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A BOUND FOR REFLECTIONS ACROSS JORDAN CURVES

SAMUEL L. KRUSHKAL

Dedicated to the memory of V. D. Kupradze
on the occasion of his 100th birthday

Abstract. The properties of logarithmic derivative of the Riemann mapping function of a quasidisk are applied to quantitative estimation of quasiconformal reflections across its boundary.

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1. Topological and Quasiconformal Reflections

By the Brouwer–Kerekjarto theorem, every periodic homeomorphism of the sphere S^2 is topologically equivalent to a rotation, or to the product of rotation and reflection across the diametral plane (see [10], [19], [48]). The first case corresponds to orientation preserving homeomorphisms (and then E consists of two points), the second one is orientation reversing, and then either the fixed point set E is empty (which is excluded in our situation) or it is a topological circle.

We are concerned with homeomorphisms reversing orientation. Such homeomorphisms of order 2 are topological involutions of S^2 with $f \circ f = \mathrm{id}$ and are called *topological reflections.*

For a discussion of the properties of periodical homeomorphisms of the sphere $S^n,\ n > 2$, we refer to [11], [48], [49].

Let us consider *quasiconformal* reflections on the sphere $S^2 = \widehat{\mathbb{C}}$. The topological circles admitting such reflections are *quasicircles*, i.e., the circles which are locally *quasi-intervals* (the images of straight line segments under quasiconformal maps of the sphere S^2). Geometrically, any quasicircle L is characterized by the uniform boundedness of cross-ratios for all ordered quadruples (z_1, z_2, z_3, z_4) of the distinct points on L or, equivalently, by the property that, for any two points z_1, z_2 on L, the ratio of the chordal distance $|z_1 - z_2|$ to the diameters of the correspoding subarcs with these endpoints is uniformly bounded (see, e.g., [2], [16], [37], [42]). Note also that, due to [3], any quasicircle admits a bi-Lipschitz reflection, which is very useful in various applications.

For an extension to higher dimensions see [39], [54].

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Quasireflections across more general sets $E \subset \widehat{\mathbb{C}}$ also appear in certain questions and are of independent interest. Those sets admitting quasireflections are called *quasiconformal mirrors*.

2. Quantitative Estimating

One defines, for each mirror E, its *reflection coefficient*

$$q_E = \inf k(f) = \inf \ \left\| \partial_z f / \partial_{\bar{z}} f \right\|_\infty \tag{1}$$

and *quasiconformal dilatation*

$$Q_E = (1 + q_E)/(1 - q_E) \geq 1;$$

the infimum in (1) is taken over all quasireflections across E provided these exist and is attained by some quasireflection f_0. When $E = L$ is a quasicircle, the corresponding quantity

$$k_E = \inf \left\{ k(f_*) : f_*(S^1) = E \right\} \tag{2}$$

and the reflection coefficient q_E can be estimated in terms of each another; moreover, due to [3], [33], we have

$$Q_E = K_E := (1 + k_E)^2/(1 - k_E)^2. \tag{3}$$

The infimum in (2) is taken over all orientation preserving quasiconformal automorphisms f_* carrying the unit circle onto L, and $k(f_*) = \|\partial_{\bar{z}} f / \partial_z f\|_\infty$.

Note that quasiconformal maps (not reflections) mean orientation preserving homeomorphic solutions of the Beltrami equation

$$\partial_{\bar{z}} w = \mu(z) \ \partial_z w$$

in a domain $D \subseteqq \widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ with $\|\mu\|_\infty < 1$, where the derivatives $\partial_{\bar{z}}$ and ∂_z are distributional and belong locally to L_2. The function μ is called the *Beltrami coefficient* of the map w; the quantities $k(w) = \|\mu\|_\infty$ and $K(f) = (1 + k(w))/(1 - k(w))$ are, respectively, its *dilatation* and *maximal dilatation.* The maps with $k(w) \leq k_0$ are called k_0-*quasiconformal* (cf., e.g., [3], [21]).

It has been established in [26], [28] that any set $E \subset S^2$, which admits an orientation reversing quasiconformal homeomorphism f of the sphere S^2 keeping this set pointwise fixed, is necessarily a subset of a quasicircle (hence any quasiconformal mirror $E \subset S^2$ obeys quasiconformal involutions of S^2 of order 2). Moreover, this quasicircle can be chosen to have the same reflection coefficient as the initial set E (cf. [27], [34]). This result gives a complete answer to a question of Kühnau of describing all sets $E \in \widehat{\mathbb{C}}$ which admit quasiconformal reflections (see, e.g., [35]); it also yields various quantitative consequences. In particular, this result provides that equality (3) is true for any subset E of S^2 (assuming $Q_E = \infty$ if E does not admit quasiconformal reflections).

