Georgian Mathematical Journal
Volume 10 (2003), Number 3, 549–560

A FACTORIZATION METHOD FOR AN INVERSE NEUMANN PROBLEM FOR HARMONIC VECTOR FIELDS

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Abstract. Extending the previous work on the corresponding inverse Dirichlet problem, we present a factorization method for the solution of an inverse Neumann boundary value problem for harmonic vector fields.

2000 Mathematics Subject Classification: 31B10, 31B20, 45A05, 45Q05, 78A30.
Key words and phrases: Potential theory, harmonic vector fields, integral equations, inverse problems, magnetostatics.

1. Introduction

Roughly speaking, inverse boundary value problems for partial differential equations consist of determining the shape of an unknown object D from a knowledge of the type of a boundary condition on the boundary ∂D and of one or several solutions to the differential equation at locations away from the boundary. They are difficult to solve since they are both nonlinear and improperly posed. Recently new solution methods have been developed which circumvent the issue of nonlinearity by introducing a parameter point z and then solving a linear operator equation to decide whether or not $z \in D$. These methods are called factorization or sampling methods and two different variants were first introduced for inverse obstacle scattering problems, i.e., for inverse boundary value problems for the Helmholtz equation, by Colton and Kirsch [3] and by Kirsch [6]. The second variant was extended to inverse boundary value problems for the Laplace equation in [5, 8, 9, 11]. The corresponding approach in inverse impedance tomography, i.e., for inverse transmission problems for the Laplace equation, was initiated in [1, 2]. An extension of Kirsch's factorization method to an inverse Dirichlet boundary value problem for harmonic vector fields, i.e., for solutions of

$$\operatorname{div} v = 0, \quad \operatorname{curl} v = 0, \tag{1.1}$$

is described in [10]. It is the aim of the present paper to extend this analysis to an inverse Neumann problem.

Assuming that D is a bounded domain in $\mathbb{R}^3$ with a connected C^2 boundary ∂D and outward unit normal ν and y is a point in $\mathbb{R}^3 \setminus \bar{D}$ and p a vector in $\mathbb{R}^3$, we consider the exterior boundary value problem to find a harmonic vector field $v(\cdot,\, y, p) \in C^1(\mathbb{R}^3 \setminus \bar{D})$ in $\mathbb{R}^3 \setminus D$ satisfying the Neumann boundary condition

$$\nu \cdot v(\cdot,\, y, p) = -\nu \cdot \operatorname{grad} \operatorname{div} \{\Phi(\cdot\,, y)p\} \quad \text{on } \partial D, \tag{1.2}$$

ISSN 1072-947X / $8.00 /

where Φ denotes the fundamental solution to the Laplace equation. Furthermore, we require that $v(x) \to 0$ as $|x| \to \infty$ uniformly in all directions. In addition, in the case of a multiply connected domain D, for simplicity, we assume that v is circulation free, i.e., v can be represented via $v = \operatorname{grad} u$ by a scalar potential u. We may interpret $v + \operatorname{grad}\operatorname{div}\{\Phi p\}$ as the total magnetostatic field of a dipole located at the point y with polarization p in the exterior of a superconductor D. Since v depends linearly on the polarization vector p we may write

$$v(\cdot\,,\cdot\,,p) = wp,$$

where w is a 3×3–matrix depending on $x, y \in \mathbb{R}^3 \setminus \bar{D}$.

Let B be an additional bounded simply connected domain with a connected C^2 boundary ∂B that contains $\bar{D}$. Then the inverse problem we are interested in is to determine the shape of D, i.e., the boundary ∂D, from the knowledge of $w(x, y)$ for all x, y in ∂B. To solve this inverse problem, we will characterize D in terms of spectral data, i.e., in terms of eigenvalues and eigenfunctions, of the integral operator W defined by

$$(Wg)(x) := -\int_{\partial B} w(x, y)g(y)\, ds(y), \quad x \in \partial B. \tag{1.3}$$

We will consider the two cases where the density g is either normal or tangential to the boundary ∂B. In the case, where $g = \tilde{g}\nu$ with some scalar function $\tilde{g}$ and ν denoting the outward normal to ∂B, we replace (1.3) by the integral operator $W_\nu : L^2(\partial B) \to L^2(\partial B)$ defined by

$$(W_\nu \tilde{g})(x) := -\int_{\partial B} w_\nu(x, y)\tilde{g}(y)\, ds(y), \quad x \in \partial B, \tag{1.4}$$

with the scalar kernel

$$w_\nu(x, y) = \nu(x) \cdot w(x, y)\nu(y), \quad x, y \in \partial B,$$

i.e., related to the direction of the dipoles we only use the normal components of the field v as data. In the second case, where g is a tangential field, we view (1.3) as an integral operator $W : L^2_t(\partial B) \to L^2_t(\partial B)$ mapping the space $L^2_t(\partial B)$ of tangential fields on ∂B into itself, i.e., we use only the tangential components of the field v as data. To distinguish between the two cases, in the sequel, we use the subscript ν for those operators that correspond to the first case.

