Georgian Mathematical Journal
Volume 10 (2003), Number 3, 543–548

ON THE ASYMPTOTICS OF SOLUTIONS OF ELLIPTIC EQUATIONS IN A NEIGHBORHOOD OF A CRACK WITH NONSMOOTH FRONT

V. A. KONDRAT'EV AND V. A. NIKISHKIN

Abstract. Two terms of asymptotics near crack are obtained for solutions of the Dirichlet boundary value problem for second-order elliptic equations in divergent form. The front of a crack is from C^{1+s} and the coefficients of the equations belong to C^s $(0.5 < s < 1)$.

2000 Mathematics Subject Classification: 35J25.
Key words and phrases: Asymptotics, solution of elliptic equation, crack, nonsmooth front, Dirichlet boundary value problem, divergent form.

A second-order elliptic equation in $G \setminus \Omega$ is considered, where $G \subset \mathbb{R}^n$ is a domain with a smooth boundary, Ω is an $(n-1)$-dimensional manifold with a boundary from C^{1+s}, $0 < s < 1$.

We study solutions of the equation

$$\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\,\frac{\partial u}{\partial x_j}\right) = 0, \qquad x \in G \setminus \Omega, \tag{1}$$

that belong to the Sobolev space $W_2^1(G)$ and satisfy the condition

$$u(x) = 0, \qquad x \in \Omega. \tag{2}$$

Let L be the boundary of Ω, r be the distance from x to L, x_L be a point on L nearest to x, φ be the polar angle in the plane normal to L and passing through x_L.

Singularities of solutions of elliptic equations near a nonsmooth boundary were studied by many authors, see, e.g., [1]–[4].

In [4], the following representation was obtained for the plane $(n-1)$-dimensional domain Ω:

$$u(x) = C(x_L)r^{1/2}\Phi(\varphi) + u_1(x),$$

where $\Phi(\varphi)$ is a smooth function,

$$|u_1| \le C_0 r^{1/2+\varepsilon}, \qquad 0 < \varepsilon < \min\{s, 0.5\},$$
$$|C(x_L)| + |C_0| \le const \cdot \|u\|_{W_2^1(G)}.$$

The following result is the main theorem of our paper.

Theorem 1. *Let $u(x) \in W_2^1(G)$ be a solution of problem* (1), (2), *and let*

$$a_{ij} \in C^s(G) \qquad (i,j = 1,\dots,n), \qquad \Omega \in C^{1+s},$$

where $0 < s < 1$.

ISSN 1072-947X / \$8.00 /

Then:
if $0 < s \leq 0.5$, *then*

$$u(x) = C(x_L)r^{1/2}\Phi(\varphi) + u_1(x),$$

where $\Phi(\varphi)$ *is a smooth function,*

$$|u_1| \leq C_0 r^{1/2+\varepsilon}, \qquad 0 < \varepsilon < s,$$
$$|C(x_L)| + |C_0| \leq const \cdot \|u\|_{W_2^1(G)};$$

if $0.5 < s < 1$, *then*

$$u(x) = C_1(x_L)r^{1/2}\Phi_1(\varphi) + C_2(x_L)r\Phi_2(\varphi) + u_1(x),$$

where $\Phi_1(\varphi)$, $\Phi_1(\varphi)$ *are smooth functions*

$$|u_1| \leq C_0 r^{1+\varepsilon}, \qquad 0 < \varepsilon < s - 0.5,$$
$$|C_1(x_L)| + |C_2(x_L)| + |C_0| \leq const \cdot \|u\|_{W_2^1(G)}.$$

Straightening of the boundary. Let P be an arbitrary point of the set L. Consider a neighborhood U of P in which Ω admits a one-to-one projection to the tangent plane. Assume that P is the origin and that in this neighborhood we have

$$\Omega = \{x \mid x_1 = F(x_2, \dots, x_n), \quad (x_2, \dots, x_n) \in \Omega_1\}, \qquad O \in \partial\Omega_1.$$

$\partial\Omega_1$ is given in a neighborhood of the origin by the equation

$$x_2 = h(x_3, \dots, x_n) \in C^{1+s}.$$

Let us extend $F(x_2, \dots, x_n) \in C^{1+s}$ in a neighborhood of the origin so that the class of smoothness be prescribed. Let $F(0), \nabla F(0) = 0$.

Introduce an averaging kernel $K(\tau)$ such that $K(\tau) \in C^\infty(R^1)$, $K(\tau)$ is even, $K(\tau) \equiv 0$ for $\mid \tau \mid \geq 1$, and

$$\int_{-1}^{1} K(\tau)d\tau = 1.$$

The straightening of the boundary consists of two steps.

