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ON THE UNIFORM CONVERGENCE AND L-CONVERGENCE OF DOUBLE FOURIER SERIES WITH RESPECT TO THE WALSH–KACZMARZ SYSTEM

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Abstract. In this paper we study the approximation by rectangular partial sums of a double Fourier series with respect to the Walsh–Kaczmarz system in the spaces C and L. From our results we obtain different criteria of the uniform convergence and L-convergence of a double Fourier–Kaczmarz series.

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1. Introduction

L. Zhizhiashvili ([13], Part. 2, Ch. 3) has established certain approximation properties of rectangular partial sums of a double trigonometric Fourier series in the spaces C and L. An analogous question for a double Fourier series with respect to the Walsh–Paley system were treated in [2], [8].

We will study the approximation of a function $f \in L^p$ in the norm of L^p for $p = 1$ or $p = \infty$ by means of rectangular partial sums of a double Fourier series with respect to the Walsh–Kaczmarz system; see Theorems 1, 2 and 3. From these theorems one can obtain different criteria for the uniform convergence and L^1-convergence of a double Fourier series with respect to the Walsh–Kaczmarz system, in particular, establish a two-dimensional version of the Dini–Lipschitz condition (see Corollaries 1 and 2).

Results of a somewhat different type can be obtained by using the variation of a function.

Jordan [7] introduced a class of functions of bounded variation and, applying it to the theory of Fourier series, he proved that if a continuous function has bounded variation, then its Fourier series converges uniformly. In 1906 Hardy [6] generalized the Jordan criterion to the double Fourier series and introduced, for the function of two variables, the notion of bounded variation. He proved that if the continuous function of two variables has bounded variation (in the sense of Hardy), then its Fourier series converges uniformly in the sense of Pringsheim (see, e.g., [13]). An analogous result for a double Walsh–Fourier series was verified by Moricz [8]. The author proves in [3] that in Hardy's theorem there is no need to require the boundedness of nuxed variation (see $V_{1,2}(f)$ below), in particular, it is proved that if f is continuous function and has bounded partial variation, then its double trigonometric Fourier series converges uniformly on

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$[0,2\pi]^2$ in the sense of Pringsheim. An analogous result for a double Walsh–Fourier series is established in [4]. In this paper we study a similar question in the case of the Walsh–Kaczmarz system (see Theorem 4).

2. Definitions and Notation

We denote the sets of all non-negative integers by $\mathbf{N}$, the set of real numbers by $\mathbf{R}$ and the set of dyadic rational numbers in the unit interval [0,1] by $\mathbf{Q}$. In particular, each element of $\mathbf{Q}$ has the form $\frac{p}{2^n}$ for some $p,n\in\mathbf{N}$, $0\le p\le 2^n$.

Let $r_0(x)$ be the function

$$r_0(x)=\begin{cases}1 & \text{if } x\in[0,1/2)\\ -1 & \text{if } x\in[1/2,1)\end{cases},\quad r_0(x+1)=r_0(x).$$

The Rademacher system is defined by $r_n(x)=r_0(2^n x)$, $n\ge 1$ and $x\in[0,1)$.

The Walsh system in the Paley enumeration $\{w_n(x):\ n\in\mathbf{N}\}$ is defined by

$$w_n(x)=\prod_{j=0}^{m}(r_j(x))^{n_j},$$

where n_j, $j=1,\dots,m$, are the binary coefficients of n.

Now we recall the definition of the Walsh–Kaczmarz system $\{\psi_n(x):\ n\in\mathbf{N}\}$. Set $\psi_0(x)=1$, while for $n\ge 1$ with the binary coefficients n_k, $k=1,\dots,m$, set

$$\psi_n(x)=r_m(x)\prod_{j=0}^{m-1}(r_{m-j-1}(x))^{n_j}.$$

Let us consider the Dirichlet kernels

$$D_n(x)=\sum_{j=0}^{n-1}\omega_j(x),\quad K_n(x)=\sum_{j=0}^{n-1}\psi_j(x)$$

for the Walsh–Paley system and for the Walsh–Kaczmarz system, respectively.

We need the well-known equality for the Dirichlet kernel of the Walsh–Paley system (see [5], p. 27)

$$D_{2^n}(x)=\begin{cases}2^n & \text{if } x\in[0,1/2^n);\\ 0 & \text{if } x\in[1/2^n,1).\end{cases}\tag{1}$$

The transformation τ_n for $x\in[0,1)$ is defined by

$$\tau_n(x)=\sum_{k=0}^{n-1}x_{n-k-1}\,2^{-(k+1)}+\sum_{j=n}^{\infty}x_j\,2^{-(j+1)},$$

where $x=\sum\limits_{k=1}^{\infty}x_k 2^{-(k+1)}$ is the dyadic expansion of x (for $x\in\mathbf{Q}$ we choose the expansion for which $x_k\to 0$ as $k\to\infty$).

