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DYNAMIC PROGRAMMING AND MEAN-VARIANCE HEDGING IN DISCRETE TIME

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Abstract. We consider the mean-variance hedging problem in the discrete time setting. Using the dynamic programming approach we obtain recurrent equations for an optimal strategy. Additionally, some technical restrictions of the previous works are removed.

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1. Introduction

Let $(\Omega, \mathcal{F}, P)$ be a probability space endowed with a filtration $F = (F_n)$, $n = 0, 1, \dots, N (F_0$ is trivial, $F_N = \mathcal{F})$ and let $X = (X_n)$ be a sequence of random variables, adapted to this filtration, such that for all n, $E((\Delta X_n)^2 | F_{n-1}) < \infty$. Denote by Π the set of sequences of predictable random variables $\pi = (\pi_n)$, $n = 1, 2, \dots, N$, such that $\sum_{i=1}^N \pi_i \Delta X_i$ is square integrable.

For a given real number c and a square integrable random variable $H \in F_N$ let us consider the optimization problem:

$$\text{minimize } E\left[\left(H - c - \sum_{i=1}^N \pi_i \Delta X_i\right)^2\right] \text{ over all } \pi \in \Pi. \tag{1.1}$$

The corresponding solution π^* will be called an optimal strategy.

(1.1) is a discrete time analogue of the mean-variance hedging problem originally introduced by Föllmer and Sondermann [2]. Here X_n can be interpreted as the price of a risky asset at time n, H as a contigent claim due at time N and c as the initial capital of an investor. The set Π is then the set of all admissible trading strategies.

This problem was solved by Schweizer [7] under some additional restrictions, namely: for an arbitrary n, $n = 1, \dots, N$, X_n is square integrable and satisfies the so-called nondegeneracy condition. Moreover, π is admissible if and only if for each n, $n = 0, 1, \dots, N$, $\pi_n \Delta X_n$ is square integrable.

Melnikov and Nechaev [6] improved Schweizer's results; they removed the non-degeneracy condition and they do not require the square integrability of $\pi_n \Delta X_n$ for all n, but they still require $X_n \in L^2(P)$, $n = 0, \dots, N$.

Unlike the approaches of those papers, in the present work the dynamic programming method is used, which enables us to derive relatively simple, one-step backward equations for random variables a_n, b_n, defining π_n^*, while Schweizer's

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or Melnikov and Nechaev's recurrent equations for quantities β_n, ρ_n defining π_n^* involve all previous β_i from $n+1$ to N. Moreover, the requirement of the square integrability of X_n, $n = 0, 1, \dots$, is removed. The dynamic programming method has already been applied to the mean-variance hedging problem in [3], [5], but they do not cover our case as [3] deals with the diffusion model, while in [5] the asset process X, which is a semimartingale, and the filtration F are supposed to be continuous.

The paper is organized as follows: in Section 2 we introduce the value function $V(n,x)$ corresponding to (1.1). Proposition 2.1 gives us the main working tool, the Bellman equation. In Section 3 we formulate and prove the main theorem. Section 4 is dedicated to the discussion of the obtained results. Finally, Appendix contains some results, used throughout the paper.

2. Backward Equation for $V(n,x)$

First let us introduce some conventions and notation:

a) If A is an empty set, then let

$$\sum_{a\in A} Y_a = 0, \quad \prod_{a\in A} Y_a = 1.$$

b) The uncertainty $\frac{0}{0} = 0$.

c) Relations between random variables are understood in the a.s. sense.

d) $\Pi(n,N)$ denotes the set of $\pi \in \Pi$ such, that $\pi_i = 0$ for $i \le n$.

Let us define the value function $V(n,x)$ corresponding to (1.1) as

$$V(n,x) = \operatorname*{essinf}_{\pi\in\Pi(n,N)} E\bigg(\bigg(H - x - \sum_{i=n+1}^{N} \pi_i \Delta X_i\bigg)^2 \;\bigg|\; F_n\bigg).$$

Note that $V(N,x) = (H-x)^2$.

Proposition 2.1. *The function $V(n,x)$ satisfies the recurrent equation*

$$V(n-1,x) = \operatorname*{essinf}_{\pi\in\Pi(n-1,N)} E\big(V(n, x + \pi_n \Delta X_n) | F_{n-1}\big) \tag{2.1}$$

with the boundary condition $V(N,x) = (H-x)^2$.

