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EXTENDING QUASI-INVARIANT MEASURES BY USING SUBGROUPS OF A GIVEN GROUP

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Abstract. A method of extending σ-finite quasi-invariant measures given on an uncountable group, by using a certain family of its subgroups, is investigated.

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Throughout this paper the symbol λ $(= \lambda_1)$ denotes the classical Lebesgue measure on the real line $\mathbf{R}$. It is well known that λ is invariant under the group of all isometric transformations of $\mathbf{R}$ and, moreover, there are invariant (under the same group) measures on $\mathbf{R}$ strictly extending λ (see, e.g., [1]–[4]).

In this context, the following problem arises naturally.

Problem 1. Give a characterization of all those sets $X \subset \mathbf{R}$ for which there exists at least one invariant measure μ on $\mathbf{R}$ extending λ and satisfying the relation $X \in \mathrm{dom}(\mu)$.

An analogous question can be posed for subsets of $\mathbf{R}$ measurable with respect to various quasi-invariant extensions of λ. More precisely, the corresponding problem is formulated as follows.

Problem 2. Give a characterization of all those sets $X \subset \mathbf{R}$ for which there exists at least one quasi-invariant measure μ on $\mathbf{R}$ extending λ and satisfying the relation $X \in \mathrm{dom}(\mu)$.

Obviously, we can formulate direct analogues of Problems 1 and 2 for the n-dimensional Euclidean space $\mathbf{R}^n$ and for the standard Lebesgue measure λ_n on $\mathbf{R}^n$. It can be shown that these two problems essentially differ from each other. Note that none of them has been solved for the time being as no reasonable approach has been found to lead to their solution.

In connection with Problems 1 and 2, let us observe that the class of all subgroups of $\mathbf{R}$ may be regarded as a class of subsets of $\mathbf{R}$ which distinguishes these problems. Namely, we can assert that:

I. For any group $G \subset \mathbf{R}$, there exists a quasi-invariant extension $\mu = \mu_G$ of λ such that $G \in \mathrm{dom}(\mu)$ (cf. Theorem 1).

At the same time we have:

II. There exists a subgroup H of $\mathbf{R}$ such that, for each invariant extension ν of λ, the relation $H \notin \mathrm{dom}(\nu)$ is valid.

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Notice that H can easily be constructed by using a Hamel basis of $\mathbf{R}$. Indeed, take an arbitrary Hamel basis $\{e_i : i \in I\}$ in $\mathbf{R}$, choose an index $i_0 \in I$ and denote by H the vector subspace of $\mathbf{R}$ (over the field $\mathbf{Q}$ of rationals) generated by $\{e_i : i \in I \setminus \{i_0\}\}$. Then H is a Vitali type subset of $\mathbf{R}$ which is nonmeasurable with respect to any $\mathbf{R}$-invariant extension of λ.

Below we will present a more general result. In order to formulate it, we first need some preliminary facts and statements.

In the sequel, the symbol ω $(= \omega_0)$ denotes the first infinite ordinal (cardinal) and ω_1 is the first uncountable ordinal (cardinal).

Let G be a group and let μ be a left G-quasi-invariant measure defined on some σ-algebra of subsets of G. We recall (see, e.g., [5]) that μ is metrically transitive if for each set $X \in \mathrm{dom}(\mu)$ with $\mu(X) > 0$ there exists a countable family $\{g_n : n < \omega\}$ of elements from G such that

$$\mu(G \setminus \cup\{g_n X : n < \omega\}) = 0.$$

We also recall that a topological group G is standard (see, e.g., [6]) if G coincides with some Borel subgroup of a Polish group.

For any group G and its subgroup H, the symbol G/H denotes, as usual, the set $\{gH : g \in G\}$ of all left translates of H in G.

