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ON FOURIER SERIES IN EIGENFUNCTIONS OF ELLIPTIC BOUNDARY VALUE PROBLEMS

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To the memory of Victor Dmitrievich Kupradze

Abstract. We consider a general elliptic formally self-adjoint problem in a bounded domain $\Omega \subset \mathbb{R}^n$ with homogeneous boundary conditions under the assumption that the boundary and coefficients are infinitely smooth. The operator in $L_2(\Omega)$ corresponding to this problem has an orthonormal basis $\{u_l\}$ of eigenfunctions, which are infinitely smooth in $\overline{\Omega}$. However, the system $\{u_l\}$ is not a basis in Sobolev spaces $H^t(\Omega)$ of high order.

We note and discuss the following possibility: for an arbitrarily large t, for each function $u \in H^t(\Omega)$ one can explicitly construct a function $u_0 \in H^t(\Omega)$ such that the Fourier series of the difference $u - u_0$ in the functions u_l converges to this difference in $H^t(\Omega)$. Moreover, the function $u(x)$ is viewed as a solution of the corresponding nonhomogeneous elliptic problem and is not assumed to be known a priori; only the right-hand sides of the elliptic equation and the boundary conditions for u are assumed to be given. These data are also sufficient for the computation of the Fourier coefficients of $u - u_0$. The function u_0 is obtained by applying some linear operator to these right-hand sides.

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1. Introduction. It is well known that the smoother a function to be expanded in a trigonometric Fourier series is, the better the series converges. Trigonometric functions are eigenfunctions of the operator $u \mapsto u''$ on the circle. More generally, the situation is similar for Fourier expansions in eigenfunctions of elliptic self-adjoint differential operators with smooth coefficients (or pseudodifferential operators of nonzero order with smooth symbols) on a smooth closed manifold (i.e., a compact manifold without boundary). This is also well known; see, e.g., [1]. Throughout the paper, except for Remark 1 in Section 3, smoothness is understood as infinite smoothness.

This is not and cannot be true for Fourier series in eigenfunctions of self-adjoint elliptic boundary value problems with homogeneous boundary conditions on a compact smooth manifold with smooth boundary. For example, the well-known papers [2] and [3] deal with the convergence or summability of these series mainly on compact sets lying in the interior of the manifold. The reason is not discussed there, but it is very simple: the eigenfunctions are subjected

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to boundary conditions, whereas the function to be expanded does not satisfy these conditions in general.

In the papers [4] and [5], additional boundary conditions are indicated that should be imposed on a function to be expanded in a Fourier series in eigenfunctions of the given elliptic boundary value problem so as to ensure that the series converges to the function in Sobolev spaces of high order. These are the homogeneous boundary conditions of the problem corresponding to an appropriate power of the operator associated with the original problem. (More precisely, these papers deal in general with nonself-adjoint problems and discuss the completeness of root functions. See also the references therein.) In [6], the orders of Sobolev spaces are discussed in which the completeness and convergence of Fourier series in eigenfunctions in domains with smooth or nonsmooth (Lipschitz) boundary are preserved.

Let us give an example. Consider the spectral boundary value problem

$$-\Delta u(x) = \lambda u(x) \quad \text{in } \Omega, \qquad Bu := \partial_\nu u(x) - \mu u(x) = 0 \quad \text{on } \Gamma. \tag{1}$$

Here and in the following, Ω is a bounded domain in $\mathbb{R}^n$ with connected $(n-1)$-dimensional smooth boundary Γ. The coefficient μ is assumed to be a given real number. By ∂_ν we denote the outward normal derivative on the boundary. This is a formally self-adjoint problem with discrete spectrum, and its eigenfunctions form an orthonormal basis in $L_2(\Omega)$. The eigenfunctions belong to $C^\infty(\overline{\Omega})$ and satisfy the boundary condition in (1). Associated with problem (1) is the operator $\mathcal{A}u = -\Delta u$ in $L_2(\Omega)$ with domain $H^2(\Omega, B)$ that is the subspace of $H^2(\Omega)$ singled out by the boundary condition in (1). Here and in the following, we use the simplest Sobolev spaces $H^t = W_2^t$. For simplicity, suppose that $\lambda = 0$ is not an eigenvalue of problem (1). Then the operator $\mathcal{A}$ maps $H^2(\Omega, B)$ onto $L_2(\Omega)$ bi-isomorphically and bicontinuously. Hence the eigenfunctions form an unconditional basis in $H^2(\Omega, B)$. (A basis is said to be unconditional if it remains a basis after any permutation of its elements.)