Proof. Due to Proposition A1 of Appendix for every fixed $\widehat{\pi} \in \Pi(n-1,N)$ we have

$$V(n-1,x) \le E\big(V(n, x + \widehat{\pi}_n \Delta X_n) | F_{n-1}\big).$$

Therefore, taking essinf, we obtain

$$V(n-1,x) \le \operatorname*{essinf}_{\pi\in\Pi(n-1,N)} E\big(V(n, x + \pi_n \Delta X_n) | F_{n-1}\big). \tag{2.2}$$

Conversely ($\widehat{\pi} = (\widehat{\pi}_n)$ is fixed),

$$\begin{aligned}
&E\big(V(n, x + \widehat{\pi}_n \Delta X_n)|F_{n-1}\big)\\
&\quad = E\Bigg(\operatorname*{essinf}_{\pi\in\Pi(n-1,N)} E\Bigg(\Bigg(H - x - \widehat{\pi}_n\Delta X_n - \sum_{i=n+1}^{N}\pi_i\Delta X_i\Bigg)^2 \,\Bigg|\, F_n\Bigg) \,\Bigg|\, F_{n-1}\Bigg)\\
&\quad \le E\Bigg(E\Bigg(\Bigg(H - x - \sum_{i=n}^{N}\widehat{\pi}_i\Delta X_i\Bigg)^2 \,\Bigg|\, F_n\Bigg) \,\Bigg|\, F_{n-1}\Bigg)\\
&\quad = E\Bigg(\Bigg(H - x - \sum_{i=n}^{N}\widehat{\pi}_i\Delta X_i\Bigg)^2 \,\Bigg|\, F_{n-1}\Bigg).
\end{aligned}$$

Taking essinf of both sides, we get

$$\begin{aligned}
&\operatorname*{essinf}_{\pi\in\Pi(n-1,N)} E\big(V(n, x + \pi_n\Delta X_n)|F_{n-1}\big)\\
&\qquad \le \operatorname*{essinf}_{\pi\in\Pi(n-1,N)} E\Bigg(\Bigg(H - x - \sum_{i=n}^{N}\pi_i\Delta X_i\Bigg)^2 \,\Bigg|\, F_{n-1}\Bigg) = V(n-1, x)
\end{aligned}$$

which together with (2.2) proves (2.1). □

3. Main Result

After above preliminary result, we are ready to formulate the main theorem.

Theorem 1. *Assume that $E((\Delta X_n)^2|F_{n-1}) < \infty$ for all $1 \le n \le N$. Then the value function $V(n,x)$ is a square trinomial in x,*

$$V(n,x) = a_n x^2 + 2b_n x + c_n,$$

where the F_n-measurable random variables a_n, b_n and c_n satisfy the backward recurrent equations

$$a_n = E(a_{n+1}|F_n) - \frac{(E(a_{n+1}\Delta X_{n+1}|F_n))^2}{E(a_{n+1}(\Delta X_{n+1})^2|F_n)}\,, \tag{3.1}$$

$$b_n = E(b_{n+1}|F_n) - \frac{E(a_{n+1}\Delta X_{n+1}|F_n)E(b_{n+1}\Delta X_{n+1}|F_n)}{E(a_{n+1}(\Delta X_{n+1})^2|F_n)}\,, \tag{3.2}$$

$$c_n = E(c_{n+1}|F_n) - \frac{(E(b_{n+1}\Delta X_{n+1}|F_n))^2}{E(a_{n+1}(\Delta X_{n+1})^2|F_n)}\,, \tag{3.3}$$

with the boundary conditions

$$a_N = 1, \quad b_N = -H, \quad c_N = H^2.$$

Moreover, an optimal strategy $\pi^* = (\pi^*_n)$ is given by

$$\pi^*_n = -\frac{E(b_n\Delta X_n|F_{n-1})}{E(a_n(\Delta X_n)^2|F_{n-1})} - \Bigg(c + \sum_{i=1}^{n-1}\pi^*_i\Delta X_i\Bigg)\frac{E(a_n\Delta X_n|F_{n-1})}{E(a_n(\Delta X_n)^2|F_{n-1})}\,. \tag{3.4}$$