In particular, we will show that W_ν and W are compact, self-adjoint, and positive semi-definite operators. Therefore their square roots $W_\nu^{1/2}$ and $W^{1/2}$ are well defined. If for some constant unit vector e and a parameter point $z \in \mathbb{R}^3$ we define the dipole field $H(\cdot\,, z) := \operatorname{grad}\operatorname{div}\{\Phi(\cdot\,, z)e\}$ in $\mathbb{R}^3 \setminus \{z\}$, then our main result is the characterization of the domain D by the property that the improperly posed linear operator equations

$$W_\nu^{1/2}\tilde{g} = \nu \cdot H(\cdot\,, z)|_{\partial B} \tag{1.5}$$

and

$$W^{1/2}g = \{\nu \times H(\cdot\,, z)|_{\partial B}\} \times \nu \tag{1.6}$$

have solutions $\tilde{g} \in L^2(\partial B)$ and $g \in L^2_t(\partial B)$, respectively, if and only if $z \in D$. With the aid of Picard's theorem this can be used numerically for the visualization of the unknown domain.

We note that in [10] the signs in (1.3) and (1.4) are reversed so that W and W_ν remain positive semi-definite. As is typical of the factorization methods, one does not need to know the boundary condition in advance, since the sign of the eigenvalues depend on whether the boundary condition is Dirichlet or Neumann.

The plan of this paper is as follows. Since the study of an inverse problem always requires a solid foundation of the corresponding direct problem, in Section 2 we summarize the classical existence and uniqueness results for the exterior Neumann problem for harmonic vector fields extended by an investigation of the solution operator as needed in the following analysis. Then, in Section 3 we introduce the inverse problem with dipoles in normal direction. Our main result is the theoretical foundation of the factorization method through Theorem 3.3 and its two corollaries. The final Section 4 is devoted to the inverse problem with dipoles in tangential directions and is shorter since the analysis is analogous to that in Section 3.

2. Direct Problem

We consider the exterior Neumann problem for harmonic vector fields in the bounded domain $D \subset \mathbb{R}^3$ with a connected C^2 boundary ∂D and exterior unit normal ν: Given a scalar function $f \in C(\partial D)$, find a vector field $v \in C^1(\mathbb{R}^3 \setminus \bar{D})$ satisfying the differential equations

$$\operatorname{div} v = 0, \quad \operatorname{curl} v = 0 \quad \text{in } \mathbb{R}^3 \setminus \bar{D}, \tag{2.1}$$

and the boundary condition

$$\nu \cdot v = f \quad \text{on } \partial D \tag{2.2}$$

in the sense of uniform convergence

$$\lim_{h \to 0} \nu(x) \cdot v(x + h\nu(x)) = f(x), \quad x \in \partial D.$$

At infinity it is required that $v(x) \to 0$ for $|x| \to \infty$ uniformly in all directions. Furthermore, for simplicity, if D and consequently $\mathbb{R}^3 \setminus \bar{D}$ is multiply connected, then we impose the condition that v is circulation free. In order to introduce notation for the subsequent analysis we briefly recall the classical results on existence and uniqueness for (2.1)–(2.2).

Since we assume that $v = \operatorname{grad} u$ for some harmonic function u, the homogeneous form of the boundary condition $\nu \cdot v = 0$ on ∂D and the behavior of harmonic vector fields at infinity via Green's integral theorem imply that $\operatorname{grad} u = 0$ and consequently $v = 0$ for any solution to the homogeneous boundary value problem (2.1)–(2.2).

If we denote by

$$\Phi(x, y) := \frac{1}{4\pi} \frac{1}{|x - y|}, \quad x \neq y,$$

the fundamental solution of Laplace's equation in $\mathbb{R}^3$, then the vector field

$$v(x) = \operatorname{grad} \int_{\partial D} \Phi(x,y)\varphi(y)\,ds(y), \quad x \in \mathbb{R}^3 \setminus \bar{D}, \tag{2.3}$$

with density $\varphi \in C(\partial D)$ is a solution of (2.1)–(2.2) if φ solves the integral equation

$$-\varphi + K^*\varphi = 2f, \tag{2.4}$$

where the integral operator $K^* : C(\partial D) \to C(\partial D)$ is given by

$$(K^*\varphi)(x) := 2 \int_{\partial D} \frac{\partial \Phi(x,y)}{\partial \nu(x)}\, \varphi(y)\,ds(y), \quad x \in \partial D.$$

Since the operator K^* is compact and $-I + K^*$ is injective (see Theorem 6.20 in [8]), by the Riesz theory, there exists a bounded inverse $(-I + K^*)^{-1} : C(\partial D) \to C(\partial D)$. This concludes the classical uniqueness and existence analysis for the exterior Neumann problem (2.1)–(2.2).