The conformal symmetry on the extended complex plane is strictly rigid and reduces to reflection $z \mapsto \bar{z}$ within conjugation by transformations $g \in PSL(2, \mathbb{C})$. The quasiconformal symmetry avoids such rigidity and is possible over quasicircles; this case turns out to be the most general one.

A somewhat different construction of quasiconformal reflections across Jordan curves was provided in [14]; it relies on the conformally natural extension of homeomorphisms of the circle introduced by Douady and Earle [12].

3. Fredholm Eigenvalues

An important problem is to provide the algorithms for calculating exact or approximate values of the reflection coefficients of particular curves and arcs. Even for polygons only special results are known (see, e.g., [31], [33], [53]).

Here, the least nontrivial Fredholm eigenvalue $\lambda_1 = \lambda_L$ plays a crucial role. The Fredholm eigenvalues are defined for a smooth closed bounded curve L to be the eigenvalues of the double-layer potential over L, in other words, of the equation

$$h(z) = \frac{\rho}{\pi} \int\limits_L h(\zeta) \frac{\partial}{\partial n_\zeta} \log \frac{1}{|\zeta - z|} \, ds_\zeta, \quad \zeta \in L,$$

where n_ζ is the outer normal and ds_ζ is the length element at $\zeta \in L$. These values are important in various questions (see, e.g., [1], [15], [33], [43], [44], [45], [46], [52]; [29], part 2).

The indicated eigenvalue ρ_L can be defined for any oriented closed Jordan curve $L \subset \widehat{\mathbb{C}}$ by the equality

$$\frac{1}{\rho_L} = \sup \frac{|\mathscr{D}_G(u) - \mathscr{D}_{G^*}(u)|}{\mathscr{D}_G(u) + \mathscr{D}_{G^*}(u)}, \tag{4}$$

where G and G^* are respectively the interior and the exterior of L; $\mathscr{D}$ denotes the Dirichlet integral, and the supremum is taken over all functions u continuous on $\widehat{\mathbb{C}}$ and harmonic on $G \cup G^*$. Due to [1],

$$q_L \geq \frac{1}{\rho_L}\,. \tag{5}$$

Denote $\Delta = \{z : |z| < 1\}$, $\Delta^* = \{z \in \widehat{\mathbb{C}} : |z| > 1\}$.

It suffices to consider the quasiconformal homeomorphisms of the sphere carrying S^1 onto L whose Beltrami coefficients $\mu_f(z) = \partial_{\bar{z}} f / \partial_z f$ have support in the unit disk Δ, and

$$f|\Delta^*(z) = z + b_0 + b_1 z^{-1} + \cdots$$

(or in the upper half-plane $U = \{\operatorname{Im} z > 0\}$) because all quantities in (5) are invariant under the action of the Möbius group $PSL(2, \mathbb{C})/\pm\mathbf{1}$. By the Kühnau–Schiffer theorem [30], [45], we have $\rho_L = 1/\varkappa(f^*)$, where

$$\varkappa(f^*) = \sup \Big| \sum_{m,n=1}^{\infty} \sqrt{mn}\ \alpha_{mn} x_m x_n \Big| \leq 1$$

is the *Grunsky constant* of the map f^*; here α_{mn} are the Grunsky coefficients obtained from the expansion of the principal branch of the function $\log[(f^*(z) - f^*(\zeta))/(z - \zeta)]$ into the double series on the bidisk $(\Delta^*)^2$, and the supremum is taken over the points $\mathbf{x} = (x_n) \in l^2$ with $\|\mathbf{x}\|^2 = 1$ (cf. [17], [33]).

Note that the function $g(z) = 1/f^*(1/z) = z + a_2 z^2 + \cdots$, which is univalent in Δ, has the same Grunsky coefficients α_{mn} as $f^*(z)$.

It is well-known that $\varkappa(f^*) \le k_0(f^*)$, but in the general case the strict inequality $\varkappa(f^*) < k_0(f^*)$ holds (see, e.g., [22], [32], [33]). A complete characterization of maps for which $\varkappa(f^*) = k_0(f^*) = q_{f^*(S^1)}$ is given in [22], [26].