Proof. For $n = N$ we have $V(N,x) = (H-x)^2$; hence $V(N,x)$ is a square trinomial in x and $a_N = 1$, $b_N = -H$, $c_N = H^2$. Suppose that for some n, $V(n,x)$ is a square trinomial in x

$$V(n,x) = a_n x^2 + 2b_n x + c_n,$$

and $0 \le a_n \le 1$. Since $V(n,x) \ge 0$, this implies that $b_n^2 \le a_n c_n$, whence

$$(\omega : a_n = 0) \subset (\omega : b_n = 0).$$

Moreover, by the Hölder inequality

$$\begin{aligned}\big(E(a_n \Delta X_n | F_{n-1})\big)^2 &= \big(E(\sqrt{a_n}\sqrt{a_n}\Delta X_n | F_{n-1})\big)^2 \\ &\le E\big(a_n | F_{n-1}\big) E(a_n (\Delta X_n)^2 | F_{n-1}\big), \end{aligned} \tag{3.5}$$

i.e.,

$$\big(\omega : E(a_n(\Delta X_n)^2 | F_{n-1}) = 0\big) \subset \big(\omega : E(a_n \Delta X_n | F_{n-1}) = 0\big). \tag{3.6}$$

If we denote by A the set $\big(\omega : E(a_n(\Delta X_n)^2 | F_{n-1}) = 0\big)$, then

$$E\big[a_n(\Delta X_n)^2 I_A\big] = E\big[E(a_n(\Delta X_n)^2 | F_{n-1}) I_A\big] = 0,$$

whence $a_n(\Delta X_n)^2 I_A = 0$. This implies that $b_n(\Delta X_n)^2 I_A = 0$, and therefore $b_n \Delta X_n I_A = 0$. Taking conditional expectation we get $E(b_n \Delta X_n | F_{n-1}) I_A = 0$, whence we finally obtain

$$\big(\omega : E(a_n(\Delta X_n)^2 | F_{n-1}) = 0\big) \subset \big(\omega : E(b_n \Delta X_n | F_{n-1}) = 0\big). \tag{3.7}$$

Therefore essinf w.r.t. $\pi \in \Pi(n-1, N)$ in

$$\begin{aligned} &E\big(V(n, x + \pi_n \Delta X_n) | F_{n-1}\big) \\ &\quad = E\big(a_n(x + \pi_n \Delta X_n)^2 + 2b_n(x + \pi_n \Delta X_n) + c_n | F_{n-1}\big) \\ &\quad = E\big(a_n x^2 + 2b_n x + c_n | F_{n-1}\big) + \pi_n^2 E\big(a_n(\Delta X_n)^2 | F_{n-1}\big) \\ &\qquad + 2\pi_n\big(x E(a_n \Delta X_n | F_{n-1}\big) + E\big(b_n \Delta X_n | F_{n-1})\big)] \end{aligned} \tag{3.8}$$

is attained for

$$\pi_n = -\frac{E(b_n \Delta X_n | F_{n-1})}{E(a_n(\Delta X_n)^2 | F_{n-1})} - x\,\frac{E(a_n \Delta X_n | F_{n-1})}{E(a_n(\Delta X_n)^2 | F_{n-1})}. \tag{3.9}$$

Note that on the set A, π_n can be chosen arbitrarily, e.g., it can be set to be 0. The agreement $\frac{0}{0} = 0$ allows one to write this in compact form (3.9). After substituting π_n defined by the above formula in (3.8) and using (2.1), we obtain

$$\begin{aligned} V(n-1, x) = &\left[E(a_n | F_{n-1}) - \frac{(E(a_n \Delta X_n | F_{n-1}))^2}{E(a_n(\Delta X_n)^2 | F_{n-1})}\right] x^2 \\ &+ 2\left[E(b_n | F_{n-1}) - \frac{E(a_n \Delta X_n | F_{n-1}) E(b_n \Delta X_n | F_{n-1})}{E(a_n(\Delta X_n)^2 | F_{n-1})}\right] x \\ &+ E(c_n | F_{n-1}) - \frac{(E(b_n \Delta X_n | F_{n-1}))^2}{E(a_n(\Delta X_n)^2 | F_{n-1})}\,. \end{aligned}$$