Using the Fredholm alternative it can be shown that the nullspace of $-I+K^*$ with respect to $L^2(\partial D)$ coincides with the nullspace with respect to $C(\partial D)$ (see also [4], p. 59). Therefore the inverse operator $(-I+K^*)^{-1} : L^2(\partial D) \to L^2(\partial D)$ also exists and is bounded.

Recall that we assume $B \subset \mathbb{R}^2$ to be a bounded, simply connected domain with connected C^2 boundary and outward unit normal ν such that $\bar{D} \subset B$. We introduce an operator

$$A_\nu : \nu \cdot v|_{\partial D} \mapsto \nu \cdot v|_{\partial B}$$

that, for solutions v to (2.1)–(2.2), maps the normal component on ∂D onto the normal component on ∂B. From the above existence analysis we have that

$$A_\nu = 2U_\nu(-I + K^*)^{-1}, \tag{2.5}$$

where $U_\nu : L^2(\partial D) \to L^2(\partial B)$ is defined by

$$(U_\nu\varphi)(x) := \int_{\partial D} \frac{\partial \Phi(x,y)}{\partial \nu(x)}\, \varphi(y)\,ds(y), \quad x \in \partial B.$$

We note that, by considering weak solutions $v \in L^2_{\text{loc}}(\mathbb{R}^3 \setminus \bar{D})$ of the Neumann problem, the operator A_ν can be extended as a bounded operator from the Sobolev space $H^{-1/2}(\partial D)$ into $L^2(\partial B)$ (see [12]).

Theorem 2.1. *The compact operator $A_\nu : L^2(\partial D) \to L^2(\partial B)$ is injective and has the dense range.*

Proof. From the above we already know that $(-I + K^*)^{-1} : L^2(\partial D) \to L^2(\partial D)$ is injective. Therefore to establish injectivity of A_ν it remains to show that $U_\nu : L^2(\partial D) \to L^2(\partial B)$ is injective. From $U_\nu\varphi = 0$ we have that the single-layer potential u with density φ solves the homogeneous Neumann problem for harmonic functions in $\mathbb{R}^3 \setminus B$. Hence we have $u = 0$ in $\mathbb{R}^3 \setminus B$, whence by analyticity $u = 0$ in $\mathbb{R}^3 \setminus \bar{D}$ follows. From this, using the potential theoretic jump-relations in the L^2 sense due to Kersten [7] (see also [8] p. 172) we obtain that $-\varphi + K^*\varphi = 0$. Hence $\varphi = 0$, since $-I + K^*$ is injective.

To establish that $A_\nu(L^2(\partial D))$ is dense in $L^2(\partial B)$ we show that the adjoint operator $A_\nu^* : L^2(\partial B) \to L^2(\partial D)$ is injective. Clearly,

$$A_\nu^* = 2(-I + K)^{-1} U_\nu^*, \tag{2.6}$$

where

$$(K\varphi)(x) = 2 \int_{\partial D} \frac{\partial \Phi(x,y)}{\partial \nu(y)}\, \varphi(y)\, ds(y), \quad x \in \partial D,$$

is the adjoint of K^* and

$$(U_\nu^* g)(x) = \int_{\partial B} \frac{\partial \Phi(x,y)}{\partial \nu(y)}\, g(y)\, ds(y), \quad x \in \partial D,$$

is the adjoint of U_ν. Again, since by the Fredholm alternative $(-I+K)^{-1} : L^2(\partial D) \to L^2(\partial D)$ is injective we only need to be concerned with the injectivity of U_ν^*. From $U_\nu^* g = 0$ we have that the double-layer potential u with density g solves the homogeneous Dirichlet problem for harmonic functions in D. Hence we have $u = 0$ in D, whence by analyticity $u = 0$ in B follows. From this, using the potential theoretic jump-relations in the L^2 sense we obtain that $-g + K_B g = 0$, where K_B denotes the operator K with D replaced by B. Hence $g = 0$, since $-I + K_B$ is injective. □

As is typical of compact operators, the ranges of A_ν and A_ν^* have more regularity than L^2. In particular, since its kernel is C^2, the operator U_ν^* is bounded from $L^2(\partial B)$ into the Hölder space $C^{1,\alpha}(\partial D)$. Furthermore, because K is compact from $C^{1,\alpha}(\partial D)$ into itself (see Theorem 3.4 in [4]), the inverse $(I - K)^{-1} : C^{1,\alpha}(\partial D) \to C^{1,\alpha}(\partial D)$ also exists and is bounded. Therefore A_ν^* maps $L^2(\partial B)$ boundedly into $C^{1,\alpha}(\partial D)$ and, consequently, also boundedly into the Sobolev space $H^1(\partial D)$.