Lemma 1. *Let G be a group equipped with a σ-finite left G-quasi-invariant measure μ and let H be a subgroup of G such that* $\mathrm{card}(G/H) > \omega$. *If μ is metrically transitive, then there exists a measure μ' on G satisfying the following relations:*

1) *μ' is left G-quasi-invariant;*
2) *μ' is metrically transitive;*
3) *μ' is an extension of μ;*
4) *H belongs to the domain of μ' and $\mu'(H) = 0$.*

Proof. We may assume, without loss of generality, that the original measure μ is nonzero and complete. If $\mu(H) = 0$, then there is nothing to prove. Let us consider the case where $\mu^*(H) > 0$ and let us verify that, in this case, for any countable family $\{g_n : n < \omega\}$ of elements from G the equality

$$\mu_*(\cup\{g_n H : n < \omega\}) = 0$$

holds true. Indeed, suppose to the contrary that

$$\mu_*(\cup\{f_n H : n < \omega\}) > 0$$

for some countable family $\{f_n : n < \omega\} \subset G$. Then, applying the metrical transitivity of μ, we can find a countable family $\{f'_n : n < \omega\} \subset G$ such that

$$\mu(G \setminus \cup\{f'_n H : n < \omega\}) = 0.$$

Since $\mathrm{card}(G/H) > \omega$, there exists an element $f' \in G$ for which we have

$$f'H \cap (\cup\{f'_n H : n < \omega\}) = \varnothing.$$

Therefore $\mu(f'H) = 0$ whence it also follows, in view of the quasi-invariance of μ, that $\mu(H) = 0$ which contradicts our assumption $\mu^*(H) > 0$. The contradiction obtained shows that the inner μ-measure of any set of the form $\cup\{g_n H : n < \omega\}$, where $\{g_n : n < \omega\} \subset G$, is equal to zero.

Denote now by $\mathcal{J}$ the σ-ideal in G generated by all sets of the above-mentioned form. Let $\mathcal{S}$ be the σ-algebra in G generated by $\mathrm{dom}(\mu) \cup \mathcal{J}$. Obviously, any set $X \in \mathcal{S}$ can be written as $X = (Y \cup Z_1) \setminus Z_2$, where $Y \in \mathrm{dom}(\mu)$ and $Z_1 \in \mathcal{J}$, $Z_2 \in \mathcal{J}$. We put

$$\mu'(X) = \mu(Y) \quad (X \in \mathcal{S}).$$

In this way we get the functional μ' on $\mathcal{S}$ (which is well defined). It can easily be shown, by applying the standard argument (see, e.g., [3] or [4]) that μ' is a left G-quasi-invariant metrically transitive measure on $\mathcal{S}$ extending μ. Moreover, since the values of μ' on all sets from $\mathcal{J}$ are equal to zero, we have $\mu'(H) = 0$. $\square$

Lemma 2. *Let G be a group equipped with a σ-finite left G-quasi-invariant measure μ and let $(H_1, H_2, \ldots, H_k)$ be a finite family of subgroups of G such that* $\mathrm{card}(G/H_i) > \omega$ *for each natural number $i \in [1, k]$. If μ is metrically transitive, then there exists a measure μ' on G for which the following relations are valid:*

1) *μ' is left G-quasi-invariant;*
2) *μ' is metrically transitive;*
3) *μ' is an extension of μ;*
4) *all subgroups H_i $(i = 1, 2, \ldots, k)$ belong to* $\mathrm{dom}(\mu')$ *and $\mu'(H_i) = 0$ $(i = 1, 2, \ldots, k)$.*

Proof. It suffices to apply Lemma 1 and induction on k. $\square$

Lemma 3. *Let Γ be a standard group equipped with a σ-finite left Γ-quasi-invariant Borel measure ν and let G be a subgroup of Γ such that* $\mathrm{card}(\Gamma/G) \leq \omega$*. Then there exists a measure ν' on Γ satisfying the following relations:*

1) *ν' is left Γ-quasi-invariant;*
2) *ν' is metrically transitive;*
3) *ν' is an extension of ν;*
4) *G belongs to the domain of ν'.*

Proof. It suffices to consider the case where ν is not identically equal to zero.