One of the standard ways of establishing the reflection coefficients q_L (respectively, the Fredholm eigenvalues ρ_L) consists of verifying whether the equality in (5) or the equality $\varkappa(f^*) = k_0(f^*)$ hold for a given curve L. There is no complete answer even for the rectangles (cf. [24], [33], [53]).

4. A Conjecture of Ahlfors

Let now f be a conformal map of the upper half-plane $U = \{z : \operatorname{Re} z > 0\}$ onto the interior domain $D = D(L)$ of a given quasicircle L. How can this map be characterized?

L. Ahlfors conjectured in [2] that it can be characterized by analytic properties of the invariant (logarithmic derivative) $b_f = f''/f'$. Starting by the well-known Becker's univalence criterion for the disk, different authors established the conditions for f''/f' which ensure quasiconformal extensions of f (see, e.g., [5], [6], [7], [18], [40], [41]).

5. General Theorem

The goal of this paper is to prove the following theorem which concerns the above conjecture and relies on the properties of the logarithmic derivative.

Theorem 5.1. *Let a function f map conformally the upper half-plane U into $\mathbb{C}$, and let the equation*

$$w''(\zeta) = t b_f(\zeta) w'(\zeta), \quad \zeta \in U, \tag{6}$$

have univalent solutions on U for all $t \in [0, t_0]$, $t_0 > 1$. Then the image $f(U)$ is a quasidisk, and the reflection coefficient of its boundary $L = f(\widehat{\mathbb{R}}) = \partial f(U)$ satisfies

$$q_L \le \frac{1}{t_0}. \tag{7}$$

The bound given by (7) cannot be improved in the general case. The equality in (7) is attained by any quasicircle which contains two $C^{1+\epsilon}$ smooth subarcs ($\epsilon > 0$) with the interior intersection angle $\alpha\pi$, where $\alpha = 1 - 1/t_0$ (under the above univalence assumption for the logarithmic derivative b_f). In this case,

$$q_L = \frac{1}{\rho_L} = \frac{1}{t_0}\,. \tag{8}$$

The exact bound for the reflection coefficient q_L follows from (7) by choosing a maximal value of t_0 admitting the indicated univalence property for all $t \in [0, t_0]$. The corresponding solution w_{t_0} of (6) for this value is also univalent on U (by the properties of holomorphic functions), but the domain $w_{t_0}(U)$ is not a quasidisk.

Proof. We map the half-plane U conformally onto the disks $\Delta_t = \{z : |z| < t\}$ by the functions

$$z = t\,\frac{\zeta - i}{\zeta + i}\,, \quad 0 < t \le t_0;$$

the inverse maps $\Delta_t \to U$ are

$$\sigma_t(z) = i(1 + tz)/(1 - tz).$$

Then

$$b_{f\circ\sigma_t}(z) = (b_f \circ \sigma_t)\sigma_t'(z) + b_{\sigma_t}(z) = b_f\Big(i\frac{1 + tz}{1 - tz}\Big)\,\frac{2it}{(1 - tz)^2} - \frac{2t}{1 - tz}\,,$$

and equation (6) is transformed to

$$u''(z) = tb_{f\circ\sigma_t}(z)u'(z). \tag{9}$$

Note that, for a fixed t, any two solutions u_*, u_{**} to (9) differ by a linear transformation $u_{**} = c_1u_* + c_2$; hence, all these solutions are simultaneously univalent on U (of course, the same is true for equation (6)).

Now observe that these both equations have only univalent solutions (in U and Δ, respectively) also for the complex values of the parameter t when $|t| \le t_0$.

Indeed, take the solution u_r of (9) normalized by $u_r(0) = 0,\ u_r'(0) = 1$. It has the form

$$u_r(z) = z + c_2(r)z^2 + c_3(r)z^2 + \cdots\,, \tag{10}$$

where the coefficients are analytic on $|r| < t_0$, i.e.,

$$c_j(r) = c_{j,0}r^j + c_{j,1}r^{j+1} + \cdots\,.$$

Define for $t = re^{i\theta}$ the function

$$u_t(z) = e^{-i\theta}u_r(e^{i\theta}z).$$

It is also univalent on Δ, and

$$\frac{u_t''(z)}{u_t'(z)} = e^{i\theta}\,\frac{u_r''(e^{i\theta}z)}{u_r'(e^{i\theta}z)} = tb_{f\circ\sigma_r}(e^{i\theta}z).$$

Taking this into account, we set

$$b_{f,t}(z) = tb_{f\circ\sigma_t}(tz)$$

and define a complex homotopy $W_t(z) : \Delta \times \Delta_t$ so that for $0 < |t| < t_0$ the map $W_t(z)$ is the solution on the unit disk $\Delta = |z| < 1$ of the differential equation

$$u''(z) = b_{f,t}(z)u'(z), \quad z \in \Delta, \quad t \in [0, t_0],$$

normalized by means of (10), while $W(z, 0) = z$.

