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ON THE DIMENSION MODULO CLASSES OF TOPOLOGICAL SPACES AND FREE TOPOLOGICAL GROUPS

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Abstract. Functions of dimension modulo a (rather wide) class of spaces are considered and the conditions are found, under which the dimension of the product of spaces modulo these classes is equal to zero. Based on these results, the sufficient conditions are established, under which spaces of free topological semigroups (in the sense of Marxen) and spaces of free topological groups (in the sense of Markov and Graev) are zero-dimensional modulo classes of compact spaces.

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The question when the product of two spaces is π-compact (i.e., has an open base, each element of which has a compact boundary) was studied in various particular cases in [17], [10] and in a general case in [5].

According to A. Lelek and J. de Groot (see, e.g., [1] and [14]), if a function of dimension type is considered modulo a class of compact spaces which is defined by the method of induction analogously to the function *ind*, then π-compact spaces turn out to be zero-dimensional (modulo a class of compact spaces).

In this paper, functions of dimension modulo a (rather wide) class of spaces are considered and the conditions are found, under which the dimension of the product of spaces modulo these classes is equal to zero. These results can therefore be regarded as a generalization of the results of [17], [10] and [5]. Moreover, based on the same results of [17], the sufficient conditions are established, under which spaces of free topological semigroups (in the sense of Marxen) and spaces of free topological groups (in the sense of Markov and Graev) are zero-dimensional modulo the class of compact spaces.

The paper consists of three parts. Part 1 contains the notation. In Part 2 the necessary definitions and statements are formulated. In Part 3 the conditions, under which the dimension modulo classes of a product of spaces is equal to zero, are established. Based on these results the structures of free topological semi-groups and free topological groups are determined.

1. Notation

All spaces considered in this paper are assumed to be Hausdorff and completely regular (Tychonoff). By $\mathfrak{M}$ (with or without an index) we denote topologically closed subclasses of the class of all completely regular spaces (that $\mathfrak{M}$

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is topologically closed means the following: if $X \in \mathfrak{M}$ and X is homeomorphic to Y, then $Y \in \mathfrak{M}$).

For each topological space X and for any of its subset U by $[U]_X$ we denote the closure of U in X, and by $Fr_X U$ the boundary of U in X.

Also, the following notation is used:

$\mathfrak{M}_c$ is a class of all compact spaces;

$\mathfrak{M}_{mc}$ is the class of all metrizable compact spaces;

$\mathfrak{M}_{dc}$ is the class of all Dugundji compacts [2];

$\mathfrak{M}_{cor}$ is the class of all Corson compacts [2];

$\mathfrak{M}_{\aleph_0}$ is the class of all countable spaces;

$\mathfrak{M}_{cn}$ is the class of spaces with countable networks in the sense of Arkhangelskii [2];

$\mathfrak{M}_{con}$ is the class of all connected spaces;

$\mathfrak{M}_{ps}$ is the class of all pseudocompact spaces;

$\mathfrak{M}_{cc}$ is the class of all countably compact spaces;

$\mathfrak{M}_{par}$ is the class of all paracompact spaces;

$\mathfrak{M}_{pp}$ is the class of all weakly paracompact spaces;

by N we denote the set of all integers ≥ -1, i.e., $N = \{-1, 0, 1, 2, \dots\}$.

2. Definitions and Necessary Auxiliary Propositions

Definition 2.1. Let $\mathfrak{M}$ be some topologically closed subclass of the class of completely regular spaces. Denote by $loc\,\mathfrak{M}$ the class of all spaces possessing the following property: $X \in loc\,\mathfrak{M}$ if and only if for each point x of the space X and each open set U_x such that $x \in U_x$ there exists an open set V_x in X such that $x \in V_x \subseteq [V_x]_X \subseteq U_x$ and $[V_x]_X \in \mathfrak{M}$.

Remark 2.1. It is obvious that for each topologically closed class $\mathfrak{M}$, $loc\,\mathfrak{M}$ is also topologically closed.

Definition 2.2. The class $\mathfrak{M}$ is called *c*-monotone if $\mathfrak{M}$ is topologically closed and any closed subspace of each space from the class $\mathfrak{M}$ also belongs to $\mathfrak{M}$.