3. Inverse Problem with Normal Dipoles

We now consider the special case of the exterior Neumann problem (2.1)–(2.2) with the boundary data given by a dipole with polarization vector p located at some point $y \in \mathbb{R}^3 \setminus \bar{D}$, i.e., the field $v = v(\cdot\,, y, p)$ is harmonic in $\mathbb{R}^3 \setminus \bar{D}$, satisfies the boundary condition (1.2), is circulation free and vanishes at infinity. The inverse problem we want to consider in this section is, given $\nu(x) \cdot v(x, y, \nu(y))$ for all $x, y \in \partial B$, to determine the shape of D. We will develop an explicit characterization of the unknown domain D in terms of spectral data of the integral operator $W_\nu : L^2(\partial B) \to L^2(\partial B)$ with kernel w_ν as defined by (1.4).

In the sequel, by $(\cdot\,,\cdot)$ we denote the inner product in $L^2(\partial B)$ and $L^2(\partial D)$. For convenience, we introduce the subspaces

$$L_0^2(\partial D) := \left\{ g \in L^2(\partial D) : (g,1) = 0 \right\} \text{ and } L_0^2(\partial B) := \left\{ g \in L^2(\partial B) : (g,1) = 0 \right\}.$$

We define the pseudodifferential operator T as the normal derivative of the double-layer potential, i.e.,

$$(T\varphi)(x) := \frac{\partial}{\partial \nu(x)} \int_{\partial D} \frac{\partial \Phi(x,y)}{\partial \nu(y)}\, \varphi(y)\, ds(y), \quad x \in \partial D,$$

that maps the Sobolev space $H^{1/2}(\partial D)$ boundedly into $H^{-1/2}(\partial D)$ (see [12]). The operator T has a one-dimensional nullspace $N(T) = \operatorname{span}\{1\}$ and it can be shown that it is self-adjoint, negative semi-definite and an isomorphism from $H^1(\partial D) \cap L^2_0(\partial D)$ onto $L^2_0(\partial D)$. By the compactness of the embedding from $H^1(\partial D)$ into $L^2(\partial D)$ this implies that the inverse T^{-1} is compact from $L^2_0(\partial D)$ into $L^2_0(\partial D)$. Furthermore, T^{-1} inherits the self-adjointness and negative semi-definiteness from T. Therefore the spectral theorem for self-adjoint compact operators can be used to define the self-adjoint and positive semi-definite operator $[-T]^{1/2}$ that is bounded from $H^{1/2}(\partial D)$ into $L^2(\partial D)$ and bounded from $L^2(\partial D)$ into $H^{-1/2}(\partial D)$. It is the square root of $-T$ in the sense that it satisfies

$$[-T]^{1/2}[-T]^{1/2} = -T$$

on $H^1(\partial D)$. For details the reader is referred to [11].

For the following theorem we note that TA_ν^* is well defined on $L^2(\partial B)$, since A_ν^* is bounded from $L^2(\partial B)$ into $H^1(\partial D)$.

Theorem 3.1. *The operators W_ν, A_ν, and T are related through the factorization*

$$W_\nu = -A_\nu T A_\nu^*. \tag{3.1}$$

Proof. The operators $P_\nu : L^2(\partial D) \to L^2(\partial B)$ and $P_\nu^* : L^2(\partial B) \to L^2(\partial D)$ given by

$$(P_\nu \varphi)(x) := \frac{\partial}{\partial \nu(x)} \int_{\partial D} \frac{\partial \Phi(x,y)}{\partial \nu(y)}\, \varphi(y)\, ds(y), \quad x \in \partial B,$$

and

$$(P_\nu^* g)(x) := \frac{\partial}{\partial \nu(x)} \int_{\partial B} \frac{\partial \Phi(x,y)}{\partial \nu(y)}\, g(y)\, ds(y), \quad x \in \partial D,$$

are adjoint. Noting that

$$\operatorname{div}_x \{\Phi(\cdot\,,y)\, g(y) \nu(y)\} = -\frac{\partial \Phi(\cdot\,,y)}{\partial \nu(y)}\, g(y)$$

for $g \in L^2(\partial B)$, from the boundary condition (1.2) we have that

$$\nu(x) \cdot v(x,y,g(y)\nu(y)) = \left(A_\nu \frac{\partial}{\partial \nu} \frac{\partial \Phi(\cdot\,,y)}{\partial \nu(y)}\, g(y) \right)(x), \quad x,y \in \partial B.$$