Now consider the eigenfunctions as elements of the spaces $H^t(\Omega)$. The boundary condition in (1) is of first order. The differentiation reduces the index of the Sobolev space by one, and the passage to the boundary reduces it further by $1/2$. Hence the boundary condition makes sense as a relation in $H^{t-3/2}(\Gamma)$, but only for $t > 3/2$. If $0 \le t < 3/2$, then the boundary condition becomes meaningless, and so the eigenfunctions satisfy no boundary conditions at all (in the sense of usual functions rather than distributions). One can show with the use of interpolation that the eigenfunctions form an unconditional basis in the space $H^t(\Omega)$ for $0 \le t < 3/2$. (See the next section for details concerning more general problems.) For $t = 3/2$, the situation is slightly more complicated, and we do not discuss it. For $3/2 < t < 7/2$, the eigenfunctions form an unconditional basis in the subspace $H^t(\Omega, B) \subset H^t(\Omega)$ singled out by the boundary condition in (1).

Let us explain how the restriction $t < 7/2$ arises. Problem (1) implies the relation

$$\partial_\nu \Delta u(x) - \mu \Delta u(x) = 0 \tag{2}$$

on Γ, since the left-hand side is equal to $\lambda[-\partial_\nu u(x)+\mu u(x)]$. The new boundary condition (2) has order 3 and makes sense for eigenfunctions provided that these are viewed as elements of $H^t(\Omega)$ with $t > 7/2$. Using the square of the operator corresponding to the original problem, one can verify that for $7/2 < t < 11/2$ the eigenfunctions form an unconditional basis in the subspace of $H^t(\Omega)$ singled out by the two boundary conditions. For $11/2 < t < 15/2$, a third boundary condition is added, and so on.

As has been noted in the abstract, our aim is to discuss the following possibility: for an arbitrarily large t, from the function $u \in H^t(\Omega)$ to be expanded we can always subtract a function $u_0 \in H^t(\Omega)$ such that the Fourier series of the difference $u - u_0$ in the eigenfunctions will converge to it in $H^t(\Omega)$. The function u_0 to be subtracted is obtained by applying some linear operator to the right-hand sides of the problem for u. For $t > n/2$, the convergence proves to be uniform. The procedure of construction and subtraction of u_0 can be viewed as a *regularization* of the Fourier series of the original function near the boundary.

If $u(x)$ is completely known, the procedure is of no interest: one can merely multiply $u(x)$ by a cut-off function equal to unity outside a boundary strip and vanishing in a smaller boundary strip, thus readily obtaining a function that lies in the domain of any power of the operator corresponding to the problem. However, our procedure is realizable if $u(x)$ is viewed as a solution of a given nonhomogeneous elliptic problem and only the right-hand sides of the elliptic equation and the boundary conditions in this problem are known. Here we also manage to compute the Fourier coefficients of $u - u_0$.

We think that this possibility is adequate to the situation and can be of some use. The latter is seen, for example, in the discussion of the physical notion of the R-matrices for the Schrödinger equation and the Dirac system, which has actually encouraged the authors to write this article; see Section 4 below and, for more details, [6] and [7].

The description and justification of our proposal is not complicated. We give precise assertions in Section 2 and indicate some possible generalizations in Section 3. The main result is Theorem 4 in Section 2.

We do not touch upon problems concerning the localization of spectral expansions and their convergence or summability almost everywhere (see [2], [3]). Recall however that the convergence almost everywhere follows from the convergence in the Sobolev space $H^\varepsilon(\Omega)$ with an arbitrarily small $\varepsilon > 0$ (see, e.g., [6], Remark 1.10.)