Definition 2.3. Let $\mathfrak{M}$ be some class of spaces. For each space X (from the class of all completely regular spaces) the small inductive dimension $in\,(X, \mathfrak{M})$ modulo $\mathfrak{M}$ is defined as follows:

(i) $in\,(X, \mathfrak{M}) = -1$ if $X \in \mathfrak{M}$;

(ii) for each $n \geq 0$, $in\,(X, \mathfrak{M}) \leq n$ if for each point x of the space X and for each neighborhood of this point U_x there exists an open neighborhood V_x in X such that $x \in V_x \subseteq U_x$ and $in\,(Fr_X V_x, \mathfrak{M}) \leq n - 1$; $in\,(X, \mathfrak{M}) = n$ if $in\,(X, \mathfrak{M}) \leq n$ and $in\,(X, \mathfrak{M}) \not\leq n - 1$ and finally, $in\,(X, \mathfrak{M}) = +\infty$ if $in\,(X, \mathfrak{M}) \not\leq n$ for each $n \in N$.

Remark 2.2. If $\mathfrak{M}$ is the empty class, then for any space X we have: $in\,(X, \mathfrak{M}) = ind\,X$ (where $ind\,X$ is the small inductive dimension of X [14]).

Remark 2.3. $in\,(X, \mathfrak{M}) \leq 0$ if and only if either 1) $X \in \mathfrak{M}$ or 2) there exists in X an open base $\{U_a\}_{a \in M}$ such that $Fr_X U_a \in \mathfrak{M}$ for each $a \in M$. In particular, for each regular space X we have: $in\,(X, \mathfrak{M}_c) \leq 0$ if and only if X is semicompact ($\equiv \pi$-compact), see [8].

Definition 2.4. The class $\mathfrak{M}$ is called stable if the following conditions are fulfilled:
1) $\mathfrak{M}$ is topologically closed;
2) $\mathfrak{M}$ is c-monotone;
3) if $X \in \mathfrak{M}$ and Y is a continuous image of the space X, then $Y \in \mathfrak{M}$.

Definition 2.5. The class $\mathfrak{M}$ is called multiplicative if the following conditions are fulfilled:
1) $\mathfrak{M}$ is topologically closed;
2) if $X \in \mathfrak{M}$ and $Y \in \mathfrak{M}$, then $X \times Y \in \mathfrak{M}$, where $X \times Y$ denotes the usual product of topological spaces X and Y.

The classes $\mathfrak{M}_c$, $\mathfrak{M}_{mc}$, $\mathfrak{M}_{dc}$, $\mathfrak{M}_{cor}$, $\mathfrak{M}_{\aleph_0}$, $\mathfrak{M}_{cn}$ are examples of stable multiplicative classes.

Definition 2.6. The class $\mathfrak{M}$ is called weakly stable if it has the following property: If $X \in \mathfrak{M}$ and $f : X \to Y$ is a continuous closed mapping "onto", then $Y \in \mathfrak{M}$.

Remark 2.4. One can easily show that if $\mathfrak{M}$ is a stable class and $f : X \to Y$ is an open continuous mapping of a space X onto a space Y and $X \in loc\,\mathfrak{M}$ then $Y \in loc\,\mathfrak{M}$.

Next, we formulate some needed definitions.

Definition 2.7 ([12])**.** Let X be a completely regular topological space. A pair $(S(X), \Theta)$ is called a free topological semigroup generated by the space X if the following conditions are fulfilled:
1) $S(X)$ is a topological semigroup the space of which is completely regular;
2) $\Theta : X \to S(X)$ is a topological embedding of the space X in $S(X)$;
3) $\Theta(X)$ generates $S(X)$ algebraically;
4) for each completely regular topological semigroup T and for each continuous mapping $\omega : X \to T$ of the space X to T, there exists a unique continuous homomorphism $\Omega : S(X) \to T$ such that the composition $\Omega\Theta$ is equal to ω.

In [12] and [4] it is shown that for the completely regular topological space X, a free topological semigroup generated by X exists and is unique.

Definition 2.8 (A. A. Markov [11])**.** Let X be a completely regular space. A Hausdorff topological group $FM(X)$ (a Hausdorff abelian topological group $AM(X)$) is called a free topological group in the sense of Markov (a free abelian topological group in the sense of Markov) of the space X if X is a subspace of $FM(X)$ (resp. of $AM(X)$) and for each continuous mapping $\Phi : X \to G$ of the space X into an arbitrary topological group (into an arbitrary abelian topological group) G there exists a unique continuous homomorphism $\Phi_F : FM(X) \to G$(resp. $\Phi_A : AM(X) \to G$) such that $\Phi_F|_X = \Phi$ (resp. $\Phi_A|_X = \Phi$).

