that

$$\lim_{\delta\to 0}\left\|g^{\epsilon,\delta}(\cdot,z)\right\|_{L^2(\Omega)}=\infty \quad \text{and} \quad \lim_{\delta\to 0}\left\|v_{g^{\epsilon,\delta}}(\cdot,z)\right\|_{H^1(D)}=\infty,$$

where $v_{g^{\epsilon,\delta}}$ is the Herglotz wave function with kernel $g^{\epsilon,\delta}$.

In the Introduction we claimed that the above analysis provides a mathematical explanation for the numerical behaviour exhibited in the implementation of the linear sampling method. A legitimate criticism of this statement is that we do not know how well the regularized solution of

$$Fg = \Phi_\infty \tag{2.36}$$

approximates the solution g of

$$Fg = \Phi_\infty^{\epsilon,\delta}, \tag{2.37}$$

where $\Phi_\infty^{\epsilon,\delta}$ is the approximation of Φ_∞ due to the approximation of the far field pattern Φ_∞ (measured by δ) and the approximation of the solution of the Helmholtz equation by a Herglotz wave function (measured by ϵ). In particular since in general a solution of (2.36) does not exist for z either in D or in $\mathbb{R}^2 \setminus \overline{D}$, it makes no sense to let ϵ and δ tend to zero, i.e. ϵ and δ are fixed parameters. However the same criticism applies in practice to any regularization scheme since noise is not a variable but rather a fixed parameter. In particular even if an exact solution to (2.36) did exist in a noise free environment (i.e. u_∞ is free of noise), in practice the kernel is noisy and one has no idea if the noise is small enough so that the regularized solution of the equation with noisy data is in fact a good approximation to the solution of (2.36) with noise free data. The only statement that can be made is what happens if the noise tends to zero. However, since the noise is fixed and nonzero, in either case the analysis leads to the same conclusion: there is a "nearby" equation (F with noisy kernel and, in the case of (2.37), also with the inexact right-hand side) whose solution behaves in a known way, and if this "nearby" equation is "close enough" to (2.36) (with u_∞ free of noise), then one expects the regularized solution to behave like the known solution. In particular, since error estimates are not available for the dependency of the regularized solution on the noise level, the remark of Lanczos is valid: "the lack of information cannot be remedied by any mathematical trickery". Nevertheless, an explanation such as that given above is valuable since it provides an explanation of the observed numerical behaviour of the regularized solution and thus an understanding of why the linear sampling method works.

2.3. **The linear sampling method for mixed boundary conditions.** In this section we show that the argument of the previous analysis can be easily applied to the more general case of the obstacle scattering problems involving partially coated obstacles or multiple scattering objects with different boundary conditions. To our knowledge the linear sampling method is the only numerically viable method for solving inverse scattering problems in the case of (unknown) mixed boundary conditions (see [2]). Here we give a mathematical justification of the method.

We consider the Lipschitz boundary Γ dissected $\Gamma = \Gamma_D \cup \Pi \cup \Gamma_I$ as described in Section 2.1, and furthermore assume that $\Gamma_I \neq \varnothing$. The direct obstacle scattering problem (2.1)–(2.4) is a special case of the exterior mixed boundary value problem

$$\Delta w + k^2 w = 0 \qquad \text{in} \qquad \mathbb{R}^2 \setminus \overline{D}, \tag{2.38}$$

$$w = f \qquad \text{on} \qquad \Gamma_D, \tag{2.39}$$

$$\frac{\partial w}{\partial \nu} + i\lambda k w = h \qquad \text{on} \qquad \Gamma_I, \tag{2.40}$$

where $\lambda \geq 0$ is a constant, and w satisfies Sommerfeld radiation condition (2.4). If $f \in H^{1/2}(\Gamma_D)$ and $h \in H^{-1/2}(\Gamma_I)$ then the exterior mixed boundary value problem has a unique solution $w \in H^1_{loc}(\mathbb{R}^2 \setminus \overline{D})$ [2], and w has the asymptotic behavior (1.2) at infinity with far field pattern w_∞. We recall that for $\Gamma_0 \subseteq \Gamma$