2. Main results. For definiteness and simplicity, we consider the elliptic boundary value problem for the $2m$th-order scalar partial differential equation

$$Au := A(x, D)u(x) = f(x) \qquad (x \in \Omega) \tag{3}$$

with the differential boundary conditions

$$B_j u := B_j(x, D)u(x) = 0 \qquad (x \in \Gamma,\ j = 1, \dots, m). \tag{4}$$

We assume that $n \geq 2$, although one can also consider ordinary differential equations. The coefficients of $A(x, D)$ and $B(x, D)$ are assumed to be smooth in $\overline{\Omega}$ and on Γ, respectively. Suppose that the boundary operators have orders $r_j < 2m$ and form a normal system, i.e., r_j are pairwise distinct and the boundary is noncharacteristic with respect to each B_j at every point of Γ. (In other words, the coefficient of the r_jth-order normal derivative in the expansion of B_j in powers of the normal derivative is an everywhere nonvanishing function.) We assume that the problem is formally self-adjoint. Then it defines a self-adjoint operator $\mathcal{A}u = f$ in $L_2(\Omega) = H^0(\Omega)$ with domain

$$H^{2m}(\Omega, B) = \{u \in H^{2m}(\Omega) : B_j u = 0,\ j = 1, \dots, m\}. \tag{5}$$

The spectrum of this operator is discrete, and the eigenvalues tend to infinity. The eigenfunctions belong to $C^\infty(\overline{\Omega})$ and form an orthonormal basis $\{u_l\}_1^\infty$ in $L_2(\Omega)$. The eigenvalues λ_l have a power asymptotics for an appropriate numbering. In particular, if the operator corresponding to the problem is bounded below, then $\lambda_l = Cl^{2m/n} + O(l^{(2m-1)/n})$, where C is a positive constant that can be expressed via the principal symbol of A in a well-known way (see [8] and references cited therein).

Shifting the spectral parameter if necessary (which does not affect the eigenfunctions), we can assume that zero is not an eigenvalue. Then the operator $\mathcal{A}$ defines a bicontinuous isomorphism of the space (5) onto $L_2(\Omega)$.

Let us give three assertions that have actually been used already in [4]–[6].

Proposition 1. *For every positive integer k, the operator $\mathcal{A}^k$ corresponds to the boundary value problem for the equation $A^k u = f$ $(x \in \Omega)$ with the boundary conditions*

$$B_j A^q u = 0 \qquad (x \in \Gamma,\ j = 1, \dots, m;\ q = 0, \dots, k-1). \tag{6}$$

These boundary operators form a normal system, and the problem is elliptic.

The ellipticity can be verified via equivalent a priori estimates, and the remaining assertions are obvious.

Let t be a positive number. (To avoid additional stipulations, we assume that it is not a half-integer.) By $H^t(\Omega, A, B) \subset H^t(\Omega)$ we denote the subspace singled out by those of the boundary conditions (6) which make sense in $H^t(\Omega)$. Namely, these are conditions (6) with $r_j + 2mq < t - 1/2$. The system formed by these conditions is still normal if it is nonempty.

Proposition 2. *The operator $\mathcal{A}^{t/2m}$ defines a bicontinuous isomorphism of the spaces $H^t(\Omega, A, B)$ and $L_2(\Omega)$.*

This follows from the results concerning the interpolation of subspaces singled out by boundary conditions in Sobolev spaces [9], [10]; cf. [4]–[6]. The space $H^t(\Omega, A, B)$ is the domain of the operator $\mathcal{A}^{t/2m}$.

Corollary. *Under the same assumptions, the eigenfunctions u_l form an unconditional basis in $H^t(\Omega, A, B)$.*

Now let $\mathcal{C} = \{\mathcal{C}_j(x, D)\}$ $(j = 0, \dots, s)$ be an arbitrary normal system of differential boundary operators of some orders $\rho_j < t - 1/2$, where t is still not a half-integer. Consider the system of boundary conditions

$$\mathcal{C}_j(x, D)v(x) = g_j(x) \qquad (x \in \Gamma,\ j = 0, \dots, s), \tag{7}$$

which is not necessarily related to an elliptic boundary value problem.