Definition 2.9 (M.N. Graev [9])**.** Let X be a completely regular space with a distinguished point e. A Hausdorff topological group $FG(X)$ (a Hausdorff abelian topological group $AG(X)$) is called a free topological group in the sense of Graev (a free abelian topological group in the sense of Graev) of the space X if the following conditions are fulfilled:

1) X is a subspace of $FG(X)$ (resp. of $AG(X)$);

2) X algebraically generates $FG(X)$ (resp. $AG(X)$);

3) for each continuous mapping $\varphi : X \to Q$ of the space X into an arbitrary topological group (an arbitrary topological abelian group) Q, which sends the point e of X to the unit (to the zero element) of the group Q, there exists a continuous homomorphism $\Phi_F : FG(X) \to Q$ (resp. $\Phi_A : AG(X) \to Q$) such that $\Phi_F|_X = \varphi$ (resp. $\Phi_A|_X = \varphi$).

A. A. Markov and M. N. Graev showed ([11], [9]) the existence and uniqueness (in the corresponding categories) of the groups $FM(X)$, $AM(X)$, $FG(X)$ and $AG(X)$.

3. On the Zero-Dimensionality by Modulo Class of Topological Spaces Topological Products, Free Topological Groups and Semigroups

The following theorem is valid:

Theorem 3.1. *If $Z = X \times Y$, where X and Y are topological spaces, Y is a nontrivial connected compact and $\mathit{in}\,(X \times Y, \mathfrak{M}) = 0$, where $\mathfrak{M}$ is a weakly stable class, then $X \in loc\,\mathfrak{M}$.*

Proof. Let us consider a point $x \in X$ and an open neighborhood O_x of x. Let U be some open in Y set with $U \neq \varnothing$ and $U \neq Y$. Take some point $y \in U$. Let $z = (x, y)$. Denote $U_z = O_x \times U \subseteq W_x$. Since $\mathit{in}\,(X \times Y, \mathfrak{M}) \leq 0$, there exists open H_z such that $z \in H_z \subseteq [H_z] \subseteq U_z$ and $\mathit{Fr}_{\,Z} H_z \in \mathfrak{M}$. Denote $T_x = \mathrm{Pr}_X(H_z)$, where $\mathrm{Pr}_X : X \times Y \to X$ is the projection. Obviously, $x \in T_x \subseteq [T_x]_X \subseteq O_x$. Show that $T_x \subseteq \mathrm{Pr}_X(\mathit{Fr}_{\,Z} H_z)$. Indeed, take any $x' \in T_x$ and consider the set $\{x'\} \times Y = A_{x'}$. Denote $A_{x'} \cap H_z = B_{x'}$. Show that $B_{x'} \neq \varnothing$. Indeed, if $B_{x'} = \varnothing$, then $\mathrm{Pr}_X(B_{x'}) = \varnothing$. But $x' \in \mathrm{Pr}_X(B_{x'})$, hence, it is not empty. On the other hand, since H_z is open in Z, the set $B_{x'}$ is open in $A_{x'}$. Now let us show that 1) $\mathit{Fr}_{\,A_{x'}} B_{x'} \subseteq \mathit{Fr}_{\,Z} H_z$ and 2) $\mathit{Fr}_{\,A_{x'}} B_{x'} \neq \varnothing$.

First let us show 1). $\mathit{Fr}_{\,A_{x'}} B_{x'} = [B_{x'}]_{A_{x'}} \backslash B_{x'}$, where $[B_{x'}]_{A_{x'}} = [B_{x'}]_Z$ (because $A_{x'}$ is closed in Z). But $[B_{x'}]_Z \subseteq [A_{x'}]_Z \cap [H_z]_Z = A_{x'} \cap [H_z]_Z$ (here we use the inclusion $[A \cap B] \subseteq [A] \cap [B]$ and the fact that $A_{x'}$ is closed in Z).