The first part of our argument is based on the fundamental Mackey theorem [7]. Let us recall that according to this theorem there exist a locally compact Polish topological group Γ' and a continuous group isomorphism

$$\phi : \Gamma' \to \Gamma,$$

such that the given measure ν turns out to be equivalent to the ϕ-image $\phi(\theta)$ of the left Haar measure θ on Γ'. In other words, the two measures $\phi(\theta)$ and ν have the same σ-ideal of sets of measure zero.

Taking this classical result into account, we may assume without loss of generality that Γ is a locally compact Polish topological group and the initial measure ν coincides with the left Haar measure on Γ. We preserve the same notation ν for the completion of the left Haar measure on Γ. Further, we may suppose that $\nu^*(G) > 0$ and G is everywhere dense in Γ (otherwise we replace Γ by $\mathrm{cl}(G) =$ the closure of G, and deal with the restriction of ν to the Borel σ-algebra of cl(G)). Now, if a subgroup G is not of outer ν-measure zero and simultaneously is everywhere dense in Γ, then G is a ν-thick set in Γ, i.e. we have the equality $\nu_*(\Gamma \setminus G) = 0$. Besides, we remember that $\mathrm{card}(\Gamma/G) \leq \omega$. Let us consider in detail only the case where $\mathrm{card}(\Gamma/G) = \omega$ (the case where $\mathrm{card}(\Gamma/G) < \omega$ can be considered analogously and is even easier). We denote by $\{Z_k : k < \omega\}$ the family of all pairwise distinct left translates of G in Γ. Let $\mathcal{S}$ stand for the family of all those subsets X of Γ which can be represented in the form:

$$X = \cup\{Y_k \cap Z_k : k < \omega\},$$

where Y_k ($k < \omega$) are some ν-measurable sets in Γ. It is not hard to verify that $\mathcal{S}$ is a left Γ-invariant σ-algebra of subsets of Γ and

$$\{Z_k : k < \omega\} \cup \mathrm{dom}(\nu) \subset \mathcal{S}.$$

Let us define a functional ν' on $\mathcal{S}$ by the formula

$$\nu'(X) = \sum_{k<\omega} (1/2)^{k+1} \nu(Y_k) \quad (X \in \mathcal{S}).$$

Then, in view of the ν-thickness of all Z_k ($k < \omega$), this functional is well defined and is a measure on Γ. A straightforward verification shows also that ν' satisfies relations 1), 3) and 4) of the lemma. It remains to observe that the metrical transitivity of ν implies the metrical transitivity of ν'. □

Now, we are able to establish the following statement.

Theorem 1. *Let Γ be a standard group equipped with a σ-finite left Γ-quasi-invariant Borel measure ν and let $(G_1, G_2, \dots, G_n)$ be a finite family of subgroups of Γ. Then there exists a left Γ-quasi-invariant measure ν' on Γ such that*

$$\{G_1, G_2, \dots, G_n\} \subset \mathrm{dom}(\nu').$$

Proof. Without loss of generality, we may suppose that

$$\mathrm{card}(\Gamma/G_1) \leq \omega,\ \mathrm{card}(\Gamma/G_2) \leq \omega, \dots,\ \mathrm{card}(\Gamma/G_k) \leq \omega,$$

$$\mathrm{card}(\Gamma/G_{k+1}) > \omega,\ \mathrm{card}(\Gamma/G_{k+2}) > \omega, \dots,\ \mathrm{card}(\Gamma/G_n) > \omega$$

for some natural number $k \in [0, n]$. Let us put

$$G = G_1 \cap G_2 \cap \dots \cap G_k.$$

Then G is a subgroup of Γ such that $\mathrm{card}(\Gamma/G) \leq \omega$. Applying Lemma 3 to Γ and G, we see that there exists a left Γ-quasi-invariant metrically transitive measure ν_G on Γ extending ν and satisfying the relation $G \in \mathrm{dom}(\nu_G)$. Since

$$\mathrm{card}(G_1/G) \leq \omega,\ \mathrm{card}(G_2/G) \leq \omega, \dots,\ \mathrm{card}(G_k/G) \leq \omega,$$

we also have

$$G_1 \in \mathrm{dom}(\nu_G),\ G_2 \in \mathrm{dom}(\nu_G), \dots,\ G_k \in \mathrm{dom}(\nu_G).$$

Now, we can apply Lemma 2 to the measure ν_G and to the finite family $(G_{k+1}, G_{k+2}, \dots, G_n)$ of subgroups of Γ. In this way we obtain the required extension ν' of ν. □

So far we have been concerned with a finite family of subgroups of the original group Γ and been able to prove that all those subgroups can be made measurable with respect to a suitable quasi-invariant extension of the initial quasi-invariant measure on Γ.