Integrating this over ∂B and using the boundedness of A_ν, we deduce that

$$W_\nu = -A_\nu P_\nu^*. \tag{3.2}$$

By the definition of the operators, we have $P_\nu = A_\nu T$, whence $P_\nu^* = TA_\nu^*$ follows. Inserting this into (3.2) completes the proof. □

As a first consequence of the factorization (3.1) we note that the operator $W : L^2(\partial B) \to L^2(\partial B)$ is compact, since $A : L^2(\partial D) \to L^2(\partial B)$ is compact and $TA_\nu^* : L^2(\partial B) \to L^2(\partial D)$ is bounded.

Theorem 3.2. *The compact operator* $W_\nu : L^2(\partial B) \to L^2(\partial B)$ *is positive semi-definite. The nullspace of* W_ν *is given by* $\operatorname{span}\{1\}$ *and there exists a complete orthonormal system* g_n, $n = 1, 2, \dots,$ *of* $L^2_0(\partial B)$ *of eigenelements of* W_ν *with positive eigenvalues* λ_n, *i.e.,*

$$W_\nu g_n = \lambda_n g_n, \quad n = 1, 2, \dots. \tag{3.3}$$

Proof. From (3.1) and the self-adjointness of T we observe that W_ν is a self-adjoint operator. For $g \in L^2(\partial B)$ we have $(W_\nu g, g) = -(TA^*_\nu g, A^*_\nu g)$ and the negative semi-definiteness of T implies that W_ν is positive semi-definite. The equality $(W_\nu g, g) = 0$ holds if and only if $TA^*_\nu g = 0$, that is, if $W_\nu g = 0$, since A_ν is injective by Theorem 2.1.

To characterize the nullspace of W_ν, assume $TA^*_\nu g = 0$, i.e., $A^*_\nu g = \text{const}$. From this, using $K1 = -1$ and (2.6) we conclude that $U^*_\nu g = \text{const}$. Therefore the double-layer potential u with density g on ∂B satisfies $u = \text{const}$ on ∂D and consequently $u = \text{const}$ in B. The jump-relations now imply $g \in \operatorname{span}\{1\}$. Reversing the arguments also shows that $A^*_\nu 1 = \text{const}$ and therefore $W_\nu 1 = 0$, and the proof is complete.

The statement on the eigenvalues and eigenfunctions are straightforward consequences of the spectral theory for self-adjoint compact operators (see Theorem 15.12 in [8]). □

Using the eigenvalues λ_n and eigenfunctions g_n of the operator W_ν we are now in a position to define the functions

$$\varphi_n := \frac{1}{\sqrt{\lambda_n}}\,[-T]^{1/2} A^*_\nu g_n, \quad n = 1, 2, \dots. \tag{3.4}$$

Since $A^*_\nu g_n \in H^1(\partial D)$, we clearly have $\varphi_n \in L^2(\partial D)$. Using (3.1) and the orthonormality of g_n, it can be seen that φ_n form an orthonormal system in $L^2(\partial D)$.

For the constant unit vector e and $z \in B$ we define

$$\Psi(\cdot\,, z) := \frac{\partial}{\partial \nu}\, \operatorname{div}\{\Phi(\cdot\,, z)e\} \quad \text{on } \partial B.$$

In view of

$$\operatorname{grad}\operatorname{div}\{\Phi(\cdot\,, z)e\} = \operatorname{curl}\operatorname{curl}\{\Phi(\cdot\,, z)e\}, \tag{3.5}$$

by Stokes' theorem we have $\Psi(\cdot\,, z) \in L^2_0(\partial B)$. Hence the Fourier series

$$\Psi(\cdot\,, z) = \sum_{n=1}^{\infty} (\Psi(\cdot\,, z), g_n) g_n \tag{3.6}$$

converges in $L^2(\partial B)$, because the g_n are complete in $L^2_0(\partial B)$.