Furthermore $[B_{x'}]_Z \backslash B_{x'} \subseteq (A_{x'} \cap [H_z]_Z) \backslash B_{x'}$ (here we use the following: if $A \subseteq B$, then for any C we have $A \backslash C \subseteq B \backslash C$). Let us show that $(A_{x'} \cap [H_z]_Z) \backslash (A_{x'} \cap H_z) \subseteq [H_z]_Z \backslash H_z$ (i.e., $(A_{x'} \cap [H_z]_Z) \backslash B_{x'} \subseteq [H_z]_Z \backslash H_z$). Indeed if $x'' \in A_{x'} \cap [H_z]_Z \backslash A_{x'} \cap H_z$, then $x'' \in A_{x'} \cap [H_z]_Z$ and $x'' \notin A_{x'} \cap H_z$. Then $x'' \in A_{x'}$, $x'' \in [H_z]_Z$ and $x'' \notin H_z$. Hence, in particular, $x'' \in [H_z]_Z \backslash H_z$. Thus $[B_{x'}]_Z \backslash B_{x'} \subseteq [H_z]_Z \backslash H_z$. So we have $\mathit{Fr}_{\,Ax'} B_{x'} = [B_{x'}]_{Ax'} \backslash B_{x'} = [B_{x'}]_Z \backslash B_{x'} \subseteq [H_z]_Z \backslash H_z = \mathit{Fr}_{\,Z} H_z$. Thus 1) is proved.

Now let us prove 2). Let us consider the mapping $\pi_{x'} : A_{x'} \to Y$ determined as follows: $\pi_{x'}(x', y) = y$. This mapping is one-to-one and surjective. The mapping $\pi_{x'}$ is continuous because it is the restriction of the continuous mapping. Since Y is compact and $\{x'\} \times Y = A_{x'}$, $A_{x'}$ is compact and so the mapping $\pi_{x'}$ is a homeomorphism (since continuous and one-to-one mapping from a compact space onto a Hausdorff space is a homeomorphism). If $Fr_{A_x} B_{x'} = \varnothing$, then $B_{x'}$ is closed and open in $A_{x'}$. Let us note that $\pi_{x'}(B_{x'}) \subseteq U$. Since $\pi_{x'}$ is a homeomorphism, the set $\pi_{x'}(B_{x'})$ is closed and open in Y. But this is impossible because $\pi_{x'}(B_{x'}) \subset U \neq Y$, $\pi_{x'}(B_{x'}) \neq \varnothing$ and Y is connected. Therefore our assumption is false. So $Fr_{A_x} B_{x'} \neq \varnothing$.

Thus we see that for every $x' \in T_x$ there exists a point $z' \in Fr_z H_z$ such that $x' = \mathrm{Pr}_x(z')$. This means that $T_x \subseteq \mathrm{Pr}_X(Fr_Z H_z)$. H_z is dense in $[H_z]$. Therefore $T_x = \mathrm{Pr}_X(H_z)$ is dense in $\mathrm{Pr}_X([H_z]_Z)$. Hence, $[T_x]_X = \mathrm{Pr}_X([H_z]_Z)$. But since Y is compact, $\mathrm{Pr}_X : X \times Y \to X$ is the closed mapping, i.e., $\mathrm{Pr}_X([H_z]_Z)$ is closed in X. On the other hand, $Fr_Z H_z \subseteq [H_z]_Z$. Thus we have $T_x \subseteq \mathrm{Pr}_X(Fr_Z H_z) \subseteq \mathrm{Pr}_X([H_z]_Z)$. But $[T_x]_X = \mathrm{Pr}_X([H_z]_Z)$, so $[T_x]_X \subseteq [\mathrm{Pr}_X(Fr_Z H_z)]_X \subseteq [\mathrm{Pr}_X[H_z]_Z]_X$. Thus $[T_x]_X = [\mathrm{Pr}_X(Fr_Z H_z)]_X = \mathrm{Pr}_X(Fr_Z H_z)$. Since $\mathrm{Pr}_X : X \times Y \to X$ is the closed mapping and $Fr_Z H_z \in \mathfrak{M}$, by weak stability, $\mathrm{Pr}_X(Fr_Z H_z) \in \mathfrak{M}$. So we have found a neighborhood T_x of the point x such that $T_x \subset [T_x]_X \subset O_x$ and $[T_x]_X \in \mathfrak{M}$. Thus $X \in loc\,\mathfrak{M}$. □

Corollary 3.1. *If $Z = X \times Y$ where X and Y are topological spaces, Y is a connected compact and* in $(X \times Y, \mathfrak{M}_{con}) = 0$, *then* $X \in loc\,\mathfrak{M}_{con}$.

Corollary 3.2. *If $Z = X \times Y$ where X and Y are topological spaces, Y is a connected compact and* in $(X \times Y, \mathfrak{M}_{ps}) = 0$, *then* $X \in loc\,\mathfrak{M}_{ps}$.

Corollary 3.3. *If $Z = X \times Y$ where X and Y are topological spaces, Y is a connected compact and* in $(X \times Y, \mathfrak{M}_{cor}) = 0$, *then* $X \in loc\,\mathfrak{M}_{cor}$.