Denoting

$$a_{n-1} = E(a_n|F_{n-1}) - \frac{(E(a_n\Delta X_n|F_{n-1}))^2}{E(a_n(\Delta X_n)^2|F_{n-1})}\,,$$
$$b_{n-1} = E(b_n|F_{n-1}) - \frac{E(a_n\Delta X_n|F_{n-1})E(b_n\Delta X_n|F_{n-1})}{E(a_n(\Delta X_n)^2|F_{n-1})}\,,$$
$$c_{n-1} = E(c_n|F_{n-1}) - \frac{(E(b_n\Delta X_n|F_{n-1}))^2}{E(a_n(\Delta X_n)^2|F_{n-1})}\,,$$

we obtain

$$V(n-1,x) = a_{n-1}x^2 + 2b_{n-1}x + c_{n-1}.$$

Using the definition of a_{n-1} and (3.5) it is easy to check that $0 \le a_{n-1} \le 1$. Therefore (3.6), (3.7) are satisfied for $k = n-1$ as well and, consequently, using the induction, for all $k = 0, \dots, N$. Hence the reasoning used to find essinf is true for all k.

Bellman's principle suggests us that a natural candidate for an optimal solution is (3.4), where a_n, b_n satisfy (3.1)–(3.2) with initial conditions $a_N = 1$, $b_N = -H$. Let us show that $X_N^{\pi^*} = \sum_{i=1}^N \pi_i^* \Delta X_i$ is square integrable (i.e., π^* is admissible). For this it is sufficient to check that $a_n(X_n^{\pi^*})^2 = a_n(\sum_{i=1}^n \pi_i^*\Delta X_i)^2$ is integrable for all n. Since $a_N = 1$, this will entail the square integrability of $X_N^{\pi^*}$. Obviously, $a_0(X_0^{\pi^*})^2$ is integrable (as $a_0 \le 1$ and $X_0^{\pi^*} = c$). If we prove the equality

$$E[a_n(X_n^{\pi^*})^2] = E[a_{n-1}(X_{n-1}^{\pi^*})^2] + E\left[\frac{(E(b_n\Delta X_n|F_{n-1}))^2}{E(a_n(\Delta X_n)^2|F_{n-1})}\right], \tag{3.10}$$

then by induction it is sufficient to show that for all n,

$$\frac{(E(b_n\Delta X_n|F_{n-1}))^2}{E(a_n(\Delta X_n)^2|F_{n-1})}$$

is integrable. But this directly follows if we note that (c_n, F_n) is a submartingale since $c_n = V(n,0)$ and for arbitrary x, $(V(n,x), F_n)$ is a submartingale, whence

$$c_n \le E(c_N|F_n) = E(H^2|F_n),$$

and since for all n we have $|b_n| \le \sqrt{a_n c_n}$, by the Hölder inequality

$$\frac{(E(b_n\Delta X_n|F_{n-1}))^2}{E(a_n(\Delta X_n)^2|F_{n-1})} \le \frac{(E(\sqrt{a_n c_n}\Delta X_n|F_{n-1}))^2}{E(a_n(\Delta X_n)^2|F_{n-1})} \le E(c_n|F_{n-1}).$$

Now let us verify (3.10). We have

$$\begin{aligned}
E\big[a_n(X_n^{\pi^*})^2\big] &= E[a_n(X_{n-1}^{\pi^*} + \pi_n^*\Delta X_n)^2\big] \\
&= E\big[a_n(X_{n-1}^{\pi^*})^2 + 2\pi_n^* a_n \Delta X_n X_{n-1}^{\pi^*} + a_n(\pi_n^*\Delta X_n)^2\big] \\
&= E\big[E(a_n|F_{n-1})(X_{n-1}^{\pi^*})^2 + 2\pi_n^* E(a_n\Delta X_n|F_{n-1})X_{n-1}^{\pi^*} \\
&\quad + (\pi_n^*)^2 E(a_n(\Delta X_n)^2|F_{n-1})\big] \\
&= E\Bigg[E(a_n|F_{n-1})(X_{n-1}^{\pi^*})^2 - 2\Bigg[\frac{E(b_n\Delta X_n|F_{n-1})E(a_n\Delta X_n|F_{n-1})}{E(a_n(\Delta X_n)^2|F_{n-1})}
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{(E(a_n\Delta X_n|F_{n-1}))^2}{E(a_n(\Delta X_n)^2|F_{n-1})} X^{\pi^*}_{n-1}\Big] X^{\pi^*}_{n-1} \\
&+ \frac{(E(b_n\Delta X_n|F_{n-1}))^2}{E(a_n(\Delta X_n)^2|F_{n-1})} + 2\frac{E(b_n\Delta X_n|F_{n-1})E(a_n\Delta X_n|F_{n-1})}{E(a_n(\Delta X_n)^2|F_{n-1})} X^{\pi^*}_{n-1} \\
&+ \frac{(E(a_n\Delta X_n|F_{n-1}))^2}{E(a_n(\Delta X_n)^2|F_{n-1})} (X^{\pi^*}_{n-1})^2\Big] \\
= {}& E\Big[(E(a_n|F_{n-1}) - \frac{(E(a_n\Delta X_n|F_{n-1}))^2}{E(a_n(\Delta X_n)^2|F_{n-1})})(X^{\pi^*}_{n-1})^2\Big] \\
&+ E\Big[\frac{(E(b_n\Delta X_n|F_{n-1}))^2}{E(a_n(\Delta X_n)^2|F_{n-1})}\Big],
\end{aligned}
$$