Proposition 3. *There exists a bounded linear operator*

$$\mathcal{T} = \mathcal{T}(\mathcal{C}) : \prod_{j \le s} H^{t-\rho_j-1/2}(\Gamma) \to H^t(\Omega)$$

that transforms each set $\{g_j\}$ of the right-hand sides in the boundary conditions (7) *into a function v in Ω satisfying these boundary conditions.*

Proof. Since one can use a partition of unity on Γ, it suffices to consider the following situation. Let x_0 be a point on Γ, and assume that it is possible to pass to local coordinates rectifying the boundary in the ball $O_R(x_0)$ of radius R with center at x_0. We need to construct a function v with support lying in $O_{R/2}(x_0) \cap \overline{\Omega}$ assuming that the supports of the right-hand sides lie in $O_{R/4}(x_0) \cap \Gamma$.

Passing to these coordinates, we can now consider the half-space $\mathbb{R}^n_+ = \{x : x_n > 0\}$ instead of Ω and the hyperplane $\mathbb{R}^{n-1} = \{x : x_n = 0\}$ instead of Γ. Moreover, we assume that the supports of g_j lie in a neighborhood $O_\varepsilon(0)$ of the origin in $\mathbb{R}^{n-1}$, and we need to construct v with support lying, say, in $O_{2\varepsilon}(0) \times [0, \varepsilon]$. Furthermore, we can assume that $s = \max \rho_j$, since the given system of boundary operators can always be supplemented by operators of the missing orders with the use of the corresponding powers of the derivative $D_n = -i\partial/\partial x_n$ in x_n with, say, zero right-hand sides in the added boundary conditions. This having been done, the operators form a so-called Dirichlet system.

The Dirichlet system is equivalent to the system of powers of D_n of orders $0, \dots, s$. This means that the column formed by the given operators arranged in ascending orders is obtained from the column formed by the powers of D_n by multiplication by a lower triangular matrix of differential operators A_{jk} of orders $j - k$ with infinitely smooth coefficients and with nowhere vanishing functions on the main diagonal. Conversely, the second column can be obtained from the first in a similar way. This reduces the problem to the case in which the system of boundary operators on $\mathbb{R}^{n-1}$ is just the column of powers of D_n. In this case, the desired function v can be constructed by explicit formulas.

To make the exposition self-contained, we recall Slobodetskii's formula [11] for the function v in this case. Let $\{\psi_k\}_0^s$ be a system of functions in $C_0^\infty(\mathbb{R})$ subject to the conditions

$$D_n^j \psi_k(0) = \delta_k^j \qquad (j, k = 0, \dots, s).$$

By F we denote the Fourier transform with respect to $x' = (x_1, \dots, x_{n-1})$. Let $t > s + 1/2$. For $g_k \in H^{t-k-1/2}(\mathbb{R}^{n-1})$, $k = 0, \dots, s$, we set

$$v(x) = \sum_{k=0}^{s} F^{-1}_{\xi' \mapsto x'} \bigl[\psi_k(x_n(1 + |\xi'|^2)^{1/2})(Fg_k)(\xi')(1 + |\xi'|^2)^{-k/2}\bigr]. \tag{8}$$

Then $v \in H^t(\mathbb{R}^n)$, and this function satisfies the desired boundary conditions

$$D_n^k v\big|_{x_n=0} = g_k(x') \qquad (k = 0, \dots, s).$$

Cf. Seeley [12]. For us, it remains to multiply the right-hand side of (8) by an appropriate cut-off function. □

Thus, the desired function can be expressed in a closed form via the right-hand sides of the given boundary conditions on Γ.

Of course, in concrete situations, other possibilities to construct u_0 can appear to be more convenient. For example, if one needs to construct a function satisfying the single Dirichlet or Neumann condition, then it is possible to use the solution of the corresponding boundary value problems for the Laplace or Helmholtz equation by means of potentials.

Now let t be an arbitrary positive number that is not a half-integer, and let u be an arbitrary function in $H^t(\Omega)$ (it need not satisfy the homogeneous boundary conditions). We set $r = [t - 1/2]$ provided that this is a nonnegative number and denote the set of right-hand sides of the nonhomogeneous boundary conditions $B_j A^q u = g_{jq}$ (cf. (6)) of order $\leq r$ by $g(u)$. (This set may well be empty.) By Proposition 3, there exists an operator $\mathcal{T}$ (we preserve this notation; it can be written out explicitly) such that the function $u - u_0$, where $u_0 = \mathcal{T} g(u)$, satisfies the same boundary conditions with zero right-hand sides. It belongs to the space $H^t(\Omega, A, B)$ and hence satisfies the following main assertion.