The function $W(z, t) = W_t(z) : \Delta \times \Delta_{t_0} \to \widehat{\mathbb{C}}$ defines *holomorphic motion* of the unit disk Δ, i.e., it satisfies: (a) the map $z \mapsto f(z, t)$ is injective on Δ for each t; (b) the ($\widehat{\mathbb{C}}$-valued) function $t \mapsto f(z, t)$ is holomorphic on Δ for each z; (c) $f(z, 0) = z$ for all $z \in \Delta$. $\square$

The basic properties of homotopies depending holomorphically on a complex parameter are presented in the following statement (see, e.g., [9], [13], [38], [47], [50]).

Extended lambda-lemma ([38], [47]). *If $f: E \times \Delta \to \widehat{\mathbb{C}}$ is holomorphic motion of a set $E \subset \widehat{\mathbb{C}}$ (containing at least three points), then:*

1) f has an extension $\widetilde{f}: \overline{E} \times \Delta \to \widehat{\mathbb{C}}$ such that $\widetilde{f}$ is holomorphic motion of the closure $\overline{E}$ of E, each $\widetilde{f}_t(z) = \widetilde{f}(z,t) : \overline{E} \to \widehat{\mathbb{C}}$ is quasiconformal, and $\widetilde{f}$ is jointly continuous in (z,t).

2) The motion f can be extended to holomorphic motion $F: \widehat{\mathbb{C}} \times \Delta \to \widehat{\mathbb{C}}$, so that $F|E \times \Delta = f$. This map determines the holomorphic map

$$t \to \mu_F = \partial_{\bar{z}} F(z,t)/\partial_z F(z,t)$$

of Δ into the unit ball $M_1 = \{\mu \in L_\infty(\mathbb{C}) : \ \|\mu\| < 1\}$ of the Beltrami coefficients, and by the Schwarz lemma,

$$\|\mu_F\|_\infty \leq |t|.$$

This bound cannot be improved in the general case.

In our case, the condition (a) is satisfied because of the univalence of solutions provided by the assumption of the theorem, and (c) holds trivially; as for (b), the general results on complex differential equations provide that the normalized solutions $W_t(z)$ are complex holomorphic functions of the parameter t by a fixed z.

Applying the lambda-lemma, one obtains that every fiber map $W_t: \ \Delta \to \widehat{\mathbb{C}}$ has quasiconformal extension $\widehat{W_t}$ to the whole sphere $\widehat{\mathbb{C}}$ whose Beltrami coefficient $\mu_{\widehat{F}_t}$ is estimated by

$$\|\mu_{\widehat{W}_t}\|_\infty \leq \left|\frac{t}{t_0}\right|$$

and depends holomorphically on t via an element of $L_\infty(\mathbb{C})$.

Returning to the half-plane U, one obtains holomorphic motion

$$\widehat{w}(\zeta, t) = \widehat{W}(\sigma^{-1}(\zeta), t/t_0) : U \times \Delta \to \widehat{\mathbb{C}}.$$

It can be renormalized by composing with the linear maps $\gamma_1(\zeta)$ and $g_2(w)$ to get the motion $\gamma_2 \circ \widehat{w}(\cdot, t) \circ g_1(\zeta)$ which includes the initial conformal map f of U, i.e., such that $\gamma_2 \circ \widehat{w}(, 1) \circ g_1(\zeta) = f(\zeta)$.

It follows from the above that we have established the existence of a quasiconformal extension $F = \gamma_2 \circ \widehat{w}(\cdot, 1/t_0) \circ g_1$ of the map f onto $\widehat{\mathbb{C}}$, whose Beltrami coefficient satisfies $\|\mu_F\|_\infty \leq 1/t_0$. Hence, the boundary curve $L = \partial f(U)$ is a quasicircle.