$$\begin{aligned} H^{1/2}(\Gamma_0) &:= \{u|_{\Gamma_0} : u \in H^{1/2}(\Gamma)\}, \\ \tilde{H}^{1/2}(\Gamma_0) &:= \{u \in H^{1/2}(\Gamma) : \operatorname{supp} u \subseteq \overline{\Gamma}_0\} \end{aligned} \tag{2.41}$$

and, moreover, $H^{-1/2}(\Gamma_0) := \big(\tilde{H}^{1/2}(\Gamma_0)\big)'$ and $\tilde{H}^{-1/2}(\Gamma_0) := \big(H^{1/2}(\Gamma_0)\big)'$.

Let B be the bounded operator from $H^{1/2}(\Gamma_D) \times H^{-1/2}(\Gamma_I)$ onto $L^2(\Omega)$ which maps the boundary data (f, h) to the far field pattern w_∞. Thus the far field operator (2.5) corresponding to this case can be written

$$(Fg) = -B(\mathcal{H}g), \tag{2.42}$$

where now the boundary trace $\mathcal{H}g$ of the Herglotz wave function is given by

$$\mathcal{H}g(x) := \begin{cases} \displaystyle\int_\Omega g(d) e^{ikx\cdot d} ds(d), & x \in \Gamma_D, \\ \displaystyle\frac{\partial}{\partial \nu_x} \int_\Omega g(d) e^{ikx\cdot d} ds(d) + ik\lambda \int_\Omega g(d) e^{ikx\cdot d} ds(d), & x \in \Gamma_I. \end{cases} \tag{2.43}$$

We define the operator $\mathcal{F} : \tilde{H}^{-1/2}(\Gamma_D) \times \tilde{H}^{1/2}(\Gamma_I) \longrightarrow L^2(\Omega)$ by

$$\begin{aligned} \mathcal{F}(\phi_D, \phi_I)(\widehat{x}) = &\int_{\Gamma_D} \phi_D(y) e^{-ik\widehat{x}\cdot y}\, ds(y) + \int_{\Gamma_I} \phi_I(y) \frac{\partial}{\partial \nu_y} e^{-ik\widehat{x}\cdot y}\, ds(y) \\ &+ ik\lambda \int_{\Gamma_I} \phi_I(y) e^{-ik\widehat{x}\cdot y}\, ds(y), \qquad \widehat{x} \in \Omega, \end{aligned} \tag{2.44}$$

and observe that for a given pair $(\phi_D, \phi_I) \in \tilde{H}^{-1/2}(\Gamma_D) \times \tilde{H}^{1/2}(\Gamma_I)$, the function $\mathcal{F}(\phi_D, \phi_I)(\widehat{x})$ is the far field pattern of the radiating solution $\gamma^{-1}\mathcal{P}(\phi_D, \phi_I)(x)$ with

$$\mathcal{P}(\phi_D, \phi_I) := \mathcal{S}\tilde{\phi}_D + \mathcal{D}\tilde{\phi}_I + ik\lambda \mathcal{S}\tilde{\phi}_I, \tag{2.45}$$

where $\tilde{\phi}_D \in H^{-1/2}(\Gamma)$ and $\tilde{\phi}_I \in H^{1/2}(\Gamma)$ are the extensions by zero of ϕ_D with support in $\overline{\Gamma}_D$ and ϕ_I with support in $\overline{\Gamma}_I$, respectively. Here $\mathcal{S}\phi$ denotes the single layer potential (2.12) and $\mathcal{D}\phi$ the double layer potential defined by

$$\mathcal{D}\phi(x) := \int_\Gamma \phi(y) \frac{\partial}{\partial \nu_y} \Phi(x, y) ds(y), \qquad x \in \mathbb{R}^2 \setminus \Gamma. \tag{2.46}$$