which is the desired result. Hence $X_N^{\pi^*}$ is square integrable.

The optimality of π^* can be easily derived using the optimality criterion (Proposition A2), equations (3.1)–(3.2) and the fact that $V(n,x)$ is a square trinomial (one should substitute the expressions for $V(n, c+\sum_{i=1}^n \pi_i^*\Delta X_i)$ and $V(n-1, c+\sum_{i=1}^{n-1}\pi_i^*\Delta X_i)$ in the hypothetical equality $E(V(n, c+\sum_{i=1}^n \pi_i^*\Delta X_i)\big|F_{n-1}) = V(n-1, c+\sum_{i=1}^{n-1}\pi_i^*\Delta X_i))$. □

4. Remarks and Applications

Remark 4.1. Let us show how the obtained results are related to the solutions previously proposed in the literature. In order to make comparisons valid, suppose that X_n is square integrable for all n. Recall from [6] that an optimal solution is given by

$$
\pi_n^* = \rho_n - \Big(c + \sum_{i=1}^{n-1} \pi_i^*\Delta X_i\Big)\beta_n, \tag{4.1}
$$

where the predictable processes $\beta = (\beta_k)$ and $\rho = (\rho_k)$ are defined by the backward equations

$$
\beta_n = \frac{E\big(\Delta X_n \prod_{i=n+1}^N (1-\beta_i\Delta X_i)\big|F_{n-1}\big)}{E\big((\Delta X_n)^2 \prod_{i=n+1}^N (1-\beta_i\Delta X_i)\big|F_{n-1}\big)}\ , \tag{4.2}
$$

$$
\rho_n = \frac{E\big(H\Delta X_n \prod_{i=n+1}^N (1-\beta_i\Delta X_i)\big|F_{n-1}\big)}{E\big((\Delta X_n)^2 \prod_{i=n+1}^N (1-\beta_i\Delta X_i)\big|F_{n-1}\big)}\ . \tag{4.3}
$$

The link between $a = (a_k)$, $b = (b_k)$ and $\beta = (\beta_k)$, $\rho = (\rho_k)$ is quite simple, namely

$$
\beta_n = \frac{E(a_n\Delta X_n\big|F_{n-1})}{E(a_n(\Delta X_n)^2\big|F_{n-1})}\ ,
$$

$$
\rho_n = -\frac{E(b_n\Delta X_n\big|F_{n-1})}{E(a_n(\Delta X_n)^2\big|F_{n-1})}\ .
$$

Indeed, since $a_N = 1$, $b_N = -H$, these relations are true for $n = N$. The same fact for general n can be easily obtained using induction. Conversely, a_n and b_n

can be expressed in terms of $\beta_{n+1}, \dots, \beta_N$ as

$$a_n = E\Big(\prod_{i=n+1}^{N} (1 - \beta_i \Delta X_i) \,\Big|\, F_n \Big),$$

$$b_n = -\Big(H \prod_{i=n+1}^{N} (1 - \beta_i \Delta X_i) \,\Big|\, F_n \Big),$$

which again can be proved by induction. As it can be seen from (4.2)–(4.3), equations (3.1)–(3.2) provide some advantages for describing an optimal strategy as they are one-step, unlike (4.2)–(4.3), which involve $\beta_{n+1}, \dots, \beta_N$. On the other hand, (4.1) seems simpler than (3.4).

Remark 4.2. Note that the requirement of the square integrability of X_n, $n = 0, \dots, N$, of [6], [7] in our setting is substituted by a weaker one:

$$E\big((\Delta X_n)^2 \big| F_{n-1}\big) < \infty$$

for all n.