In dealing with countable families of subgroups of Γ, we come to a significantly different situation. For example, it is not difficult to show that there exists a countable family of subgroups of $\mathbf{R}$ such that the Lebesgue measure λ cannot be extended to an $\mathbf{R}$-quasi-invariant measure whose domain includes all these subgroups.

The next result generalizes the above-mentioned fact.

Theorem 2. *Let Γ be an uncountable divisible commutative group. Then there exists a countable family $\{G_i\ :\ i \in I\}$ of subgroups of Γ such that:*

1) *for each $i \in I$, we have* $\mathrm{card}(\Gamma/G_i) > \omega$*;*

2) $\cup\{G_i\ :\ i \in I\} = \Gamma$.

In particular, for any probability Γ-quasi-invariant measure μ on Γ, at least one group G_i is nonmeasurable with respect to μ.

Proof. Here we essentially employ the classical result from the theory of groups, stating that every divisible commutative group can be represented as a direct sum of a family of groups each of which is isomorphic either to $\mathbf{Q}$ (the group of all rationals) or to the quasi-cyclic group of type p^∞, where p is a prime number (see, e.g., [8]). Thus our group Γ is representable as a direct sum

$$\Gamma = \sum_{j \in J} \Gamma_j,$$

where J is some uncountable set of indices and every Γ_j is a group of the above-mentioned type. Now it can be easily verified that, for each $j \in J$, we have

$$\Gamma_j = \cup\{H_{j,n} : n < \omega\}$$

where $\{H_{j,n} : n < \omega\}$ is an increasing (by inclusion) countable family of proper subgroups of Γ_j. For any $n < \omega$ let us put

$$G_n = \sum_{j \in J} H_{j,n}.$$

Then it is not difficult to verify that the family of groups

$$\{G_i\ :\ i \in I\} = \{G_n\ :\ n < \omega\}$$

is the required one. $\square$

Remark 1. Obviously, in Theorem 2 any uncountable vector space over $\mathbf{Q}$ can be taken as Γ (in particular, we may put $\Gamma = \mathbf{R}^n$ where $n \geq 1$). Also, we may put $\Gamma = \mathbf{S}_1^\kappa$, where $\mathbf{S}_1$ denotes the one-dimensional unit torus and κ is an arbitrary nonzero cardinal.

Remark 2. Let Γ be a commutative group and let G be a subgroup of Γ such that $\mathrm{card}(\Gamma/G) > \omega$. It can be proved that G is a Γ-absolutely negligible subset of Γ (for detailed information about this notion, cf. [3] or [4]). We thus claim that each subgroup G_i of the preceding theorem turns out to be a Γ-absolutely negligible subset of Γ. Therefore, for a given $i \in I$, every probability Γ-quasi-invariant measure μ on Γ can be extended to a probability Γ-quasi-invariant measure μ' on Γ satisfying the relation $G_i \in \mathrm{dom}(\mu')$. However, there is no nonzero σ-finite Γ-quasi-invariant measure on Γ whose domain contains all sets G_i $(i \in I)$.

Remark 3. It would be interesting to extend Theorem 2 to a more general class of uncountable groups Γ (not necessarily divisible or commutative). In this connection, let us point out that the assertion of this theorem fails to be true for some uncountable groups. In particular, if Γ is uncountable and contains no proper uncountable subgroup, then the above-mentioned theorem is obviously false for Γ. On the other hand, by starting with the result formulated in this theorem, it is not difficult to construct an uncountable noncommutative nondivisible group Γ with a countable family $(G_i)_{i \in I}$ of its subgroups such that each G_i $(i \in I)$ is a Γ-absolutely negligible set and, for any left Γ-quasi-invariant probability measure μ on Γ, at least one G_i is nonmeasurable with respect to μ.