Since $\phi_D|_{\Gamma_I} \equiv 0$ and $\phi_I|_{\Gamma_D} \equiv 0$, the trace theorem and jump relations of the single- and double- layer potentials on the boundary Γ yield

$$\begin{pmatrix} \mathcal{P}(\phi_D, \phi_I)|_{\Gamma_D} \\ \left(\frac{\partial}{\partial \nu} + ik\lambda\right)\mathcal{P}(\phi_D, \phi_I)|_{\Gamma_I} \end{pmatrix} = \frac{1}{2} M \begin{pmatrix} \phi_D \\ \phi_I \end{pmatrix}, \tag{2.47}$$

where the operator M is given by

$$M = \begin{pmatrix} S_{DD}, & K_{DI} + ik\lambda S_{DI} \\ K'_{ID} + ik\lambda S_{ID}, & -k^2\lambda^2 S_{II} + ik\lambda(K'_{II} + K_{II}) + T_{II} \end{pmatrix}. \tag{2.48}$$

Here S, K, K', T denote the four basic boundary integral operators defined by

$$S\phi(x) := 2\int_\Gamma \phi(y)\Phi(x,y)ds_y, \qquad K\phi(x) := 2\int_\Gamma \phi(y)\frac{\partial}{\partial \nu_y}\Phi(x,y)ds_y,$$

$$K'\phi(x) := 2\int_\Gamma \phi(y)\frac{\partial}{\partial \nu_x}\Phi(x,y)ds_y, \qquad T\phi(x) := 2\frac{\partial}{\partial \nu_x}\int_\Gamma \psi(y)\frac{\partial}{\partial \nu_y}\Phi(x,y)ds_y,$$

the operator S_{ID} is the operator S applied to a function ϕ with $\operatorname{supp}\phi \subseteq \overline{\Gamma}_D$ and evaluated on Γ_I, with analogous definitions for S_{DD}, S_{DI}, S_{II}, K_{DI}, K'_{ID}, K_{II}, K'_{II} and T_{II}. In [2] it is proved that provided $\lambda > 0$ and $\Gamma_I \neq \varnothing$ the operator

$$A = M \cdot \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} : \tilde{H}^{-1/2}(\Gamma_D) \times \tilde{H}^{1/2}(\Gamma_I) \longrightarrow H^{1/2}(\Gamma_D) \times H^{-1/2}(\Gamma_I),$$

with I the identity operator, is bijective. Hence, if we define $H := \tilde{H}^{-1/2}(\Gamma_D) \times \tilde{H}^{1/2}(\Gamma_I)$ and its dual $H^* := H^{1/2}(\Gamma_D) \times H^{-1/2}(\Gamma_I)$, then M is an isomorphism from H onto H^*.

Obviously, the following relation holds:

$$\mathcal{F}(\phi_D, \phi_I) = -\gamma BM(\phi_D, \phi_I), \tag{2.49}$$

and hence the far field equation can be written as

$$\mathcal{F}M^{-1}(\mathcal{H}g) = -e^{-ik\widehat{x}\cdot z}. \tag{2.50}$$

Theorem 2.3. *Assume that $\lambda > 0$ and $\Gamma_I \neq \varnothing$. Then the operator $\mathcal{F} : H \longrightarrow L^2(\Omega)$ defined by (2.44) is injective and has dense range.*

Proof. The proof of the theorem proceeds in exactly the same way as in Theorem 2.1. We note that in this case the dual operator $\mathcal{F}^\top : L^2(\Omega) \longrightarrow H^*$ is given by

$$(\mathcal{F}^\top g)(y) := \begin{cases} \displaystyle\int_\Omega g(\widehat{x})e^{-ik\widehat{x}\cdot y}ds(\widehat{x}), & y \in \Gamma_D, \\ \displaystyle\frac{\partial}{\partial \nu_x}\int_\Omega g(d)e^{-ik\widehat{x}\cdot y}ds(\widehat{x}) + ik\lambda\int_\Omega g(\widehat{x})e^{-ik\widehat{x}\cdot y}ds(\widehat{x}), & y \in \Gamma_I. \end{cases}$$

Moreover, in the proof of this theorem the bijective operator M given by (2.48) plays the role of the operator S in Theorem 2.1. $\square$