Theorem 4. *The Fourier series of the function $u - \mathcal{T} g(u)$ in the eigenfunctions u_l converges to this function in the norm of the space $H^t(\Omega)$.*

Moreover, the convergence is unconditional.

In particular, by Sobolev's embedding theorem, the convergence will be uniform up to the boundary provided that $t > n/2$. By increasing t, one can ensure the uniform convergence of the series obtained by k term-by-term derivations for an arbitrary given k.

Now we assume that only the following is known about the function u: it is the solution of the (uniquely solvable) elliptic boundary value problem

$$Au = f \quad (x \in \Omega), \qquad B_j u = g_j \quad (x \in \Gamma,\ j = 1, \dots, m), \tag{9}$$

and we know only the right-hand sides. We claim that then one can find the function u_0 and the Fourier coefficients of the difference $u - u_0$. Of course, the eigenfunctions u_l and the eigenvalues λ_l are assumed to be known.

To this end, it suffices to make the following three remarks.

1. Let u be a smooth function. Once Au is known, we can find $A^2 u, \dots,$ $B_j A u, \dots$.

2. The function u_0 is constructed from the right-hand sides g_{jq} of the boundary conditions, and it is the result of the application of some linear operator to the right-hand sides of problem (9) for u.

3. Let u_0 satisfy all boundary conditions in (9). Then, after subtracting u_0 from u, we obtain a function whose Fourier coefficients can be determined in the usual manner from those of the right-hand side of the equation $A(u-u_0)=f_1$, where $f_1 = f - Au_0$. Namely,

$$(u-u_0,u_l)_\Omega = \lambda_l^{-1}(u-u_0,Au_l)_\Omega = \lambda_l^{-1}(f_1,u_l)_\Omega.$$

Note that here, if $u-u_0$ belongs to the domain of $\mathcal{A}^2$ and $A^2(u-u_0)=f_2$, then

$$(f_1,u_l)_\Omega = \lambda_l^{-1}(f_2,u_l)_\Omega,$$

etc.

3. Generalizations. Here we indicate some possible generalizations. In doing so, we neither dwell on their combinations nor waste space on precise statements.

1. One can assume that the boundary and the coefficients are of finite smoothness. Then t in the assertions of the preceding section is naturally bounded above.

2. One can consider matrix operators A and B_j. In the simplest case, the principal part of the square matrix $A(x,D)$ is homogeneous with respect to D. However, one can also consider Douglis–Nirenberg elliptic systems under the following restriction: the higher orders of the operators on the main diagonal in $A(x,D)$ are the same (cf. [4], [5]). Under this restriction, a positive integer power of the operator $\mathcal{A}$ corresponds to an elliptic problem of the same structure. Let us comment this as follows. If the orders of all diagonal operators are equal to m, then the multiplication of $A(x,D)$ by itself increases the orders of all matrix entries by m, and the same happens with the matrix of boundary operators if we multiply it by $A(x,D)$ on the right. Here we do not touch more general Douglis–Nirenberg elliptic systems.

3. One can consider nonself-adjoint boundary value problems. In this case, one can discuss completeness, i.e., the possibility of an arbitrarily accurate approximation to the function $u-\mathcal{T}g(u)$ in the space $H^t(\Omega,A,B)$ by linear combinations of root functions, as well as the summability of the Fourier series of this function in root functions by the Abel–Lidskii method with parentheses in this space. In some cases, one has the unconditional convergence of the Fourier series with parentheses of the function $u-\mathcal{T}g(u)$ in root functions in the same space (cf. [1] and [4]–[6]).

