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ON THE ASYMPTOTICS OF SOLUTIONS OF ELLIPTIC EQUATIONS IN A NEIGHBORHOOD OF A CRACK WITH NONSMOOTH FRONT

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Abstract. Two terms of asymptotics near crack are obtained for solutions of the Dirichlet boundary value problem for second-order elliptic equations in divergent form. The front of a crack is from C^{1+s} and the coefficients of the equations belong to C^s $(0.5 < s < 1)$.

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A second-order elliptic equation in $G \setminus \Omega$ is considered, where $G \subset \mathbb{R}^n$ is a domain with a smooth boundary, Ω is an $(n-1)$-dimensional manifold with a boundary from C^{1+s}, $0 < s < 1$.

We study solutions of the equation

$$\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) = 0, \qquad x \in G \setminus \Omega, \tag{1}$$

that belong to the Sobolev space $W_2^1(G)$ and satisfy the condition

$$u(x) = 0, \qquad x \in \Omega. \tag{2}$$

Let L be the boundary of Ω, r be the distance from x to L, x_L be a point on L nearest to x, φ be the polar angle in the plane normal to L and passing through x_L.

Singularities of solutions of elliptic equations near a nonsmooth boundary were studied by many authors, see, e.g., [1]–[4].

In [4], the following representation was obtained for the plane $(n-1)$-dimensional domain Ω:

$$u(x) = C(x_L)r^{1/2}\Phi(\varphi) + u_1(x),$$

where $\Phi(\varphi)$ is a smooth function,

$$|u_1| \le C_0 r^{1/2+\varepsilon}, \qquad 0 < \varepsilon < \min\{s, 0.5\},$$
$$|C(x_L)| + |C_0| \le const \cdot \|u\|_{W_2^1(G)}.$$

The following result is the main theorem of our paper.

Theorem 1. *Let $u(x) \in W_2^1(G)$ be a solution of problem* (1), (2), *and let*

$$a_{ij} \in C^s(G) \qquad (i,j = 1,\dots,n), \qquad \Omega \in C^{1+s},$$

where $0 < s < 1$.

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Then:
if $0 < s \leq 0.5$, *then*

$$u(x) = C(x_L)r^{1/2}\Phi(\varphi) + u_1(x),$$

where $\Phi(\varphi)$ *is a smooth function,*

$$|u_1| \leq C_0 r^{1/2+\varepsilon}, \qquad 0 < \varepsilon < s,$$
$$|C(x_L)| + |C_0| \leq const \cdot \|u\|_{W_2^1(G)};$$

if $0.5 < s < 1$, *then*

$$u(x) = C_1(x_L)r^{1/2}\Phi_1(\varphi) + C_2(x_L)r\Phi_2(\varphi) + u_1(x),$$

where $\Phi_1(\varphi)$, $\Phi_1(\varphi)$ *are smooth functions*

$$|u_1| \leq C_0 r^{1+\varepsilon}, \qquad 0 < \varepsilon < s - 0.5,$$
$$|C_1(x_L)| + |C_2(x_L)| + |C_0| \leq const \cdot \|u\|_{W_2^1(G)}.$$

Straightening of the boundary. Let P be an arbitrary point of the set L. Consider a neighborhood U of P in which Ω admits a one-to-one projection to the tangent plane. Assume that P is the origin and that in this neighborhood we have

$$\Omega = \{x \mid x_1 = F(x_2, \dots, x_n), \quad (x_2, \dots, x_n) \in \Omega_1\}, \qquad O \in \partial\Omega_1.$$

$\partial\Omega_1$ is given in a neighborhood of the origin by the equation

$$x_2 = h(x_3, \dots, x_n) \in C^{1+s}.$$

Let us extend $F(x_2, \dots, x_n) \in C^{1+s}$ in a neighborhood of the origin so that the class of smoothness be prescribed. Let $F(0), \nabla F(0) = 0$.

Introduce an averaging kernel $K(\tau)$ such that $K(\tau) \in C^\infty(R^1)$, $K(\tau)$ is even, $K(\tau) \equiv 0$ for $|\tau| \geq 1$, and

$$\int_{-1}^{1} K(\tau)d\tau = 1.$$

The straightening of the boundary consists of two steps.

The first transformation of the coordinates has the form:

$$x_1 = x_1' + H(x'), \qquad x_2 = x_2', \quad \dots, \quad x_n = x_n',$$

where

$$H(x') = \int_{R^{n-1}} F(t) \prod_{l=2}^{n} \left(\frac{1}{|x_1|} K\left(\frac{t_l - z_l'}{|x_1|} \right) \right) dt.$$

The second transformation of the coordinates is the same as that in [4].