Theorem 3.3. *The point* $z \in B$ *belongs to* D *if and only if*

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n}\, |(\Psi(\cdot\,, z), g_n)|^2 < \infty. \tag{3.7}$$

Proof. Let $z \in D$. Then the field

$$H(\cdot\,, z) := \operatorname{grad}\operatorname{div}\{\Phi(\cdot\,, z)e\}$$

is harmonic in $\mathbb{R}^3 \setminus \{z\}$, circulation free and vanishes at infinity. Therefore, setting

$$f(\cdot\,, z) := \frac{\partial}{\partial \nu}\ \operatorname{div}\{\Phi(\cdot\,, z)e\} \quad \text{on } \partial D$$

we have

$$A_\nu f = \Psi.$$

Because by (3.5) and Stokes' theorem clearly $f \in L^2_0(\partial D)$, using the invertibility of T as discussed above, there exists a function $\psi \in H^1(\partial D)$ such that $T\psi = f$. Hence we have $\Psi = A_\nu f = A_\nu T\psi$ and consequently

$$g = A_\nu[-T]^{1/2}\varphi,$$

where $\varphi = -[-T]^{1/2}\psi$ is in $L^2(\partial D)$. Since (3.4) is an orthonormal system, Bessel's inequality implies that

$$\sum_{n=1}^{\infty} |(\varphi, \varphi_n)|^2 \le \|\varphi\|^2. \tag{3.8}$$

From (3.1) and (3.4) we find that

$$\sqrt{\lambda_n}\,(\varphi, \varphi_n) = (\varphi, [-T]^{1/2}A_\nu^* g_n) = (A_\nu[-T]^{1/2}\varphi, g_n) = (g, g_n), \quad n = 1, 2, \dots,$$

and inserting this into (3.8) yields the convergence of series (3.7).

Conversely, assume that the series (3.7) converges. Then,

$$\varphi := \sum_{n=1}^{\infty} \frac{1}{\sqrt{\lambda_n}}\ (\Psi(\cdot\,, z), g_n)\varphi_n$$

defines a function $\varphi \in L^2(\partial D)$. With the aid of (3.6), the boundedness of the operators $[-T]^{1/2} : L^2(\partial D) \to H^{-1/2}(\partial D)$ and $A_\nu : H^{-1/2}(\partial D) \to L^2(\partial B)$ and the relation

$$A_\nu[-T]^{1/2}\varphi_n = \sqrt{\lambda_n}\, g_n, \quad n = 1, 2, \dots,$$

we deduce that φ satisfies $A_\nu[-T]^{1/2}\varphi = \Psi(\cdot\,, z)$. By the uniqueness for the Neumann problem in the exterior of B and analyticity, this implies that for the weak solution $v \in L^2_{\mathrm{loc}}(\mathbb{R}^3 \setminus \bar{D})$ to the Neumann problem in the exterior of D with normal derivative $[-T]^{1/2}\varphi \in H^{-1/2}(\partial D)$ we have $v = H(\cdot\,, z)$. Therefore the point z cannot belong to $B \setminus D$, since $H(\cdot\,, z) \notin L^2_{\mathrm{loc}}(\mathbb{R}^3 \setminus \bar{D})$ if $z \in B \setminus D$. □

Since the operator W_ν is compact, self-adjoint, and positive semi-definite the square root operator $W_\nu^{1/2} : L^2(\partial B) \to L^2(\partial B)$ is well defined by

$$W_\nu^{1/2} g = \sum_{n=1}^{\infty} \sqrt{\lambda_n}\,(g, g_n) g_n. \tag{3.9}$$

In terms of this operator, by using Picard's theorem (see Theorem 15.18 in [8]) we can reformulate Theorem 3.3 as the following corollary.

Corollary 3.4. *The point $z \in B$ belongs to D if and only if the operator equation*

$$W_\nu^{1/2} g = \Psi(\cdot\,, z)$$

has a solution $g \in L^2(\partial B)$.

Corollary 3.5. *The domain D is uniquely determined by the knowledge of $\nu(x) \cdot v(x, y, \nu(y))$ for $x, y \in \partial B$.*

Proof. This is an immediate consequence of Theorem 3.3. □

4. Inverse Problem with Tangential Dipoles

We now turn to the case of the inverse problem using tangential dipole fields $g \in L^2_t(\partial B)$ and the tangential components of $v(x, y, g(y))$ as data. Here, by $L^2_t(\partial B)$ we denote the space of square integrable tangential fields on ∂B. Again we will develop an explicit characterization of the unknown domain D in terms of spectral data of the integral operator $W : L^2_t(\partial B) \to L^2_t(\partial B)$ with kernel w as defined by (1.3). For doing so, we need to replace the operator A_ν mapping the normal components of harmonic vector fields on ∂D onto the normal components on ∂B by the operator A that maps the normal components on ∂D onto the tangential components on ∂B, i.e., $A : \nu\cdot v|_{\partial D} \mapsto \{\nu \times v|_{\partial B}\} \times \nu$. Then (2.5) has to be replaced by

$$A = 2U(-I + K^*)^{-1}, \tag{4.1}$$

where $U : L^2(\partial D) \to L^2_t(\partial B)$ is now given by

$$(U\varphi)(x) := \operatorname{Grad} \int_{\partial D} \Phi(x, y)\varphi(y)\, ds(y), \quad x \in \partial B.$$

Here Grad denotes the surface gradient on ∂B. Analogously to A_ν, the operator A can be extended as a bounded operator from the Sobolev space $H^{-1/2}(\partial D)$ into $L^2_t(\partial B)$.