We apply the transformation τ_n also for integers $p \geq 0$ given by

$$\tau_n(p) = \sum_{j=0}^{n-1} x_{n-j-1}\, 2^j,$$

where

$$p = \sum_{j=0}^{n-1} x_j\, 2^j.$$

Given $n \geq 0$ and $0 \leq p < 2^n$, we set

$$I_n(p) = [p\, 2^{-n}, (p+1)\, 2^{-n}).$$

It is evident that the transformation $\tau_n(x)$ maps the segment $I_n(p)$ on the segment $I_n(\tau_n(p))$.

It is known [11] that

$$K_n(x) = D_{2^k}(x) + r_k(x) \quad D_m(\tau_k(x)) \quad \text{for} \quad n = 2^k + m, \;\; 0 \leq m < 2^k, \tag{2}$$

and

$$|D_m(\tau_k(x))| \leq \frac{2^k}{\tau_k(p)} \quad \text{for} \quad x \in I_k(p), \quad p = 1, 2, \ldots, 2^k - 1. \tag{3}$$

We consider the double system $\{\psi_n(x) \times \psi_m(y) : n, m \in \mathbf{N}\}$ on the unit square $I^2 = [0,1) \times [0,1)$. If $f \in L(I^2)$, then

$$\hat{f}(n,m) = \int_0^1 \int_0^1 f(x,y)\, \psi_n(x) \psi_m(y)\, dx\, dy$$

is the (n,m)-th Fourier coefficient of f.

The rectangular partial sums of double Fourier series with respect to the Walsh–Kaczmarz system are defined by

$$S_{M,N}(f; x, y) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \hat{f}(m,n)\, \psi_m(x) \psi_n(y).$$

As usual, denote by $L(I^2)$ the set of all measurable functions defined on I^2, for which $\|f\|_1$, the integral of $|f(x,y)|$ on I^2, is finite. Furthermore, let $C(I^2)$ be the set of all functions $f : I^2 \to R$ that are uniformly continuous from the dyadic topology of I^2 to the usual topology of R with the norm (see [9], pp. 9–11)

$$\|f\|_C = \sup_{x,y \in I^2} |f(x,y)| \quad \left(f \in C\left(I^2\right)\right).$$

The total modulus of continuity and the total integrated modulus of continuity are respectively defined by

$$\omega(f; \delta_1, \delta_2)_C = \sup\left\{ \left\| f(x \oplus u, y \oplus v) - f(x,y) \right\|_C : \; 0 \leq u < \delta_1, \; 0 \leq v < \delta_2 \right\},$$

$$\omega(f; \delta_1, \delta_2)_1 = \sup\left\{ \left\| f(x \oplus u, y \oplus v) - f(x,y) \right\|_1 : \; 0 \leq u < \delta_1, \; 0 \leq v < \delta_2 \right\},$$

where $\oplus$ denotes the dyadic addition (see, e.g., [5], [9]), while the partial moduli of continuity and the partial integrated moduli of continuity are respectively defined by

$$\omega_1(f;\delta)_C = \omega(f;\delta,0)_C \quad \text{and} \quad \omega_2(f;\delta)_C = \omega(f;0,\delta)_C, \qquad f \in C\left(I^2\right),$$
$$\omega_1(f;\delta)_1 = \omega(f;\delta,0)_1 \quad \text{and} \quad \omega_2(f;\delta)_1 = \omega(f;0,\delta)_1, \qquad f \in L\left(I^2\right).$$

We also use the notion of the mixed modulus of continuity, and the mixed integrated modulus of continuity which are respectively defined as follows:

$$\omega_{1,2}(f;\delta_1,\delta_2)_C = \sup\left\{\left\|f(x\oplus u, y\oplus v) - f(x\oplus u, y)\right.\right.$$
$$\left.\left.-f(x, y\oplus v) + f(x,y)\right\|_C : 0 \le u < \delta_1,\ 0 \le v < \delta_2\right\},\ \ f \in C\left(I^2\right),$$
$$\omega_{1,2}(f;\delta_1,\delta_2)_1 = \sup\left\{\left\|f(x\oplus u, y\oplus v) - f(x\oplus u, y)\right.\right.$$
$$\left.\left.-f(x, y\oplus v) + f(x,y)\right\|_1 : 0 \le u < \delta_1,\ 0 \le v < \delta_2\right\},\ \ f \in L\left(I^2\right).$$