Let us give some details. Suppose that a given problem has been obtained from a formally self-adjoint problem by adding some terms of order $\le s < 2m$ to $A(x,D)$. (Here we restrict ourselves to this case.) We set $p = 2m/n$ (this is the exponent of the power growth of the eigenvalues) and $q = s/2m$ (this is the relative order of the perturbation of the self-adjoint part of $A(x,D)$ by the added terms). Consider the number

$$p(1-q) = (2m-s)/n. \tag{10}$$

One can readily verify that it does not change if one passes to powers of the operator corresponding to the problem. It follows from theorems on weak perturbations of self-adjoint operators that if number (10) is not less than unity, then the difference $u - u_0$ can be expanded in a Fourier series in root functions, which is unconditionally convergent in $H^t(\Omega)$ after some arrangement of parentheses independent of u. If number (10) is less than unity, then this series can be summed in $H^t(\Omega)$ by the Abel–Lidskii method (with parentheses) of an arbitrary order greater than $p^{-1} - (1-q) = [n - (2m-s)]/2m$. Here the function u_0 is the same as above.

4. Some applications. Consider the problem

$$-\Delta u(x) + V(x)u(x) = \lambda u(x) \quad \text{in } \Omega, \qquad \partial_\nu u(x) - \mu u(x) = g(x) \quad \text{on } \Gamma \tag{11}$$

with a spectral parameter λ. For simplicity, we assume that the potential $V(x)$ is smooth and real and μ is a real number. Let the problem be uniquely solvable for the given λ. The operator $g \mapsto u|_\Gamma$ is referred to as the *R-matrix* by physicists. See [13] and references therein. To construct it, they use the following algorithm. One expands the solution u in $L_2(\Omega)$ into a Fourier series in the orthonormal system $\{u_l\}_1^\infty$ of eigenfunctions of problem (11) with $g = 0$. The computation of the coefficients (with the use of integration by parts) shows that the series has the form

$$u(x) = \sum_1^\infty \frac{(g, u_l|_\Gamma)_\Gamma}{\lambda_l - \lambda} u_l(x), \tag{12}$$

where λ_l are the corresponding eigenvalues. In this series, one passes to the boundary, and this gives the following formula for the R-matrix:

$$u(x)\big|_\Gamma = \sum_1^\infty \frac{(g, u_l|_\Gamma)_\Gamma}{\lambda_l - \lambda} u_l(x)\big|_\Gamma. \tag{13}$$

For this passage to be nonformal, one should assume that $u \in H^t(\Omega)$ with $1/2 < t < 3/2$ and $g \in H^{t-1/2}(\Gamma)$, and then the series (13) converges in $H^{t-1/2}(\Gamma)$. However, physicists prefer to use uniform convergence. To ensure the uniform convergence, for $n = 3$ one should assume that $3/2 < t < 5/2$, but then the difference $u - u_0$, rather than u, is to be expanded in the functions u_l, where $u_0 \in H^t(\Omega)$ is some function satisfying the same nonhomogeneous boundary condition. For example, one can take a function with the Cauchy data $u_0|_\Gamma = 0$, $\partial_\nu u_0|_\Gamma = g$. Then the regularized series (13) has the form

$$u(x)\big|_\Gamma = \sum_1^\infty \left[\frac{(g, u_l|_\Gamma)_\Gamma}{\lambda_l - \lambda} - (u_0, u_l)_\Omega\right] u_l(x)\big|_\Gamma,$$

and the regularization is essentially performed by the subtraction of a series with zero sum from (13) (see [6]).

Regularization is even more important in the case of the boundary value problem considered in [13] and [7] for the 4×4 homogeneous Dirac system (which is of first order). If u and v are the upper and lower halves, respectively, of the column vector $w(x)$ of unknown functions in the Dirac system, then the boundary condition has the form

$$b(x)v(x)\big|_\Gamma - \mu u(x)\big|_\Gamma = g,$$

where $b(x)$ is some nondegenerate 2×2 matrix function. The operator $g \mapsto u|_\Gamma$ is called the *R-matrix* (again see [13] and references therein). To construct it, one uses an orthonormal basis formed in the space $L_2(\Omega)$ (of two-dimensional vector functions) by the parts u_l of the eigenfunctions w_l. The passage to the boundary is performed again but cannot be justified without subtracting an appropriate function w_0 from the solution. It is convenient to take w_0 with zero upper component and with the boundary condition $b(x)v_0|_\Gamma = g$. See details in [7].

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