We are now ready to analyze the far field equation written in the form (2.50). We proceed for $\mathcal{F}$ given by (2.44) and $\mathcal{H}$ given by (2.43) in the same way as in Section 2.2. Note that the role of a weak solution $\mathcal{S}\phi_z \in H^1(D)$ and the boundary operator S is now replaced by a weak solution $\mathcal{P}(\phi_D, \phi_I) \in H^1(D)$ and the boundary operator M, respectively. Hence we conclude that Theorem 2.2 is also valid for the far field operator F corresponding to the scattering problem (2.1)–(2.4) provided $\lambda > 0$ and $\Gamma_I \neq \varnothing$:

Theorem 2.4. *Assume that Γ is Lipschitz having a Lipschitz dissection $\Gamma = \Gamma_D \cup \Pi \cup \Gamma_I$ with $\Gamma_I \neq \varnothing$, and $\lambda > 0$. Then if F is the far field operator corresponding to* (2.1)–(2.4) *we have that*

1) *if $z \in D$, then for every $\epsilon > 0$ there exists a solution $g^\epsilon(\cdot, z) \in L^2(\Omega)$ of the inequality*

$$\left\| Fg^\epsilon(\cdot, z) - \gamma e^{-ik\widehat{x}\cdot z} \right\|_{L^2(\Omega)} < \epsilon$$

such that

$$\lim_{z\to\Gamma} \left\| g^\epsilon(\cdot, z) \right\|_{L^2(\Omega)} = \infty \quad \textit{and} \quad \lim_{z\to\Gamma} \left\| v_{g^\epsilon}(\cdot, z) \right\|_{H^1(D)} = \infty,$$

where v_{g^ϵ} is the Herglotz wave function with kernel g^ϵ, and

2) *if $z \in \mathbb{R}^2 \setminus \overline{D}$, then for every $\epsilon > 0$ and $\delta > 0$ there exists a solution $g^{\epsilon,\delta}(\cdot, z) \in L^2(\Omega)$ of the inequality*

$$\left\| Fg^{\epsilon,\delta}(\cdot, z) - \gamma e^{-ik\widehat{x}\cdot z} \right\|_{L^2(\Omega)} < \epsilon + \delta$$

such that

$$\lim_{\delta\to 0} \left\| g^{\epsilon,\delta}(\cdot, z) \right\|_{L^2(\Omega)} = \infty \quad \textit{and} \quad \lim_{\delta\to 0} \left\| v_{g^{\epsilon,\delta}}(\cdot, z) \right\|_{H^1(D)} = \infty,$$

where $v_{g^{\epsilon,\delta}}$ is the Herglotz wave function with kernel $g^{\epsilon,\delta}$.

We remark that if $\lambda = 0$, i.e. the Neumann boundary condition is assumed on Γ_I, the same conclusion remains valid provided k^2 is not an eigenvalue for the interior homogeneous mixed boundary value problem in D.

In the particular case of the impedance problem, i.e. $\Gamma_D = \varnothing$, $\lambda > 0$, we obtain the same result as in Theorem 2.4 by an appropriate modification of the previous analysis.

3. The Linear Sampling Method for the Inverse Medium Problem

We now turn our attention to the scattering of a plane wave by a penetrable inhomogeneous medium of compact support. In particular, consider the direct scattering problem of finding $u \in C^2(\mathbb{R}^2 \setminus \Gamma) \cup C^1(\mathbb{R}^2)$ such that

$$\Delta u + k^2 n(x) u = 0 \qquad \text{in} \qquad \mathbb{R}^2 \setminus \Gamma, \tag{3.1}$$

$$u(x) = e^{ikx\cdot d} + u^s(x), \tag{3.2}$$

$$\lim_{r\to\infty} \left(\frac{\partial u^s}{\partial r} - iku^s \right) = 0. \tag{3.3}$$

The index of refraction n is assumed to be piecewise continuously differentiable such that $m := 1 - n$ has a compact support $\overline{D} \subset \mathbb{R}^2$ where the complement of $\overline{D}$ is connected and $\overline{D}$ has a smooth boundary Γ with unit outward normal ν. More specifically, we assume that n is smooth except for jump discontinuity across Γ. We will further restrict ourselves to the case where $\operatorname{Im} n(x) \geq c > 0$ for $x \in D$ and c is a constant (absorbing medium). However, if we have that $\operatorname{Im} n(x) = 0$ for $x \in D$ (nonabsorbing medium), then the analysis that follows remains valid if we assume that k is not a transmission eigenvalue (c.f. [6] for the definition of a transmission eigenvalue).