Example 1. Consider an arbitrary nonzero σ-finite $\mathbf{R}$-quasi-invariant measure ν on $\mathbf{R}$. In view of Theorem 2, there exists a subgroup of $\mathbf{R}$ nonmeasurable with respect to ν. Moreover, by applying an argument similar to the proof of Theorem 2, it can be shown that there exists a vector subspace of $\mathbf{R}$ (over rationals) which is nonmeasurable with respect to ν (cf. Remark 4 below).

In our further consideration, we will present an analogue of Theorem 2 for an arbitrary uncountable commutative group Γ. To obtain this analogue, we need some techniques developed in [9] and [10].

Lemma 4. *Let G be any commutative group of cardinality ω_1. Then there exists a countable family $\{G_i : i \in I\}$ of subgroups of G such that, for every nonzero σ-finite diffused measure μ on G, at least one group G_i is nonmeasurable with respect to μ.*

The proof can be found in [10]. Since every nonempty set can be endowed with the structure of a commutative group, Lemma 4 generalizes the well-known purely set-theoretic result of Ulam [11], which states that the cardinal ω_1 is not real-valued measurable.

Example 2. For noncommutative groups of cardinality ω_1, Lemma 4 fails to be true. Indeed, take any group G of the same cardinality, whose all proper subgroups are at most countable (recall that the existence of such a group was first established by Shelah [12]). It is easy to define a G-invariant probability measure μ on G such that all countable subsets of G are of μ-measure zero. Evidently, G does not contain a subgroup nonmeasurable with respect to μ.

Lemma 5. *Let E_1 and E_2 be any two sets and let ϕ be an arbitrary mapping from E_1 into E_2. Suppose also that $\{Y_i : i \in I\}$ is a family of subsets of E_2 such that, for every diffused probability measure μ on E_2, at least one set Y_i is nonmeasurable with respect to μ. Then the family $\{X_i : i \in I\} = \{\phi^{-1}(Y_i) :$*

$i \in I\}$ of subsets of E_1 has the following property: for every nonzero σ-finite measure ν on E_1 such that

$$\nu(\phi^{-1}(y)) = 0 \quad (y \in E_2),$$

at least one set X_i is nonmeasurable with respect to ν.

Proof. Let ν be a nonzero σ-finite measure on E_1 satisfying the relation $\nu(\phi^{-1}(y)) = 0$ for all elements $y \in E_2$. Replacing ν by an equivalent measure, we may assume, without loss of generality, that $\nu(E_1) = 1$. Suppose to the contrary that $\{X_i : i \in I\} \subset \mathrm{dom}(\nu)$, and denote

$$\mathcal{S} = \{Y \subset E_2 : \phi^{-1}(Y) \in \mathrm{dom}(\nu)\}.$$

Then $\mathcal{S}$ is a σ-algebra of subsets of E_2, containing all singletons in E_2 and satisfying the inclusion $\{Y_i : i \in I\} \subset \mathcal{S}$. Clearly, we can define a diffused probability measure μ on $\mathcal{S}$ by putting

$$\mu(Y) = \nu(\phi^{-1}(Y)) \quad (Y \in \mathcal{S}).$$

But, according to the definition of $\{Y_i : i \in I\}$, at least one set Y_i must be nonmeasurable with respect to μ.

The obtained contradiction finishes the proof of Lemma 5. □

Now we can establish the following statement.