The first transformation of the coordinates has the form:

$$x_1 = x_1' + H(x'), \qquad x_2 = x_2', \quad \dots, \quad x_n = x_n',$$

where

$$H(x') = \int\limits_{R^{n-1}} F(t) \prod_{l=2}^{n} \left(\frac{1}{|x_1|} K\left(\frac{t_l - z_l'}{|x_1|}\right)\right) dt.$$

The second transformation of the coordinates is the same as that in [4].

Under the above transformations, equation (1) becomes an equation in divergent form with coefficients in C^s.

Dirichlet problem in a dihedral angle for an equation with constant coefficients. Let G_0 be a dihedral angle

$$G_0 = \{x \mid 0 < x_1^2 + x_2^2 < \infty,\ \ 0 < \varphi < \omega\},$$

where φ is the polar angle in the plane (x_1, x_2).

Set

$$\rho = \sqrt{\sum_{i=1}^{n} x_i^2}\,, \qquad r' = \frac{\sqrt{x_1^2 + x_2^2}}{\rho}\,.$$

We need the weighted Sobolev spaces $\dot{W}^0_{\alpha,\beta}$ and $\dot{W}^1_{\alpha,\beta}$ in which the norms are defined as follows:

$$\|u\|^2_{\dot{W}^0_{\alpha,\beta}} = \int\limits_{G_0} u^2 \rho^{\alpha} (r')^{\beta} dx,$$

$$\|u\|^2_{\dot{W}^1_{\alpha,\beta}} = \int\limits_{G_0} \rho^{\alpha} (r')^{\beta} \operatorname{grad}^2 u dx + \int\limits_{G_\circ} u^2 \rho^{\alpha-2} (r')^{\beta-2} dx.$$

Let $u(x) \in W_2^1(G_0)$ be a generalized solution (here and below, in the sense of distributions) of the following Dirichlet problem:

$$\Delta u(x) = f_0(x) + \sum_{i=1}^{n} \frac{\partial f_i(x)}{\partial x_i}, \qquad x \in G_0, \tag{3}$$

$$u(x) = 0, \qquad x \in \partial G_0. \tag{4}$$

The following two assertions can be proved by the method developed in [1].

Theorem 2. *Let $u(x) \in W_2^1(G_0)$ be a generalized solution of problem* (3), (4)*, where*

$$f_0 \in \dot{W}^0_{\alpha,\beta}(G_0), \qquad f_i \in \dot{W}^0_{\alpha-2,\beta-2}(G_0) \qquad (i = 1, \dots, n),$$
$$\alpha + 2\left(\frac{\pi}{\omega} - 2\right) + n > 0, \qquad \beta + 2\left(\frac{\pi}{\omega} - 2\right) + n - 1 > 0.$$

Then $u \in \dot{W}^1_{\alpha-2,\beta-2}(G_0)$.

Theorem 3. *Let $u(x) \in W_2^1(G_0)$ be a generalized solution of problem* (3), (4)*, where*

$$f_0 \in \dot{W}^0_{\alpha,\beta}(G_0), \quad f_i \in \dot{W}^0_{\alpha-2,\beta-2}(G_0) \quad (i = 1, \dots, n),$$
$$\alpha + 2\left(\frac{2\pi}{\omega} - 2\right) + n < 0, \quad \beta + 2\left(\frac{\pi}{\omega} - 2\right) + n - 1 > 0,$$
$$\alpha + 2\left(\frac{3\pi}{\omega} - 2\right) + n > 0.$$

Then $u(x)$ can be represented in the form

$$u(x) = C_1 \rho^{\frac{\pi}{\omega}} \Phi_1(\theta) + C_2 \rho^{\frac{2\pi}{\omega}} \Phi_2(\theta) + u_1,$$

where θ are the coordinates on the unit sphere, $\Phi_1(\theta)$,$\Phi_1(\theta)$ are the eigenfunctions of the Beltrami operator, and

$$u_1 \in \dot{W}^1_{\alpha-2,\beta-2}(G_0).$$

Dirichlet problem in a dihedral angle for an equation with variable coefficients. Let $u(x) \in W^1_2(G_0)$ be a generalized solution of the Dirichlet problem

$$\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\,\frac{\partial u(x)}{\partial x_j}\right) = f_0(x) + \sum_{i=1}^{n} \frac{\partial f_i(x)}{\partial x_i}, \qquad x \in G_0, \tag{5}$$

$$u(x) = 0, \qquad x \in \partial G_0, \tag{6}$$

where $a_{ij} \in C^s(\overline{G}_0)$, $a_{ij}(0) = \delta_{ij}$ (without loss of generality) $(i, j = 1, \ldots, n)$.