Under the above transformations, equation (1) becomes an equation in divergent form with coefficients in C^s.

Dirichlet problem in a dihedral angle for an equation with constant coefficients. Let G_0 be a dihedral angle

$$G_0 = \{x \mid 0 < x_1^2 + x_2^2 < \infty, \ \ 0 < \varphi < \omega\},$$

where φ is the polar angle in the plane (x_1, x_2).

Set

$$\rho = \sqrt{\sum_{i=1}^{n} x_i^2}\,, \qquad r' = \frac{\sqrt{x_1^2 + x_2^2}}{\rho}\,.$$

We need the weighted Sobolev spaces $\dot{W}^0_{\alpha,\beta}$ and $\dot{W}^1_{\alpha,\beta}$ in which the norms are defined as follows:

$$\|u\|^2_{\dot{W}^0_{\alpha,\beta}} = \int\limits_{G_0} u^2 \rho^{\alpha} (r')^{\beta} dx,$$

$$\|u\|^2_{\dot{W}^1_{\alpha,\beta}} = \int\limits_{G_0} \rho^{\alpha} (r')^{\beta} \operatorname{grad}^2 u dx + \int\limits_{G_\circ} u^2 \rho^{\alpha-2} (r')^{\beta-2} dx.$$

Let $u(x) \in W_2^1(G_0)$ be a generalized solution (here and below, in the sense of distributions) of the following Dirichlet problem:

$$\Delta u(x) = f_0(x) + \sum_{i=1}^{n} \frac{\partial f_i(x)}{\partial x_i}, \qquad x \in G_0, \tag{3}$$

$$u(x) = 0, \qquad x \in \partial G_0. \tag{4}$$

The following two assertions can be proved by the method developed in [1].

Theorem 2. *Let $u(x) \in W_2^1(G_0)$ be a generalized solution of problem* (3), (4), *where*

$$f_0 \in \dot{W}^0_{\alpha,\beta}(G_0), \qquad f_i \in \dot{W}^0_{\alpha-2,\beta-2}(G_0) \qquad (i = 1, \ldots, n),$$

$$\alpha + 2\left(\frac{\pi}{\omega} - 2\right) + n > 0, \qquad \beta + 2\left(\frac{\pi}{\omega} - 2\right) + n - 1 > 0.$$

Then $u \in \dot{W}^1_{\alpha-2,\beta-2}(G_0)$.

Theorem 3. *Let $u(x) \in W_2^1(G_0)$ be a generalized solution of problem* (3), (4), *where*

$$f_0 \in \dot{W}^0_{\alpha,\beta}(G_0), \quad f_i \in \dot{W}^0_{\alpha-2,\beta-2}(G_0) \quad (i = 1, \ldots, n),$$

$$\alpha + 2\left(\frac{2\pi}{\omega} - 2\right) + n < 0, \quad \beta + 2\left(\frac{\pi}{\omega} - 2\right) + n - 1 > 0,$$

$$\alpha + 2\left(\frac{3\pi}{\omega} - 2\right) + n > 0.$$

Then $u(x)$ can be represented in the form

$$u(x) = C_1 \rho^{\frac{\pi}{\omega}} \Phi_1(\theta) + C_2 \rho^{\frac{2\pi}{\omega}} \Phi_2(\theta) + u_1,$$

where θ are the coordinates on the unit sphere, $\Phi_1(\theta)$,$\Phi_1(\theta)$ are the eigenfunctions of the Beltrami operator, and

$$u_1 \in \dot{W}^1_{\alpha-2,\beta-2}(G_0).$$

Dirichlet problem in a dihedral angle for an equation with variable coefficients. Let $u(x) \in W^1_2(G_0)$ be a generalized solution of the Dirichlet problem

$$\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u(x)}{\partial x_j}\right) = f_0(x) + \sum_{i=1}^{n}\frac{\partial f_i(x)}{\partial x_i}, \qquad x \in G_0, \tag{5}$$

$$u(x) = 0, \qquad x \in \partial G_0, \tag{6}$$

where $a_{ij} \in C^s(\overline{G}_0)$, $a_{ij}(0) = \delta_{ij}$ (without loss of generality) $(i, j = 1, \ldots, n)$.

Lemma 1. *Let $u(x) \in W^1_2(G_0)$ be a generalized solution of problem* (5), (6), *where*

$$f_0 \in \dot{W}^0_{2-2ks+\varepsilon_0,2}(G_0), \quad f_i \in \dot{W}^0_{-2ks+\varepsilon_0,0}(G_0) \quad (i = 1, \ldots, n),$$

$$2 - 2ks + \varepsilon_0 + 2\left(\frac{\pi}{\omega} - 2\right) + n > 0, \quad 2 - 2(k+1)s + \varepsilon_0 + 2\left(\frac{\pi}{\omega} - 2\right) + n < 0,$$

k is a nonnegative integer, and $\varepsilon_0 > 0$ is sufficiently small.