A function $f : I^2 \to R$ is said to be of bounded variation in the sense of Hardy($f \in HBV(I^2)$) if there exists a constant K such that for any partition

$$\Delta_1 : 0 \le x_0 < x_1 < x_2 < \cdots < x_n \le 1,$$
$$\Delta_2 : 0 \le y_0 < y_1 < y_2 < \cdots < y_m \le 1,$$

we have

$$V_{1,2}(f) = \sup_{\Delta_1\times\Delta_2} \sum_{i=0}^{n-1}\sum_{j=0}^{m-1} \Big| f(x_i,y_j) - f(x_{i+1},y_j)$$
$$- f(x_i,y_{j+1}) + f(x_{i+1},y_{j+1}) \Big| \le K,$$
$$V_1(f) = \sup_y \sup_{\Delta_1} \sum_{i=0}^{n-1} \Big| f(x_i,y) - f(x_{i+1},y) \Big| \le K,$$
$$V_2(f) = \sup_x \sup_{\Delta_2} \sum_{j=0}^{m-1} \Big| f(x,y_j) - f(x,y_{j+1}) \Big| \le K.$$

Definition 1 ([4])**.** We say that the function $f : I^2 \to R$ is of bounded partial variation ($f \in PBV(I^2)$) if $V_1(f)$ and $V_2(f)$ are finite.

Given a function $f(x,y)$, periodic in both variables with period 1, for $0 \le j < 2^m$ and $0 \le i < 2^n$ and integers $m, n \ge 0$ we set

$$\Delta_j^m f(x,y)_1 = f\left(x \oplus 2j2^{-m-1}, y\right) - f\left(x \oplus (2j+1)\,2^{-m-1}, y\right),$$
$$\Delta_i^n f(x,y)_2 = f\left(x, y \oplus 2i2^{-n-1}\right) - f\left(x, y \oplus (2i+1)\,2^{-n-1}\right),$$
$$\Delta_{ji}^{mn} f(x,y) = \Delta_i^n\left(\Delta_j^m f(x,y)_1\right)_2 = \Delta_j^m\left(\Delta_i^n f(x,y)_2\right)_1$$
$$= f\left(x \oplus 2j2^{-m-1}, y \oplus 2i2^{-n-1}\right) - f\left(x \oplus (2j+1)\,2^{-m-1}, y \oplus 2i2^{-n-1}\right)$$
$$- f\left(x \oplus 2j2^{-m-1}, y \oplus (2i+1)\,2^{-n-1}\right)$$
$$+ f\left(x \oplus (2j+1)\,2^{-m-1}, y \oplus (2i+1)\,2^{-n-1}\right).$$

Furthermore, set $\lambda_0^m = 1$ and $\lambda_j^m = (\tau_m(j))^{-1}$ for $1 \le j < 2^m$ and

$$W_m^{(1)}(f;x,y) = \sum_{j=0}^{2^m-1} \lambda_j^m \left|\Delta_j^m f(x,y)_1\right|,$$

$$W_n^{(2)}(f;x,y) = \sum_{i=0}^{2^n-1} \lambda_i^n \left|\Delta_i^n f(x,y)_2\right|,$$

$$W_{mn}(f;x,y) = \sum_{j=0}^{2^m-1} \sum_{i=0}^{2^n-1} \lambda_j^m \lambda_i^n \left|\Delta_{ji}^{mn} f(x,y)\right|.$$

2.1. **Main Results.**

Theorem 1. *Let M, N be positive integers such that $M = 2^m + j$, $0 \le j < 2^m$, and $N = 2^n + i$, $0 \le i < 2^n$, for some integers $m, n \ge 0$. If $f \in C(I^2)$, then*

$$\|S_{M,N}(f) - f\|_C \le \omega\left(f; \frac{1}{2^m}, \frac{1}{2^n}\right)_C + \frac{1}{2}\left\|W_m^{(1)}(f)\right\|_C + \frac{1}{2}\left\|W_n^{(2)}(f)\right\|_C + \frac{1}{4}\|W_{mn}(f)\|_C.$$

Theorem 2. *Let M, N be positive integers such that $M = 2^m + j$, $0 \le j < 2^m$, and $N = 2^n + i$, $0 \le i < 2^n$, for some integers $m, n \ge 0$. If $f \in C(I^2)$; then*

$$\|S_{M,N}(f) - f\|_C \le c\left\{\omega_1\left(f; \frac{1}{2^m}\right)_C m + \omega_2\left(f; \frac{1}{2^n}\right)_C n + \omega_{1,2}\left(f; \frac{1}{2^m}, \frac{1}{2^n}\right)_C mn\right\}.$$ [1]

Corollary 1. *Let $f \in C(I^2)$ and*

$$\omega_1\left(f; \frac{1}{2^m}\right)_C m \to 0 \quad as \quad m \to \infty,$$

$$\omega_2\left(f; \frac{1}{2^n}\right)_C n \to 0 \quad as \quad n \to \infty,$$

$$\omega_{1,2}\left(f; \frac{1}{2^m}, \frac{1}{2^n}\right)_C mn \to 0 \quad as \quad n, m \to \infty;$$

then the double Fourier series with respect to Walsh–Kaczmarz system converges uniformly on I^2.