Under these assumptions it is known [6] that there exists a unique solution u to (3.1)–(3.3) and u has the asymptotic behavior (1.2) at infinity with the far field pattern u_∞ given by

$$u_\infty(\widehat{x}; d) = -\gamma \int\limits_D e^{-ik\widehat{x}\cdot y} m(y) u(y)\, dy, \qquad \widehat{x} \in \Omega. \tag{3.4}$$

The scattering problem (3.1)–(3.3) can be written as the integral equation

$$e^{ikx\cdot d} = u(x,d) + k^2 T u(\cdot, d)(x) \tag{3.5}$$

with the operator $T : L^2(D) \longrightarrow L^2(D)$ defined by

$$(T\varphi)(x) := \int\limits_D \Phi(x,y) m(y) \varphi(y)\, dy \qquad \text{for} \qquad x \in D, \tag{3.6}$$

where $\Phi(x,y)$ is given by (1.4). It is known [6] that the operator $I + k^2 T$ is an isomorphism from $L^2(D)$ onto itself.

The inverse medium scattering problem we discuss in this section is to determine the support D of $m = 1 - n$ from the knowledge of $u_\infty(\widehat{x}, d)$ for $\widehat{x}$ and d on the unit circle Ω and fixed wave number k. We use the linear sampling method to solve the inverse medium problem. In other words, our aim is to show again in this case that there exists an approximate solution $g(\cdot, z)$ of the far field equation (1.3) such that $\|g(\cdot, z)\|_{L^2(\Omega)}$ and $\|v_g(\cdot, z)\|_{L^2(D)}$, become unbounded as z tends to Γ and remain such for $z \in \mathbb{R}^2 \setminus D$.

If v_g is a Herglotz wave function with kernel g, then by virtue of (3.4) the far field equation (2.5) takes the form

$$-\gamma \int\limits_D e^{-ik\widehat{x}\cdot y} m(y) (I + k^2 T)^{-1} v_g(y)\, dy = \gamma e^{-ik\widehat{x}\cdot z}, \qquad z \in \mathbb{R}^2, \tag{3.7}$$

where γ is given by (2.7). Denote by $\overline{H}$ the closure of the set of Herglotz wave functions in $L^2(D)$ and define the operator $\mathcal{F} : L^2(D) \longrightarrow L^2(\Omega)$ by

$$(\mathcal{F}\varphi)(\widehat{x}) := \gamma \int\limits_D e^{-ik\widehat{x}\cdot y} m(y) \varphi(y)\, dy, \qquad \widehat{x} \in \Omega. \tag{3.8}$$

Hence, in terms of the operator $\mathcal{F}$ the far field equation (3.7) can be written as

$$\mathcal{F}(I + k^2 T)^{-1} v_g = -e^{-ik\widehat{x}\cdot z}. \tag{3.9}$$

We note that $\mathcal{F}(I + k^2 T)^{-1}$ is clearly a bounded operator from $L^2(D)$ onto itself.

For the following analysis we recall the *interior transmission problem* of finding a pair of functions v, w such that

$$\Delta w + k^2 n(x) w = 0, \qquad \Delta v + k^2 v = 0 \qquad \text{in} \quad D, \tag{3.10}$$

and v, w satisfy the transmission conditions

$$w - v = \Phi(\cdot, z), \qquad \frac{\partial w}{\partial \nu} - \frac{\partial v}{\partial \nu} = \frac{\partial}{\partial \nu}\Phi(\cdot, z) \qquad \text{on} \quad \Gamma. \tag{3.11}$$

For $z \in D$, this problem has a unique weak solution $v \in \overline{H}$ and $w \in L^2(D)$, (for the definition of a weak solution and the proof of this statement see [4] or Theorem 10.24, Theorem 10.25 in [6]).