Theorem 3. *Let $(\Gamma, +, 0)$ be an arbitrary uncountable commutative group. There exists a countable family $\{H_i : i \in I\}$ of subgroups of Γ such that, for every nonzero σ-finite Γ-quasi-invariant measure ν on Γ, at least one subgroup H_i is nonmeasurable with respect to ν.*

Proof. According to the well-known result from the general theory of infinite commutative groups (see, e.g., [8]), we can represent Γ in the form

$$\Gamma = \cup\{G_n : n < \omega\},$$

where $\{G_n : n < \omega\}$ is an increasing (by inclusion) countable family of subgroups of Γ and each G_n is representable as a direct sum of cyclic groups. Since $\mathrm{card}(\Gamma) \geq \omega_1$, we may also assume that all subgroups G_n are uncountable. Therefore we can write

$$G_n = G'_n + G''_n \quad (n < \omega),$$

where the groups G'_n and G''_n satisfy the relations

$$\mathrm{card}(G'_n) = \omega_1, \quad G'_n \cap G''_n = \{0\}.$$

Further, we know (see Lemma 4) that each group G'_n admits a countable family $\{G'_{n,k} : k < \omega\}$ of its subgroups such that, for any diffused probability measure μ on G'_n, at least one of these subgroups is nonmeasurable with respect to μ. We define

$$G_{n,k} = G'_{n,k} + G''_n \quad (n < \omega,\ k < \omega)$$

and put

$$\{H_i : i \in I\} = \{G_n : n < \omega\} \cup \{G''_n : n < \omega\} \cup \{G_{n,k} : n < \omega,\ k < \omega\}.$$

Let us verify that the family $\{H_i : i \in I\}$ is the required one, i.e. for any nonzero σ-finite Γ-quasi-invariant measure ν on Γ, at least one group from this family is nonmeasurable with respect to ν. Suppose to the contrary that

$$\{H_i : i \in I\} \subset \mathrm{dom}(\nu).$$

In view of the equality $\Gamma = \cup\{G_n : n < \omega\}$, we must have $\nu(G_m) > 0$ for some $m < \omega$. We may assume, without loss of generality, that $\nu(G_m) = 1$. Further, since there are uncountably many translates of G''_m in G_m, the equality $\nu(G''_m) = 0$ must be valid in virtue of the quasi-invariance of ν. Moreover, we have $\nu(g + G''_m) = 0$ for all elements $g \in G_m$. Let now

$$\phi_m \;:\; G_m \to G'_m$$

be the canonical projection of G_m onto G'_m (we recall that G_m is the direct sum of its subgroups G'_m and G''_m). Then

$$\nu(\phi_m^{-1}(g)) = \nu(g + G''_m) = 0 \quad (g \in G'_m),$$
$$\phi_m^{-1}(G'_{m,k}) = G_{m,k} \quad (k < \omega).$$

We thus see that Lemma 5 can be applied in this situation and, according to it, at least one subgroup $G_{m,k}$ must be nonmeasurable with respect to ν. This contradicts the inclusion $\{H_i : i \in I\} \subset \mathrm{dom}(\nu)$. The obtained contradiction ends the proof. $\square$

Remark 4. It is easy to see that direct analogues of Lemma 4 and Theorem 3 are valid for vector spaces (e.g., over the field $\mathbf{Q}$ of all rationals) instead of commutative groups. In particular, if V is a vector space over $\mathbf{Q}$ with $\mathrm{card}(V) = \omega_1$, then there exists a countable family $(V_i)_{i \in I}$ of vector subspaces of V such that, for any nonzero σ-finite diffused measure μ on V, at least one subspace V_i is nonmeasurable with respect to μ.

Similarly, if E is an uncountable vector space (over $\mathbf{Q}$), then there exists a countable family $(E_i)_{i \in I}$ of vector subspaces of E such that, for any nonzero σ-finite E-quasi-invariant measure ν on E, at least one subspace E_i is nonmeasurable with respect to ν.

In connection with the results presented in this paper, the following two problems seem to be of interest.

Problem 3. Let Γ be an uncountable commutative group. Does there exist a countable family $\{G_i : i \in I\}$ of subgroups of Γ such that $\mathrm{card}(\Gamma/G_i) > \omega$ for all $i \in I$ and, for any nonzero σ-finite Γ-quasi-invariant measure μ on Γ, at least one subgroup G_i is nonmeasurable with respect to μ?

Problem 4. Let Γ be a commutative group whose cardinality is not real-valued measurable, and let μ be a nonzero σ-finite diffused measure on Γ. Does there exist a subgroup of Γ nonmeasurable with respect to μ?

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