Theorem 3. *Let M, N be positive integers such that $M = 2^m + j$, $0 \le j < 2^m$, and $N = 2^n + i$, $0 \le i < 2^n$, for some integers $m, n \ge 0$. If $f \in L(I^2)$; then*

$$\|S_{M,N}(f) - f\|_1 \le c\left\{\omega_1\left(f; \frac{1}{2^m}\right)_1 m + \omega_2\left(f; \frac{1}{2^n}\right)_1 n + \omega_{1,2}\left(f; \frac{1}{2^m}, \frac{1}{2^n}\right)_1 mn\right\}.$$

[1]Here and below the constant c is an absolute constant and may be different in different places.

Corollary 2. *Let $f \in L(I^2)$ and*

$$\omega_1\left(f;\frac{1}{2^m}\right)_1 m \to 0 \quad as \quad m \to \infty,$$

$$\omega_2\left(f;\frac{1}{2^n}\right)_1 n \to 0 \quad as \quad n \to \infty,$$

$$\omega_{1,2}\left(f;\frac{1}{2^m},\frac{1}{2^n}\right)_1 mn \to 0 \quad as \quad n,m \to \infty;$$

then the double Fourier series with respect to Walsh–Kaczmarz system converges to f in L-norm.

Theorem 4. *Let $f \in C(I^2) \cap PBV(I^2)$. Then the double Fourier series with respect to Walsh–Kaczmarz system converges uniformly on I^2.*

2.2. **Proof of the Main Results.**

Proof of Theorem 1. By (2) we write

$$\begin{aligned}
&S_{M,N}(f;x,y) - f(x,y)\\
&= \int_0^1\int_0^1 [f(x\oplus u, y\oplus v) - f(x,y)]\, K_M(u)\, K_N(v)\, du\, dv\\
&= \int_0^1\int_0^1 [f(x\oplus u, y\oplus v) - f(x,y)]\, D_{2^m}(u)\, D_{2^n}(v)\, du\, dv\\
&\quad + \int_0^1\int_0^1 [f(x\oplus u, y\oplus v) - f(x,y)]\, D_{2^m}(u)\, r_n(v)\, D_i(\tau_n(v))\, du\, dv\\
&\quad + \int_0^1\int_0^1 [f(x\oplus u, y\oplus v) - f(x,y)]\, D_{2^n}(v)\, r_m(u)\, D_j(\tau_m(u))\, du\, dv\\
&\quad + \int_0^1\int_0^1 [f(x\oplus u, y\oplus v) - f(x,y)]\, r_n(v)\, r_m(u)\, D_j(\tau_m(u))\, D_i(\tau_n(v))\, dudv\\
&= I + II + III + IV. \qquad (4)
\end{aligned}$$

We find by (1) that

$$\begin{aligned}
\|I\|_C &= 2^{n+m}\left\| \int_0^{1/2^m}\int_0^{1/2^n} [f(x\oplus u, y\oplus v) - f(x,y)]\, du\, dv \right\|_C\\
&\le \omega\left(f;\frac{1}{2^m},\frac{1}{2^n}\right)_C. \qquad (5)
\end{aligned}$$

It is well-known (see, e.g., [5]) that

a) $w_{2^m}(u) = \begin{cases} 1 & \text{if } u \in I_{m+1}(2j) \\ -1 & \text{if } u \in I_{m+1}(2j+1), \end{cases} \quad m \in N, \quad 0 \le j < 2^m;$

b) $t = u \oplus 2^{-m-1}$ is a one-to-one mapping of $I_{m+1}(2j)$ onto $I_{m+1}(2j+1)$.