Lemma 1. *Let $u(x) \in W^1_2(G_0)$ be a generalized solution of problem* (5), (6), *where*

$$f_0 \in \dot{W}^0_{2-2ks+\varepsilon_0,2}(G_0), \quad f_i \in \dot{W}^0_{-2ks+\varepsilon_0,0}(G_0) \quad (i = 1, \ldots, n),$$

$$2 - 2ks + \varepsilon_0 + 2\left(\frac{\pi}{\omega} - 2\right) + n > 0, \quad 2 - 2(k+1)s + \varepsilon_0 + 2\left(\frac{\pi}{\omega} - 2\right) + n < 0,$$

k is a nonnegative integer, and $\varepsilon_0 > 0$ is sufficiently small.

Then

$$u \in \dot{W}^1_{-2ks+\varepsilon_0,0}(G_0).$$

The proof of Lemma 1 follows from Theorem 2 by induction on k_1 $(0 \le k_1 \le k)$ and is based on the representation of equation (5) in the form

$$\Delta u(x) = f_0(x) + \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(f_i(x) - \sum_{j=1}^{n}(a_{ij}(x) - \delta_{ij})\frac{\partial u}{\partial x_j}\right).$$

Using this representation, Lemma 1, and Theorem 3, we obtain the following assertion.

Lemma 2. *Let $u(x) \in W^1_2(G_0)$ be a generalized solution of problem* (5), (6), *where*

$$f_0 \in \dot{W}^0_{\alpha,2}(G_0), \qquad f_i \in \dot{W}^0_{\alpha-2,0}(G_0) \qquad (i = 1, \ldots, n),$$

$\alpha = -\frac{4\pi}{\omega} - n + 4 - \varepsilon_1$, $0 < \varepsilon_1 < 2s - \frac{2\pi}{\omega}$.

Then $u(x)$ can be represented in the form

$$u(x) = C_1\rho^{\frac{\pi}{\omega}}\Phi_1(\theta) + C_2\rho^{\frac{2\pi}{\omega}}\Phi_2(\theta) + u_1,$$

where θ are the coordinates on the unit sphere, $\Phi_1(\theta)$,$\Phi_1(\theta)$ are the eigenfunctions of the Beltrami operator, and

$$u_1 \in \dot{W}^1_{\alpha-2,0}(G_0).$$

Remark 1. One can readily see that all conditions of Theorems 2 and 3 are satisfied for the weights in Lemmas 1 and 2 for $0 < \varepsilon_1 < 2s - \frac{2\pi}{\omega}$.

Bounds for $|u_1|$. Consider the cones K and $\widehat{K}$

$$K = \{x \mid x_3^2 + c \cdots + x_n^2 \geq k^2(x_1^2 + x_2^2)\},$$
$$\widehat{K} = \{x \mid x_3^2 + \cdots + x_n^2 \geq \widehat{k}^2(x_1^2 + x_2^2)\}$$

and the domains

$$G_1 = G_0 \setminus K, \qquad \widehat{G}_1 = G_0 \setminus \widehat{K}.$$

Obviously, $\widehat{G}_1 \Subset G_1$ for $\widehat{k} < k$.

Lemma 3. *Suppose that, in addition to the assumptions of Lemma* 2, *the following inequalities hold in the domain* G_1*:*

$$|f_0(x)| \leq C_0 \rho^{\frac{2\pi}{\omega} - 2 + \varepsilon}, \qquad |f_i(x)| \leq C_0 \rho^{\frac{2\pi}{\omega} - 1 + \varepsilon} \qquad (i = 1 \ldots, n),$$

and let $f_0(x)$ *and* $f_i(x)$ *be continuous in* G_1, $0 < \varepsilon < s - \frac{\pi}{\omega}$.

Then

$$|u_1| \leq C_1 \rho^{\frac{2\pi}{\omega} + \varepsilon},$$
$$|\operatorname{grad} u_1(x)| \leq C \rho^{\frac{2\pi}{\omega} - 1 + \varepsilon}$$

in $\widehat{G}_1$.

The proof of Lemma 3 is the same as that in [4].

Remark 2. The proof of Theorem 1 is obtained on the basis of the above assertions.

Acknowledgement

The work was carried out according to Project No. 01-00 of A. M. Lyapunov French-Russian Institute.

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(Received 27.03.2003)

Authors' addresses:

V. A. Kondrat'ev
Department of Mechanics and Mathematics
Moscow State University,
Vorob'evy gory, Moscow 119899
Russia

V. A. Nikishkin
Department of Mathematics
Moscow State University of Economics, Statistics and Informatics,
7, Nezhinskaya, Moscow 119501
Russia