This extension defines a quasireflection

$$\widetilde{F}(\zeta) = F \circ \overline{F^{-1}(\zeta)}$$

across L; moreover, the dilatations of both maps $\widetilde{F}$ and F are equal because of the conformality of F on U. Thereby one obtains bound (7).

To examine the case of the equality in (7), we combine (5) with the angle inequality of Kühnau [33] which asserts that if any closed curve $L \subset \widehat{\mathbb{C}}$ contains two analytic arcs with the interior intersection angle $\pi\alpha$, then its reflection coefficient satisfies

$$1/\rho_L \geq |1 - \alpha|. \tag{11}$$

This provides the equalities (8) for quasicircles with analytic subarcs.

By an appropriate approximation, one obtains that the inequality (11) is extended to the curves with $C^{1+\epsilon}$-smooth subarcs. This completes the proof of the theorem.

6. Geometric Features

The following theorem announced in [28] shows that the bound (6) given by Theorem 1 is achieved for unbounded convex or concave domains and their fractional linear images.

Theorem 2. *For every unbounded convex domain $D \subset \mathbb{C}$ with piecewise $C^{1+\epsilon}$-smooth boundary L ($\epsilon > 0$), the equalities*

$$q_L = 1/\rho_L = \varkappa(g) = \varkappa(g^*) = k_0(g) = k_0(g^*) = 1 - |\alpha| \tag{12}$$

hold, where g and g^* denote the appropriately normalized conformal maps $\Delta \to D$ and $\Delta^* \to \widehat{\mathbb{C}} \setminus \overline{D}$, respectively; $k_0(g)$ and $k_0(g^*)$ are the minimal dilatations of their quasiconformal extensions to $\widehat{\mathbb{C}}$, and $\pi|\alpha|$ is the opening of the least interior angle between the boundary arcs $L_j \subset L$. Here $0 < \alpha < 1$ if the corresponding vertex is finite and $-1 < \alpha < 0$ for the angle at the vertex at infinity.

The same is true for the unbounded concave domains which do not contain ∞; for these one must replace the last term by $|\beta| - 1$, where $\pi|\beta|$ is the opening of the largest interior angle of D.

The univalence of solutions of equation (6) for $0 \leq t \leq 1/(1 - |\alpha|)$ is established for such domains approximating $f : U \to D$ by conformal maps f_n of the half-plane onto the rectilinear polygons P_n which are chosen to be also unbounded and convex or concave simultaneously with the original domain D. These maps f_n are represented by the Schwarz–Christoffel integral (cf., e.g., [20]).

The main point is that if the least interior angle of P_n equals $\pi|\alpha_n|$ (taking the negative sign for the angle at infinity), then for any $t \in [0, 1/(1 - |\alpha_n|)]$ the corresponding fiber map $w_t(z)$ is again a conformal map of the half-plane onto a well-defined rectilinear polygon.

The basic equalities $q_{\partial P_n} = 1/\rho_{\partial P_n} = 1 - |\alpha_n|$ follow now from Theorem 1, while the remaining equalities in (12) are obtained by combining this theorem with the Kühnau–Schiffer theorem mentioned above.

Theorems 1 and 2 have various important consequences. For example, for any closed unbounded curve L with the convex interior, which is $C^{1+\epsilon}$-smooth at all finite points and has, at infinity, the asymptotes approaching the interior angle $\pi\alpha < 0$, we have

$$q_L = 1/\rho_L = 1 - |\alpha|. \tag{13}$$

More generally, let $L = \gamma_1 \cup \gamma_2 \cup \gamma_3$, where

$$\gamma_1 = [a_1, \infty], \quad \gamma_2 = e^{i\pi\alpha}[a_2, \infty], \quad a_1 \geq a_2 > 0, \quad 0 < \alpha \leq 1/2,$$

and

$$\gamma_3 = \{(x, y) : \ y = h(x), \ a_2 \cos \pi\alpha \leq x \leq a_1\}$$

with a decreasing convex piecewise $C^{1+\epsilon}$-smooth function h such that

$$h(a_2 \cos \pi\alpha) = a_2 \sin \pi|\alpha|, \ h(a_1) = 0.$$

The equalities (13) hold for any such curve.

The above geometric assumptions on the domains are essential. In particular, the assertion of Theorem 2 extends neither to arbitrary unbounded nonconvex and nonconcave domains nor to arbitrary bounded convex domains without some additional assumptions.