Thus, by (1) and (a)–(b) ,

$$\begin{aligned}
II &= 2^m \int_{I_m(0)} \int_0^1 [f(x \oplus u, y \oplus v) - f(x,y)] \, r_n(v) \, D_i(\tau_n(v)) \, du \, dv \\
&= 2^m \sum_{r=0}^{2^n-1} \int_{I_m(0)} \Big(\int_{I_n(r)} [f(x \oplus u, y \oplus v) - f(x,y)] \, r_n(v) \, D_i(\tau_n(v)) \, dv \Big) du \\
&= 2^m \sum_{r=0}^{2^n-1} \int_{I_m(0)} \Big(\int_{I_{n+1}(2r)} [f(x \oplus u, y \oplus v) - f(x,y)] \, D_i(\tau_n(v)) \, dv \\
&\quad - \int_{I_{n+1}(2r+1)} [f(x \oplus u, y \oplus v) - f(x,y)] \, D_i(\tau_n(v)) \, dv \Big) du \\
&= 2^m \sum_{r=0}^{2^n-1} \int_{I_m(0)} \int_{I_{n+1}(2r)} \left[f(x \oplus u, y \oplus v) - f\left(x \oplus u, y \oplus v \oplus 2^{-n-1}\right) \right] \\
&\qquad \times D_i(\tau_n(v)) \, du \, dv \\
&= 2^m \int_{I_m(0)} \int_{I_{n+1}(0)} \left[f(x \oplus u, y \oplus v) - f\left(x \oplus u, y \oplus v \oplus 2^{-n-1}\right) \right] \\
&\qquad \times D_i(\tau_n(v)) \, du \, dv \\
&\quad + 2^m \sum_{r=1}^{2^n-1} D_i(\tau_n(r)) \int_{I_m(0)} \int_{I_{n+1}(0)} \left[f\left(x \oplus u, y \oplus v \oplus 2r2^{-n-1}\right) \right. \\
&\quad \left. - f\left(x \oplus u, y \oplus v \oplus (2r+1) \, 2^{-n-1}\right) \right] du \, dv.
\end{aligned}$$

From (3) we have

$$\begin{aligned}
|II| &\le 2^m \, i \int_{I_m(0)} \int_{I_{n+1}(0)} \left| \Delta_0^n f(x \oplus u, y \oplus v)_2 \right| du \, dv \\
&\quad + 2^{m+n} \int_{I_m(0)} \int_{I_{n+1}(0)} \sum_{r=1}^{2^n-1} \frac{1}{\tau_n(r)} \left| \Delta_r^n f(x \oplus u, y \oplus v)_2 \right| du \, dv.
\end{aligned}$$

Consequently,

$$\|II\|_C \le \frac{1}{2} \left\| W_n^{(2)}(f) \right\|_C . \tag{6}$$

The estimation of III is analogous to that of II and we have

$$\|III\|_C \leq \frac{1}{2}\left\|W_m^{(1)}(f)\right\|_C. \tag{7}$$

Following a similar pattern for the case of II, by (a)–(b) we obtain

$$\begin{aligned}
IV &= \sum_{s=0}^{2^m-1}\sum_{r=0}^{2^n-1} D_i(\tau_n(r))\, D_j(\tau_m(s)) \Bigg(\int\limits_{I_{m+1}(2s)} \int\limits_{I_{n+1}(2r)} - \int\limits_{I_{m+1}(2s)} \int\limits_{I_{n+1}(2r+1)} \\
&\quad - \int\limits_{I_{m+1}(2s+1)} \int\limits_{I_{n+1}(2r)} + \int\limits_{I_{m+1}(2s+1)} \int\limits_{I_{n+1}(2r+1)} \Bigg) [f(x\oplus u, y\oplus v) - f(x,y)]\, du\, dv \\
&= \sum_{s=0}^{2^m-1}\sum_{r=0}^{2^n-1} D_i(\tau_n(r))\, D_j(\tau_m(s)) \\
&\quad \times \int\limits_{I_{m+1}(2s)} \int\limits_{I_{n+1}(2r)} \Big[f\left(x\oplus u\oplus 2^{-m-1}, y\oplus v\oplus 2^{-n-1}\right) \\
&\qquad\qquad - f\left(x\oplus u\oplus 2^{-m-1}, y\oplus v\right) \\
&\qquad\qquad - f\left(x\oplus u, y\oplus v\oplus 2^{-n-1}\right) + f(x\oplus u, y\oplus v)\Big]\, du\, dv \\
&= \sum_{s=0}^{2^m-1}\sum_{r=0}^{2^n-1} D_i(\tau_n(r))\, D_j(\tau_m(s)) \\
&\quad \times \int\limits_{I_{m+1}(0)} \int\limits_{I_{n+1}(0)} \Big[f\left(x\oplus u\oplus (2s+1)\,2^{-m-1},\ \ y\oplus v\oplus (2r+1)\,2^{-n-1}\right) \\
&\qquad\qquad - f\left(x\oplus u\oplus (2s+1)\,2^{-m-1}, y\oplus v\oplus 2r2^{-n-1}\right) \\
&\qquad\qquad - f\left(x\oplus u\oplus 2s2^{-m-1}, y\oplus v\oplus (2r+1)\,2^{-n-1}\right) \\
&\qquad\qquad + f\left(x\oplus u\oplus 2s2^{-m-1}, y\oplus v\oplus 2r2^{-n-1}\right)\Big]\, du\, dv.
\end{aligned}$$