Theorem 4.1. *The compact operator $A : L^2(\partial D) \to L^2_t(\partial B)$ is injective on the subspace $L^2_0(\partial D)$.*

Proof. Using $K1 = -1$, it can be seen that $(-I+K^*)^{-1}$ maps $L^2_0(\partial D)$ into itself. Hence we only need to be concerned with the injectivity of U on $L^2_0(\partial D)$. From $U\varphi = 0$ we have that the single-layer potential u with density φ is constant $u = u_0$ on ∂B. Then, employing the jump-relations, by Green's theorem and using $\varphi \in L^2_0(\partial D)$ we have that

$$\int_{\partial B} u\, \frac{\partial u}{\partial \nu}\, ds = u_0 \int_{\partial D} \frac{\partial u_+}{\partial \nu}\, ds = u_0 \int_{\partial D} \frac{\partial u_-}{\partial \nu}\, ds = 0,$$

where by the subscripts $+$ and $-$ we distinguish the limits obtained by approaching ∂D from within $\mathbb{R}^3 \setminus \bar{D}$ and D, respectively. From this we can conclude that $u = 0$ first in $\mathbb{R}^3 \setminus \bar{D}$ and then also in D and the jump-relations yield $\varphi = 0$.

The L^2 adjoint $A^* : L^2_t(\partial B) \to L_t(\partial D)$ is given by

$$A^* = 2(-I + K)^{-1}U^*, \tag{4.2}$$

where

$$(U^*g)(x) = -\operatorname{div} \int_{\partial B} \Phi(x,y) g(y)\, ds(y), \quad x \in \partial D,$$

is the adjoint $U^* : L^2_t(\partial B) \to L^2(\partial D)$ of U. This, as in Section 2, can be used to show that A^* maps $L^2_t(\partial B)$ boundedly into $L^2(\partial D)$. □

Theorem 4.2. *The operators W, A, and T are related through the factorization*

$$W = -ATA^*. \tag{4.3}$$

Proof. The operators $P : L^2(\partial D) \to L^2_t(\partial B)$ and $P^* : L^2_t(\partial B) \to L^2(\partial D)$ defined by

$$(P\varphi)(x) := \operatorname{Grad} \int_{\partial D} \frac{\partial \Phi(x,y)}{\partial \nu(y)}\, \varphi(y)\, ds(y), \quad x \in \partial B,$$

and

$$(P^*g)(x) := -\frac{\partial}{\partial \nu(x)} \operatorname{div} \int_{\partial B} \Phi(x,y) g(y)\, ds(y), \quad x \in \partial D,$$

are adjoint. Integrating

$$v(x,y,g(y)) = -\left(A\, \frac{\partial}{\partial \nu} \operatorname{div}\{\Phi(\cdot\,,y) g(y)\} \right)(x), \quad x,y \in \partial B,$$

over ∂B and using the boundedness of A, we can deduce that

$$W = -AP^*. \tag{4.4}$$

By the definition of the operators, we have $P = AT$, whence $P^* = TA^*$ follows. Inserting this into (4.4) completes the proof. □

Consider the single-layer operator $S : L^2(\partial B) \to H^1(\partial B)$ given by

$$(S\varphi)(x) := \int_{\partial B} \Phi(x,y)\, \varphi(y)\, ds(y), \quad x \in \partial B.$$

Then, by interchanging orders of integration for functions $\varphi, \psi \in C^{1,\alpha}(\partial B)$ and using a denseness argument it can be seen that

$$(\operatorname{Grad} S\varphi, \psi) = -(\varphi, \operatorname{Grad} S\psi) \tag{4.5}$$

for all $\varphi, \psi \in L^2(\partial B)$. We denote

$$L^2_{t,div}(\partial B) := \{ g \in L^2_t(\partial B) : \operatorname{Div} g \in L^2(\partial B) \},$$

where Div is the surface divergence of a tangential field on ∂B in the weak sense.

Theorem 4.3. *The compact operator $W : L^2(\partial B) \to L^2_t(\partial B)$ is positive semi-definite. The nullspace of W is given by*

$$L^{2,0}_{t,div}(\partial B) := \{ g \in L^2_{t,div}(\partial B) : \operatorname{Div} g = 0 \}$$

and there exists an orthonormal system $g_n \in L^2_t(\partial B)$ of eigenelements of W with positive eigenvalues λ_n, i.e.,

$$W g_n = \lambda_n g_n, \quad n = 1, 2, \dots. \tag{4.6}$$

The eigenelements are complete in the orthogonal complement of $L^{2,0}_{t,div}(\partial B)$.