Theorem 3.1. *The operator* $\mathcal{F}(I + k^2 T)^{-1} : \overline{H} \subset L^2(D) \longrightarrow L^2(\Omega)$ *is injective and has dense range.*

Proof. Let $\mathcal{F}(I + k^2 T)^{-1} v = 0$ for a $v \in \overline{H}$ and set $w = (I + k^2 T)^{-1} v$. Then the pair $v \in \overline{H}$ and $w \in L^2(D)$ is a solution of the homogenous interior transmission problem, and by uniqueness $v \equiv 0$. Hence $\mathcal{F}(I + k^2 T)^{-1}$ is injective.

Now, let us suppose that there exists $g \in L^2(\Omega)$ such that for all $v \in \overline{H}$ we have

$$\big(\mathcal{F}(I + k^2 T)^{-1} v, g\big) = 0. \tag{3.12}$$

Then the far field patterns corresponding to all Herglotz wave functions as incident fields are orthogonal to g. But the far field patterns corresponding to the incident fields $e^{ikx\cdot d}$, $d \in \Omega$, are complete in $L^2(\Omega)$ (Theorem 8.12 in [6]) whence the far field patterns corresponding to all Herglotz functions are complete in $L^2(\Omega)$. Hence $g \equiv 0$, and therefore the range $\mathcal{F}(I + k^2 T)^{-1}(\overline{H})$ is dense in $L^2(\Omega)$. □

Now let us consider the far field equation (3.9). If $z \in D$, we have that $e^{-ik\hat{x}\cdot z}$ is in the range of $\mathcal{F}(I + k^2 T)^{-1}$. Then, let $v(\cdot, z) \in \overline{H}$ and $w(\cdot, z) = (I + k^2 T)^{-1} v(\cdot, z) \in L^2(D)$ be a unique weak solution of the interior transmission problem (3.10)–(3.11). In this case it can be shown that (see Theorem 10.26 in [6]) for every $\epsilon > 0$ the kernel $g(\cdot, z) \in L^2(\Omega)$ of the Herglotz wave function $v_g(\cdot, z)$ which approximates $v(\cdot, z) \in \overline{H}$, i.e., $\big\|v(\cdot, z) - v_g(\cdot, z)\big\|_{L^2(\Omega)} < \epsilon'$ for sufficiently small $\epsilon' > 0$, solves the inequality

$$\big\|\mathcal{F}(I + k^2 T)^{-1} v_g(\cdot, z) + e^{-ik\hat{x}\cdot z}\big\|_{L^2(\Omega)} < \epsilon, \qquad z \in D, \tag{3.13}$$

and, moreover, satisfies

$$\lim_{z\to\Gamma} \big\|g(\cdot, z)\big\|_{L^2(\Gamma)} = \infty \qquad \text{and} \qquad \lim_{z\to\Gamma} \big\|v_g(\cdot, z)\big\|_{L^2(D)} = \infty. \tag{3.14}$$

If $z \in \mathbb{R}^2 \setminus \overline{D}$, then $e^{-ik\hat{x}\cdot z}$ is not in the range of $\mathcal{F}(I + k^2 T)^{-1}$. But, from Theorem 3.1, by using Tikhonov regularization we can construct a regularized solution $v^\alpha(\cdot, z) \in \overline{H}$ depending on the regularization parameter α such that

$$\big\|\mathcal{F}(I + k^2 T)^{-1} v^\alpha(\cdot, z) + e^{-ik\hat{x}\cdot z}\big\|_{L^2(\Omega)} < \delta, \qquad z \in \mathbb{R}^2 \setminus \overline{D}, \tag{3.15}$$

for arbitrary small $\delta > 0$, and

$$\lim_{\alpha\to 0} \left\| v^{\alpha}(\cdot, z) \right\|_{L^2(D)} = \infty. \tag{3.16}$$