By (3) we obtain

$$\begin{aligned}
|IV| &\leq ij \int\limits_{I_{m+1}(0)} \int\limits_{I_{n+1}(0)} \left|\Delta_{00}^{mn} f(x\oplus u, y\oplus v)\right| du\, dv \\
&\quad + j2^n \int\limits_{I_{m+1}(0)} \int\limits_{I_{n+1}(0)} \sum_{r=1}^{2^n-1} \frac{1}{\tau_n(r)} \left|\Delta_{0r}^{mn} f(x\oplus u, y\oplus v)\right| du\, dv \\
&\quad + i2^m \int\limits_{I_{m+1}(0)} \int\limits_{I_{n+1}(0)} \sum_{s=1}^{2^m-1} \frac{1}{\tau_m(s)} \left|\Delta_{s0}^{mn} f(x\oplus u, y\oplus v)\right| du\, dv
\end{aligned}$$

$$+\,2^{n+m}\int\limits_{I_{m+1}(0)}\int\limits_{I_{n+1}(0)}\sum_{r=1}^{2^n-1}\sum_{s=1}^{2^m-1}\frac{1}{\tau_n\left(r\right)}\frac{1}{\tau_m\left(s\right)}\left|\Delta_{sr}^{mn}f\left(x\oplus u,y\oplus v\right)\right|du\,dv$$

$$\leq 2^{n+m}\int\limits_{I_{m+1}(0)}\int\limits_{I_{n+1}(0)}W_{mn}f\left(f;x\oplus u,y\oplus v\right)du\,dv.$$

Consequently,

$$\left\|IV\right\|_C\leq\frac{1}{4}\left\|W_{mn}\left(f\right)\right\|_C. \tag{8}$$

Combining (4)–(8), we complete the proof of Theorem 1. □

Proof of Theorem 2. Since

$$\sum_{r=0}^{2^n-1}\lambda_r^n=1+\sum_{r=1}^{2^n-1}\frac{1}{\tau_n\left(r\right)}=1+\sum_{r=1}^{2^n-1}\frac{1}{r}\leq cn$$

and

$$W_m^{(1)}\left(f;x,y\right)\leq\omega_1\left(f;\frac{1}{2^m}\right)_C\sum_{r=0}^{2^m-1}\lambda_r^m\leq c\omega_1\left(f;\frac{1}{2^m}\right)m,$$

$$W_n^{(2)}\left(f;x,y\right)\leq\omega_2\left(f;\frac{1}{2^n}\right)_C\sum_{i=0}^{2^n-1}\lambda_i^n\leq c\omega_2\left(f;\frac{1}{2^n}\right)n,$$

$$W_{mn}\left(f;x,y\right)\leq\omega_{1,2}\left(f;\frac{1}{2^m},\frac{1}{2^n}\right)_C\sum_{r=0}^{2^n-1}\sum_{i=0}^{2^m-1}\lambda_r^n\lambda_i^m\leq c\omega_{1,2}\left(f;\frac{1}{2^m},\frac{1}{2^n}\right)nm,$$

the validity of Theorem 2 follows from Theorem 1. □

Calculations similar to those that were performed in the proofs of Theorems 1, 2 and the application of the Minkowski inequality yield the validity of Theorem 3.

Proof of Theorem 4. On the basis of Theorem 1 it suffices to show that

$$\begin{aligned}&\left\|W_m^{(1)}\left(f\right)\right\|_C\to 0\quad\text{as}\quad m\to\infty,\\&\left\|W_n^{(2)}\left(f\right)\right\|_C\to 0\quad\text{as}\quad n\to\infty,\\&\left\|W_{mn}\left(f\right)\right\|_C\to 0\quad\text{as}\quad n,m\to\infty.\end{aligned}$$

Let

$$A_r^m=\left\{j:j=\overline{1,2^m-1},2^r\leq\tau_m\left(j\right)<2^{r+1}\right\},\quad r=0,1,\ldots,m-1.$$