For example, any rectangle P is fractional-linearly equivalent to a circular 4-gon P^* with one vertex at infinity, whose boundary consists of two infinite straight line intervals and two circular arcs; these four parts of ∂P^* are mutually orthogonal. By the results of [31] and [53], we have $q_{\partial P} > 1/2$ for any rectangle whose conformal module is greater than 2.76, though the interior angles of P^* are equal to $\pm\pi/2$.

On the other hand, the examples from [32] and [22] provide the bounded convex domains $D \subset \mathbb{C}$ whose Grunsky constants $\varkappa(g^*)$ and the minimal dilatation $k_0(g^*)$ are related by $\varkappa(g^*) < k_0(g^*) = q_{\partial D}$, in contrast to the situation in Theorem 2.

This shows also that the univalence of solutions w_t for all $t \in [0, t_0]$ in Theorem 1 is essential.

Note that for a few special curves, similar equalities were established in [31], [33], [53]) by applying geometric constructions giving explicitly the extremal quasireflections.

7. An Obstruction to Connectedness of Generic Intervals

Theorem 1 closely relates to the structure of holomorphic embeddings of the universal Teichmüller space. An old question posed already in [8] in a collection of unsolved problems for Teichmüller spaces and Kleinian groups is whether for an arbitrary finitely or infinitely generated Fuchsian group Γ the Bers embedding of its Teichmüller space $\mathbf{T}(\Gamma)$ is starlike.

The Bers embedding of $\mathbf{T}(\Gamma)$ can be modelled as a bounded domain in the complex Banach space $\mathbf{B}(\Delta^*, \Gamma)$ of holomorphic Γ-automorohic forms of the degree -4 (quadratic differentials) ψ on the disk Δ^*, with the norm

$$\|\psi\| = \sup_{\Delta^*} (|z|^2 - 1)^2 |\psi(z)|.$$

This domain is filled by the Schwarzian derivatives

$$S_f(z) = b_f'(z) - \frac{b_f(z)^2}{2}$$

of univalent and nonvanishing functions

$$f(z) = z + a_1 z^{-1} + \cdots \tag{14}$$

in Δ^* admitting quasiconformal extensions to the whole sphere $\widehat{\mathbb{C}}$ which are compatible with the group Γ (i.e., such that the conjugation fGf^{-1} produces a quasifuchsian group). The functions (14) form the well-known class Σ.

A complete answer for the universal Teichmüller space (in the negative) was given in [23]. Denote

$$\mathbf{S} = \big\{S_f \in \mathbf{B} : \ f \text{ univalent in } \Delta^*\big\}.$$

It was shown in [23] that the space $\mathbf{T}$ is not starlike with respect to any of its points, and there exist points $\varphi \in \mathbf{T}$ for which the line interval

$$I_\varphi = \big\{t\varphi : 0 < t < 1\big\} \subset \mathbf{B} \tag{15}$$

contains the points from $\mathbf{B} \setminus \mathbf{S}$. In other words, the corresponding functions f with $S_f = \varphi$ in Δ^* are only locally univalent on Δ^*.

The proof of this important fact relies on Thurston's theorem on the existence of conformally rigid domains which correspond to isolated components of $\mathbf{S} \setminus \mathbf{T}$ ([51], see also [4]).

The logarithmic derivatives $\beta_f = f''/f'$ determine the Becker embedding of $\mathbf{T}$ as a domain $\mathbf{D}_1$ in the Banach space $\mathbf{B}_1$ of holomorphic functions ψ on Δ^* with the norm $\|\psi\| = \sup_{\Delta^*}(|z|^2 - 1)|z\psi(z)|$.

Using the known estimate $\|S_f\| \le 41\|b_f\|$ (see, e.g., [6]), one concludes that the domain $\mathbf{D}_1$ is also not starlike.

In particular, it was shown in [23] that there are functions $f \in \Sigma$ mapping Δ^* onto the domains G with analytic boundaries so that the corresponding intervals (15) for $\varphi = S_f$ (belonging to $\mathbf{T}$) contain the points from $\mathbf{B} \setminus \mathbf{S}$. This yields intervals $\{t\varphi : 0 < t < 1\} \subset \mathbf{B}_1$ which do not lie entirely in $\mathbf{D}_1$, i.e., the intersections of these intervals with the domain $\mathbf{D}_1$ are not connected.

The domains G can be approximated by polygons for which the above intervals have a similar property.

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Author's address:

Research Institute for Mathematical Sciences
Department of Mathematics and Statistics
Bar-Ilan University, 52900 Ramat-Gan
Israel