Proof. As in the proof of Theorem 3.2, from (4.3) we deduce that W is self-adjoint and that equality holds in $(Wg, g) \geq 0$ if and only if $Wg = 0$. In view of Theorem 4.1 and (4.4) we have that $N(W) = N(P^*)$.

To characterize the nullspace of P^* let $P^*g = 0$. Then

$$u(x) := \operatorname{div} \int_{\partial B} \Phi(x, y) g(y)\, ds(y), \quad x \in \mathbb{R}^3 \setminus \partial B, \tag{4.7}$$

has a vanishing normal derivative on ∂D. This implies that $u = u_0 = \text{const}$ in D and, by analyticity, $u = u_0$ in B. Since the single-layer operator S is an isomorphism from $L^2(\partial B)$ onto $H^1(\partial B)$ (see [12]), there exists $g_0 \in L^2(\partial B)$ such that $u_0 = Sg_0$. Now let $f \in H^1(\partial B)$ and, correspondingly, $\varphi \in L^2(\partial B)$ such that $f = S\varphi$. Then, using (4.5) and the self-adjointness of S, we obtain

$$(\operatorname{Grad} f, g) = (\operatorname{Grad} S\varphi, g) = -(\varphi, \operatorname{Grad} Sg) = -(\varphi, u_0) = -(f, g_0)$$

i.e., $g \in L^2_{t,div}(\partial B)$ with $\operatorname{Div} g = g_0$. Hence we can transform (4.7) into

$$u(x) = \int_{\partial B} \Phi(x, y) g_0(y)\, ds(y), \quad x \in \mathbb{R}^3 \setminus \partial B. \tag{4.8}$$

From this and the fact that $u = u_0$ in B, by the L^2 jump relations we conclude that

$$g_0 + K_B g_0 = 0.$$

Since the nullspace of $I + K_B$ has dimension one with $N(I + K_B) = \operatorname{span}\{\psi_0\}$ for some ψ_0 with $\int_{\partial B} \psi_0\, ds = 1$ (see Theorem 6.20 in [8]), the observation

$$\int_{\partial B} g_0\, ds = \int_{\partial B} \operatorname{Div} g\, ds = 0$$

now implies that $\operatorname{Div} g = 0$, i.e., $g \in L^{2,0}_{t,div}(\partial B)$.

Conversely, from (4.8) and the definition of P^* it is obvious that for $g \in L^2_{t,div}(\partial B)$ with $\operatorname{Div} g = 0$ we have that $P^*g = 0$. Now as in the proof of Theorem 3.2 the statement on the eigenelements and eigenvalues follows from the spectral theorem. □

For a constant unit vector e and $z \in B$ we define

$$\chi(\cdot\,, z) := \operatorname{Grad} \operatorname{div}\{\Phi(\cdot\,, z) e\} \quad \text{on } \partial B.$$

Since

$$(\chi(\cdot\,, z), g) = (\operatorname{Grad} \operatorname{div}\{\Phi(\cdot\,, z) e\}, g) = -(\operatorname{div}\{\Phi(\cdot\,, z) e\}, \operatorname{Div} g)$$

for all $g \in L^2_{t,div}(\partial B)$, it is obvious that $\chi(\cdot\,, z)$ is orthogonal to the nullspace $L^{2,0}_{t,div}(\partial B)$ of W. Therefore we have the Fourier series

$$\chi(\cdot\,, z) = \sum_{n=1}^{\infty} (\Psi(\cdot\,, z), g_n) g_n. \tag{4.9}$$

With the aid to this expansion the proof of the following theorem is completely analogous to that of Theorem 3.3. Only instead of the uniqueness for the

Neumann problem in the exterior of B the uniqueness for the Dirichlet problem has to used.

Theorem 4.4. *The point $z \in B$ belongs to D if and only if*

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n} \, |(\chi(\cdot\,, z), g_n)|^2 < \infty. \tag{4.10}$$

Corollary 4.5. *The point $z \in B$ belongs to D if and only if the operator equation*

$$W^{1/2} g = \chi(\cdot\,, z)$$

has a solution $g \in L^2(\partial B)$.

Corollary 4.6. *The domain D is uniquely determined by the knowledge of $\nu(x) \times v(x, y, p)$ for all $x, y \in \partial B$ and $p \in \mathbb{R}^3$ with $p \cdot \nu(y) = 0$.*

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(Received 2.04.2003)

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