Then it is evident that

$$\left|A_r^m\right|<2^r \tag{9}$$

and

$$\bigcup_{r=0}^{m-1}A_r^m=\left\{1,2,\ldots,2^m-1\right\}. \tag{10}$$

By the condition of the theorem and (9), (10) we get

$$\begin{aligned}
W_m^{(1)}(f;x,y) &= \left|\Delta_0^m f(x,y)_1\right| + \sum_{j=1}^{2^m-1} \frac{1}{\tau_m(j)} \left|\Delta_j^m f(x,y)_1\right| \\
&\le \omega_1\left(f;\frac{1}{2^m}\right)_C + \sum_{r=0}^{m-1} \sum_{j\in A_r^m} \frac{1}{\tau_m(j)} \left|\Delta_j^m f(x,y)_1\right| \\
&\le \omega_1\left(f;\frac{1}{2^m}\right)_C + \sum_{r=0}^{m-1} \frac{1}{2^r} \sum_{j\in A_r^m} \left|\Delta_j^m f(x,y)_1\right| \\
&\le \omega_1\left(f;\frac{1}{2^m}\right)_C + \sum_{r=0}^{\eta(m)-1} \frac{1}{2^r} \sum_{j\in A_r^m} \left|\Delta_j^m f(x,y)_1\right| \\
&\quad + \sum_{r=\eta(m)}^{m-1} \frac{1}{2^r} \sum_{j\in A_r^m} \left|\Delta_j^m f(x,y)_1\right| \\
&\le \omega_1\left(f;\frac{1}{2^m}\right)_C + \omega_1\left(f;\frac{1}{2^m}\right)_C \sum_{r=0}^{\eta(m)-1} \frac{1}{2^r}\left|A_r^m\right| + V_1(f) \sum_{r=\eta(m)}^{m-1} \frac{1}{2^r} \\
&\le c\left\{\omega_1\left(f;\frac{1}{2^m}\right)_C \eta(m) + \frac{1}{2^{\eta(m)}}\right\} \to 0 \qquad \text{as} \quad m\to\infty,
\end{aligned}$$

where

$$\min_{1\le\eta\le 2^m-1}\left\{\omega_1\left(f;\frac{1}{2^m}\right)_C \eta + \frac{1}{2^\eta}\right\} = \omega_1\left(f;\frac{1}{2^m}\right)_C \eta(m) + \frac{1}{2^{\eta(m)}};$$

consequently,

$$\left\|W_m^{(1)}(f)\right\|_C \to 0 \qquad \text{as} \quad m\to\infty. \tag{11}$$

Analogously,

$$\left\|W_n^{(2)}(f)\right\|_C \to 0 \qquad \text{as} \quad n\to\infty. \tag{12}$$

We write

$$\begin{aligned}
W_{mn}(f;x,y) &= \left|\Delta_{00}^{mn} f(x,y)\right| + \sum_{i=1}^{2^n-1} \frac{1}{\tau_n(i)} \left|\Delta_{0i}^{mn} f(x,y)\right| \\
&+ \sum_{j=1}^{2^m-1} \frac{1}{\tau_m(j)} \left|\Delta_{j0}^{mn} f(x,y)\right| + \sum_{j=1}^{2^m-1} \sum_{i=1}^{2^n-1} \frac{1}{\tau_n(i)} \frac{1}{\tau_m(j)} \left|\Delta_{ji}^{mn} f(x,y)\right| \\
&= I + II + III + IV.
\end{aligned} \tag{13}$$

It is evident that

$$\left|\Delta_{00}^{mn} f(x,y)\right| \le \omega_{1,2}\left(f;\frac{1}{2^m},\frac{1}{2^n}\right)_C \to 0 \quad \text{as} \quad n,m\to\infty. \tag{14}$$

Since

$$\left|\Delta_{j0}^{mn} f(x,y)\right| \le \left|\Delta_j^m f(x,y)_1\right| + \left|\Delta_j^m f\left(x, y\oplus 2^{-n-1}\right)_1\right|,$$

from (11) we get

$$\|II\|_C \leq 2\left\|W_n^{(2)}(f)\right\|_C \to 0 \quad \text{as} \quad m \to \infty. \tag{15}$$

Analogously,

$$\|III\|_C \to 0 \quad \text{as} \quad m \to \infty. \tag{16}$$

From (9) and (10) we get

$$\begin{aligned} IV &= \sum_{r=0}^{m-1}\sum_{s=0}^{n-1}\sum_{j\in A_r^m}\sum_{i\in A_s^n}\frac{1}{\tau_n(i)}\frac{1}{\tau_m(j)}\left|\Delta_{ji}^{mn}f(x,y)\right| \\ &\leq \sum_{r=0}^{m-1}\frac{1}{2^r}\sum_{s=0}^{n-1}\frac{1}{2^s}\sum_{j\in A_r^m}\sum_{i\in A_s^n}\left|\Delta_{ji}^{mn}f(x,y)\right|. \end{aligned} \tag{17}$$