Then

$$u \in \dot{W}^1_{-2ks+\varepsilon_0,0}(G_0).$$

The proof of Lemma 1 follows from Theorem 2 by induction on k_1 $(0 \le k_1 \le k)$ and is based on the representation of equation (5) in the form

$$\Delta u(x) = f_0(x) + \sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(f_i(x) - \sum_{j=1}^{n}(a_{ij}(x) - \delta_{ij})\frac{\partial u}{\partial x_j}\right).$$

Using this representation, Lemma 1, and Theorem 3, we obtain the following assertion.

Lemma 2. *Let $u(x) \in W^1_2(G_0)$ be a generalized solution of problem* (5), (6), *where*

$$f_0 \in \dot{W}^0_{\alpha,2}(G_0), \qquad f_i \in \dot{W}^0_{\alpha-2,0}(G_0) \qquad (i = 1, \ldots, n),$$

$\alpha = -\frac{4\pi}{\omega} - n + 4 - \varepsilon_1$, $0 < \varepsilon_1 < 2s - \frac{2\pi}{\omega}$.

Then $u(x)$ can be represented in the form

$$u(x) = C_1\rho^{\frac{\pi}{\omega}}\Phi_1(\theta) + C_2\rho^{\frac{2\pi}{\omega}}\Phi_2(\theta) + u_1,$$

where θ are the coordinates on the unit sphere, $\Phi_1(\theta)$,$\Phi_1(\theta)$ are the eigenfunctions of the Beltrami operator, and

$$u_1 \in \dot{W}^1_{\alpha-2,0}(G_0).$$

Remark 1. One can readily see that all conditions of Theorems 2 and 3 are satisfied for the weights in Lemmas 1 and 2 for $0 < \varepsilon_1 < 2s - \frac{2\pi}{\omega}$.

Bounds for $|u_1|$. Consider the cones K and $\widehat{K}$

$$K = \{x \mid x_3^2 + c \cdots + x_n^2 \geq k^2(x_1^2 + x_2^2)\},$$
$$\widehat{K} = \{x \mid x_3^2 + \cdots + x_n^2 \geq \widehat{k}^2(x_1^2 + x_2^2)\}$$

and the domains

$$G_1 = G_0 \setminus K, \qquad \widehat{G}_1 = G_0 \setminus \widehat{K}.$$

Obviously, $\widehat{G}_1 \Subset G_1$ for $\widehat{k} < k$.

Lemma 3. *Suppose that, in addition to the assumptions of Lemma* 2, *the following inequalities hold in the domain* G_1*:*

$$|f_0(x)| \leq C_0 \rho^{\frac{2\pi}{\omega}-2+\varepsilon}, \qquad |f_i(x)| \leq C_0 \rho^{\frac{2\pi}{\omega}-1+\varepsilon} \qquad (i = 1 \dots, n),$$

and let $f_0(x)$ *and* $f_i(x)$ *be continuous in* G_1, $0 < \varepsilon < s - \frac{\pi}{\omega}$.

Then

$$|u_1| \leq C_1 \rho^{\frac{2\pi}{\omega}+\varepsilon},$$
$$|\operatorname{grad} u_1(x)| \leq C \rho^{\frac{2\pi}{\omega}-1+\varepsilon}$$

in $\widehat{G}_1$.

The proof of Lemma 3 is the same as that in [4].

Remark 2. The proof of Theorem 1 is obtained on the basis of the above assertions.

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References

1. V. A. Kondrat'ev, Boundary value problems for elliptic equations in domains with conical or angular points. (Russian) *Trudy Moscow. Mat. Obshch.* **16**(1967), 209–292.
2. G. M. Verzhbinskiĭ and V. G. Maz'ja, Asymptotic behavior of the solutions of second order elliptic equations near the boundary. II. (Russian) *Sibirsk. Mat. Zh.* **13**(1972), 1239–1271.
3. V. A. Kondrat'ev, I. Kopachek, and O. A. Oleinik, Best Hölder exponents for generalized solutions of the Dirichlet problem for a second-order elliptic uquation. *Mat. Sb.* **131**(1986), 113–125.
4. V. A. Kondrat'ev and V. A. Nikishkin, On the behavior of solutions of elliptic equations in a neighborhood of a crack with nonsmooth front. *Russian J. Math. Phys.* **9**(2002), No. 1, 61–65.

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