Remark 4.3. It is interesting to determine the value of the shortfall

$$R^* = E\bigg[\Big(H - c - \sum_{i=1}^{N} \pi_i^* \Delta X_i\Big)^2\bigg]$$

associated with the optimal strategy π^*. Noting that

$$E\bigg[\Big(H - c - \sum_{i=1}^{N} \pi_i^* \Delta X_i\Big)^2\bigg] = V(0, c),$$

we obtain

$$R^* = a_0 c^2 + 2 b_0 c + c_0.$$

Remark 4.4. When hedging a contigent claim, we usually are also interested in determining the price of this claim, i.e. we consider the problem

$$\text{minimize } E\Big[\Big(H - x - \sum_{i=1}^{N} \pi_i \Delta X_i\Big)^2\Big] \quad \text{over all} \quad (x, \pi) \in \mathbf{R} \times \Pi.$$

For every fixed c and arbitrary $\pi \in \Pi$ we have

$$E\Big[\Big(H - c - \sum_{i=1}^{N} \pi_i^*(c) \Delta X_i\Big)^2\Big] \le E\Big[\Big(H - c - \sum_{i=1}^{N} \pi_i \Delta X_i\Big)^2\Big],$$

where $\pi^*(c)$ is defined by (3.1)–(3.4). Let us minimize the left-hand side by c. Since it is equal to

$$V(0, c) = a_0 c^2 + 2 b_0 c + c_0,$$

the minimum is attained for $-b_0/a_0$ (of course, if $a_0 \neq 0$) and a minimal shortfall is equal to

$$R^* = -\frac{b_0^2 - a_0 c_0}{a_0}\ .$$

If $a_0 = 0$, then $b_0 = 0$ too (see the proof of Theorem 1), which means that an optimal strategy $\pi^*(c)$ does not depend on the initial capital c; therefore the price can be set to 0 and it can be again written as $c^* = -\frac{b_0}{a_0}$. The shortfall in this case equals c_0. Note that this result due to Remark 4.1 coincides with the solution proposed in [6]. $\square$

Appendix

Lemma A1. *The family of random variables* $\Lambda_n^\pi = E((H - x - \sum_{i=1}^n \widehat{\pi}_i \Delta X_i - \sum_{i=n+1}^N \pi_i \Delta X_i)^2 \big| F_n)$, $\pi \in \Pi(\widehat{\pi}, n, N)$, *possesses the ε-lattice property (for $\varepsilon = 0$). Here $\Pi(\widehat{\pi}, n, N)$ denotes the set of $\pi \in \Pi$ such that $\pi_i = \widehat{\pi}_i$ for $i = 0, 1, \dots, n$.*

Proof. Let $\pi^1, \pi^2 \in \Pi(\widehat{\pi}, n, N)$ and let us define $\pi^3 \in \Pi(\widehat{\pi}, n, N)$ as

$$\pi_i^3 = \pi_i^1 I_B + \pi_i^2 I_{B^c},$$

where $B = (\omega : \Lambda_n^{\pi^1} \le \Lambda_n^{\pi^2})$. Then since $B \in F_n$,

$$\begin{aligned}
\Lambda_n^{\pi^3} &= E\Bigg(\Bigg(H - x - \sum_{i=1}^N \pi_i^3 \Delta X_i\Bigg)^2 \Bigg| F_n\Bigg) \\
&= E\Bigg(\Bigg(H - x - I_B \sum_{i=1}^N \pi_i^1 \Delta X_i - I_{B^c} \sum_{i=1}^N \pi_i^2 \Delta X_i\Bigg)^2 \Bigg| F_n\Bigg) \\
&= I_B E\Bigg(\Bigg(H - x - \sum_{i=1}^N \pi_i^1 \Delta X_i\Bigg)^2 \Bigg| F_n\Bigg) \\
&\quad + I_{B^c} E\Bigg(\Bigg(H - x - \sum_{i=1}^N \pi_i^2 \Delta X_i\Bigg)^2 \Bigg| F_n\Bigg) \\
&= E\Bigg(\Bigg(H - x - \sum_{i=1}^N \pi_i^1 \Delta X_i\Bigg)^2 \Bigg| F_n\Bigg) \wedge E\Bigg(\Bigg(H - x - \sum_{i=1}^N \pi_i^2 \Delta X_i\Bigg)^2 \Bigg| F_n\Bigg)
\end{aligned}$$

and therefore the family Λ_n^π has the ε-lattice property. $\square$

Proposition A1. *The sequence $V(n, x + \sum_{i=1}^n \widehat{\pi}_i \Delta X_i), F_n)$ is a submartingale for arbitrary fixed $\widehat{\pi} \in \Pi, x \in \mathbf{R}$.*