Since

$$\sum_{j\in A_r^m}\sum_{i\in A_s^n}\left|\Delta_{ji}^{mn}f(x,y)\right| \leq 2\left|A_r^m\right|\sup_{x\in[0,1]}\sum_{i\in A_s^n}\left|\Delta_i^n f(x,y)_2\right|,$$

$$\sum_{j\in A_r^m}\sum_{i\in A_s^n}\left|\Delta_{ji}^{mn}f(x,y)\right| \leq 2\left|A_s^n\right|\sup_{y\in[0,1]}\sum_{j\in A_r^m}\left|\Delta_j^m f(x,y)_1\right|,$$

we have

$$\begin{aligned} \sum_{j\in A_r^m}\sum_{i\in A_s^n}\left|\Delta_{ji}^{mn}f(x,y)\right| &= \Bigg(\sum_{j\in A_r^m}\sum_{i\in A_s^n}\left|\Delta_{ji}^{mn}f(x,y)\right|\Bigg)^{1/2} \\ &\times\Bigg(\sum_{j\in A_r^m}\sum_{i\in A_s^n}\left|\Delta_{ji}^{mn}f(x,y)\right|\Bigg)^{1/2} \leq 2\Bigg[\left|A_r^m\right|\left|A_s^n\right| \\ &\times\sup_{x\in[0,1]}\sum_{i\in A_s^n}\left|\Delta_i^n f(x,y)_2\right|\sup_{y\in[0,1]}\sum_{j\in A_r^m}\left|\Delta_j^m f(x,y)_1\right|\Bigg]^{1/2}. \end{aligned} \tag{18}$$

After substituting (18) in (17), we obtain by (9) and the condition of the theorem that

$$\begin{aligned} IV &\leq 2\sum_{r=0}^{m-1}\frac{1}{2^{r/2}}\Bigg(\sup_{y\in[0,1]}\sum_{j\in A_r^m}\left|\Delta_j^m f(x,y)_1\right|\Bigg)^{1/2} \\ &\times\sum_{s=0}^{n-1}\frac{1}{2^{s/2}}\Bigg(\sup_{x\in[0,1]}\sum_{i\in A_s^n}\left|\Delta_i^n f(x,y)_2\right|\Bigg)^{1/2} \\ &= 2\Bigg\{\sum_{r=0}^{\varphi(m)-1}\frac{1}{2^{r/2}}\Bigg(\sup_{y\in[0,1]}\sum_{j\in A_r^m}\left|\Delta_j^m f(x,y)_1\right|\Bigg)^{1/2} \\ &+\sum_{r=\varphi(m)}^{m-1}\frac{1}{2^{r/2}}\Bigg(\sup_{y\in[0,1]}\sum_{j\in A_r^m}\left|\Delta_j^m f(x,y)_1\right|\Bigg)^{1/2}\Bigg\} \end{aligned}$$

$$\times \left\{ \sum_{s=0}^{\psi(n)-1} \frac{1}{2^{s/2}} \left(\sup_{x\in[0,1]} \sum_{i\in A_s^n} \left|\Delta_i^n f(x,y)_2\right| \right)^{1/2} + \sum_{s=\psi(n)}^{n-1} \frac{1}{2^{r/2}} \left(\sup_{x\in[0,1]} \sum_{i\in A_s^n} \left|\Delta_i^n f(x,y)_2\right| \right)^{1/2} \right\}$$

$$\leq c\left\{ \sqrt{\omega_1\left(f;\frac{1}{2^m}\right)_C \varphi(m)} + \frac{1}{2^{\varphi(m)/2}} \right\} \times \left\{ \sqrt{\omega_2\left(f;\frac{1}{2^n}\right)_C \psi(n)} + \frac{1}{2^{\psi(n)/2}} \right\} \to 0 \quad \text{as} \quad m,n\to\infty, \tag{19}$$

where

$$\min_{1\leq\varphi\leq 2^m-1} \left\{ \sqrt{\omega_1\left(f;\frac{1}{2^m}\right)_C \varphi} + \frac{1}{2^{\varphi/2}} \right\} = \sqrt{\omega_1\left(f;\frac{1}{2^m}\right)_C \varphi(m)} + \frac{1}{2^{\varphi(m)/2}}$$

and

$$\min_{1\leq\psi\leq 2^n-1} \left\{ \sqrt{\omega_2\left(f;\frac{1}{2^n}\right)_C \psi} + \frac{1}{2^{\psi/2}} \right\} = \sqrt{\omega_2\left(f;\frac{1}{2^n}\right)_C \psi(n)} + \frac{1}{2^{\psi(n)/2}}.$$

Combining (13)–(16) and (19) we have

$$\|W_{mn}(f)\|_C \to 0 \quad \text{as} \quad n,m\to\infty. \tag{20}$$

From (11), (12) and (20) we complete the proof of Theorem 4. □

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