By the definition of the space $\overline{H}$ we can approximate $v^{\alpha}(\cdot, z) \in \overline{H}$ by a Herglotz wave function $v_{g_\alpha}(\cdot, z)$ with kernel $g_\alpha(\cdot, z)$, i.e., for every $\epsilon' > 0$ there exists $g_\alpha(\cdot, z) \in L^2(\Omega)$ such that

$$\left\| v^{\alpha}(\cdot, z) - v_{g_\alpha}(\cdot, z) \right\|_{L^2(D)} < \epsilon'. \tag{3.17}$$

Then, by the continuity of $\mathcal{F}(I + k^2 T)^{-1}$, we have that

$$\left\| \mathcal{F}(I + k^2 T)^{-1} v^{\alpha}(\cdot, z) - \mathcal{F}(I + k^2 T)^{-1} v_{g_\alpha}(\cdot, z) \right\|_{L^2(D)} < \bar{c}\epsilon' \tag{3.18}$$

for some positive constant $\bar{c}$. If we now combine the inequalities (3.15)–(3.18), we get

$$\left\| \mathcal{F}(I + k^2 T)^{-1} v_{g_\alpha}(\cdot, z) + e^{-ik\widehat{x}\cdot z} \right\|_{L^2(D)} < \epsilon + \delta, \qquad z \in \mathbb{R}^2 \setminus \overline{D}, \tag{3.19}$$

with $\epsilon = \bar{c}\epsilon'$. Moreover, both inequalities (3.16)–(3.17) imply that

$$\lim_{\alpha\to 0} \left\| v_{g_\alpha}(\cdot, z) \right\|_{L^2(D)} = \infty, \tag{3.20}$$

whence

$$\lim_{\alpha\to 0} \left\| g_\alpha(\cdot, z) \right\|_{L^2(\Omega)} = \infty. \tag{3.21}$$

Noting that $\alpha \to 0$ as $\delta \to 0$, we have proved the following theorem for the far field operator F corresponding to the scattering problem (3.1)–(3.3) provided $\operatorname{Im} n(x) \geq c > 0$ for $x \in D$:

Theorem 3.2. *Assume that Γ is smooth and $\operatorname{Im} n(x) \geq c > 0$ for $x \in D$. Then if F is the far field operator corresponding to (3.1)–(3.3), we have that*

1) if $z \in D$, then for every $\epsilon > 0$ there exists a solution $g^{\epsilon}(\cdot, z) \in L^2(\Omega)$ of the inequality

$$\left\| F g^{\epsilon}(\cdot, z) - \gamma e^{-ik\widehat{x}\cdot z} \right\|_{L^2(\Omega)} < \epsilon$$

such that

$$\lim_{z\to\Gamma} \left\| g^{\epsilon}(\cdot, z) \right\|_{L^2(\Omega)} = \infty \quad \text{and} \quad \lim_{z\to\Gamma} \left\| v_{g^{\epsilon}}(\cdot, z) \right\|_{L^2(D)} = \infty,$$

where v_{g^ϵ} is the Herglotz wave function with kernel g^{ϵ}, and

2) if $z \in \mathbb{R}^2 \setminus \overline{D}$, then for every $\epsilon > 0$ and $\delta > 0$ there exists a solution $g^{\epsilon,\delta}(\cdot, z) \in L^2(\Omega)$ of the inequality

$$\left\| F g^{\epsilon,\delta}(\cdot, z) - \gamma e^{-ik\widehat{x}\cdot z} \right\|_{L^2(\Omega)} < \epsilon + \delta$$

such that

$$\lim_{\delta\to 0} \left\| g^{\epsilon,\delta}(\cdot, z) \right\|_{L^2(\Omega)} = \infty \quad \text{and} \quad \lim_{\delta\to 0} \left\| v_{g^{\epsilon,\delta}}(\cdot, z) \right\|_{L^2(D)} = \infty,$$

where $v_{g^{\epsilon,\delta}}$ is the Herglotz wave function with kernel $g^{\epsilon,\delta}$.

We end by noting that the assumption on the smoothness of the refraction index n and of the boundary Γ can be weakened. One can consider, for example, that n is a complex valued Lipschitz function except for jump discontinuity across a Lipschitz curve Γ and the above analysis remains valid.

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