Proof. We must prove that for any n

$$E\Bigg(V\Bigg(n, x + \sum_{i=1}^n \widehat{\pi}_i \Delta X_i\Bigg) \Bigg| F_{n-1}\Bigg) \ge V\Bigg(n-1, x + \sum_{i=1}^{n-1} \widehat{\pi}_i \Delta X_i\Bigg).$$

Taking into account the previous lemma and Lemma 16.A.5 of [1], we obtain

$$E\Bigg(\operatorname*{essinf}_{\pi \in \Pi(\widehat{\pi}, n, N)} E\Bigg(\Bigg(H - x - \sum_{i=1}^n \widehat{\pi}_i \Delta X_i - \sum_{i=n+1}^N \pi_i \Delta X_i\Bigg)^2 \Big| F_n\Bigg) \Bigg| F_{n-1}\Bigg)$$

$$= \operatorname*{essinf}_{\pi\in\Pi(\widehat{\pi},n,N)} E\biggl(E\biggl(\biggl(H - x - \sum_{i=1}^{n}\widehat{\pi}_i\Delta X_i - \sum_{i=n+1}^{N}\pi_i\Delta X_i)^2\Big|F_n\biggr)\ \Big|\ F_{n-1}\biggr)\biggr)$$

$$= \operatorname*{essinf}_{\pi\in\Pi(\widehat{\pi},n,N)} E\biggl(\biggl(H - x - \sum_{i=1}^{n}\widehat{\pi}_i\Delta X_i - \sum_{i=n+1}^{N}\pi_i\Delta X_i\biggr)^2\ \Big|\ F_{n-1}\biggr)$$

$$\geq \operatorname*{essinf}_{\pi\in\Pi(\widehat{\pi},n-1,N)} E\biggl(\biggl(H - x - \sum_{i=1}^{n-1}\widehat{\pi}_i\Delta X_i - \sum_{i=n}^{N}\pi_i\Delta X_i\biggr)^2\ \Big|\ F_{n-1}\biggr)$$

$$= V\biggl(n-1, x + \sum_{i=k}^{n-1}\widehat{\pi}_i\Delta X_i\biggr),$$

which is the desired result. □

Proposition A2 (Optimality Principle). $\pi^* \in \Pi$ *is optimal if and only if the sequence* $(V(n, c + \sum_{i=1}^{n}\pi_i^*\Delta X_i), F_n)$ *is a martingale.*

Proof. Necessity. Suppose that $\pi^* = (\pi_n^*)$ is optimal. Then since $(V(n, c + \sum_{i=1}^{n}\pi^*\Delta X_i), F_n)$ is a submartingale, due to Lemma 6.6 of [4] it is sufficient to check the equality

$$E\biggl[V\biggl(N, c + \sum_{i=1}^{N}\pi_i^*\Delta X_i\biggr)\biggr] = E[V(0,c)].$$

We have

$$E\biggl[V\biggl(N, c + \sum_{i=1}^{N}\pi_i^*\Delta X_i\biggr)\biggr] = E\biggl[\biggl(H - c - \sum_{i=1}^{N}\pi_i^*\Delta X_i\biggr)^2\biggr].$$

Noting that the optimality of π^* means

$$V(0,c) = E\biggl[\biggl(H - c - \sum_{i=1}^{N}\pi_i^*\Delta X_i\biggr)^2\biggr],$$

we finally obtain

$$E\biggl[V\biggl(N, c + \sum_{i=1}^{N}\pi_i^*\Delta X_i\biggr)\biggr] = E[V(0,c)].$$

Sufficiency. Suppose that $(V(n, c + \sum_{i=1}^{n}\pi_i^*\Delta X_i), F_n)$ is a martingale. Then

$$E\biggl(V\biggl(N, c + \sum_{i=1}^{N}\pi_i^*\Delta X_i\biggr)\ \Big|\ F_0\biggr) = V(0,c),$$

whence

$$E\biggl[\biggl(H - c - \sum_{i=1}^{N}\pi_i^*\Delta X_i\biggr)^2\biggr] = V(0,c),$$

which means that π^* is optimal. □

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