Corollary 3.4. *If $Z = X \times Y$, where X and Y are topological spaces, Y is a connected compact and* in $(X \times Y, \mathfrak{M}_{par}) = 0$, *then* $X \in loc\,\mathfrak{M}_{par}$.

Corollary 3.5. *If $Z = X \times Y$, where X and Y are topological spaces, Y is a connected compact and* in $(X \times Y, \mathfrak{M}_{pp}) = 0$, *then* $X \in loc\,\mathfrak{M}_{pp}$.

Corollary 3.6. *If $Z = X \times Y$, where X and Y are topological spaces, Y is a connected compact and* in $(X \times Y, \mathfrak{M}_{c}) = 0$, *then* $X \in loc\,\mathfrak{M}_{c}$.

As it is shown in [1], the following theorem holds.

Theorem 3.2. *If $X \times Y$ is a topological product of the spaces X and Y such that* in $(X \times Y, \mathfrak{M}) \leq 0$, *where $\mathfrak{M}$ is a stable class and* ind $X > 0$, *then* $Y \in loc\,\mathfrak{M}$.

In the sequel we shall need the following propositions which are easy to prove.

Proposition 3.1. *If $\mathfrak{M}$ is a c-monotone class and $X \in loc\,\mathfrak{M}$, then* in $(X, \mathfrak{M}) \leq 0$.

Proof. Let $x \in X$ and O_x be some open neighborhood of x in X. Since $X \in \mathit{loc}\,\mathfrak{M}$, there is an open neighborhood V_x such that $x \in V_x \subseteq [V_x]_X \subseteq O_x$ and $[V_x]_X \in \mathfrak{M}$. But $\mathit{Fr}_X V_x \subseteq [V_x]_X$ and since $[V_x]_X \in \mathfrak{M}$ and $\mathit{Fr}_X V_x$ is closed in X (and therefore in $[V_x]_X$ too), then, by c-monotonity, $\mathit{Fr}_X V_x \in \mathfrak{M}$, which means that $\mathit{in}\,(X, \mathfrak{M}) \le 0$. □

Proposition 3.2. *If $\mathfrak{M}$ is a multiplicative class, then the class $\mathit{loc}\,\mathfrak{M}$ is also multiplicative.*

Proof. $X \in \mathit{loc}\,\mathfrak{M}$ means that for every point x of X and for every its open neighborhood O_x there exists an open neighborhood V_x of x such that $x \in V_x \subseteq [V_x]_X \subseteq O_x$ and $[V_x]_X \in \mathfrak{M}$. Also, $Y \in \mathit{loc}\,\mathfrak{M}$ means that for every point $y \in Y$ and any its open neighborhood O_y there is an open neighborhood V_y of y such that $y \in V_y \subseteq [V_y]_Y \subseteq O_y$ and $[V_y]_Y \in \mathfrak{M}$. Denote $Z = X \times Y$. Consider the point $z = (x, y) \in Z$ and its open neighborhood $O_z = O_x \times O_y$. We have $z = (x, y) \in V_x \times V_y \subseteq [V_x \times V_y]_Z = [V_x]_X \times [V_y]_Y \subseteq O_x \times O_y$. Since $\mathfrak{M}$ is multiplicative, $[V_x]_X \times [V_y]_Y \in \mathfrak{M}$, i.e., $[V_x \times V_y]_Z \in \mathfrak{M}$. Since the point $z = (x, y)$ was selected arbitrarily in $(X \times Y)$ and since the family of neighborhoods of the form $O_z = O_x \times O_y$ is the base of the space $X \times Y$ at the point $z = (x, y)$, then $X \times Y \in \mathit{loc}\,\mathfrak{M}$, i.e., $\mathit{loc}\,\mathfrak{M}$ is multiplicative. □

Proposition 3.3. *If $\mathfrak{M}$ is a c-monotone class, A is closed in X and $\mathit{in}\,(X, \mathfrak{M}) \le 0$, then $\mathit{in}\,(A, \mathfrak{M}) \le 0$.*

Proof. Let us consider a point $x \in A$ and an open in A neighborhood O_x of x. Then there exists an open neighborhood O'_x of x in X such that $O'_x \cap A = O_x$. Then there is V'_x, open in X, such that $x \in V'_x \subseteq O'_x$ and $\mathit{Fr}_X V'_x \in \mathfrak{M}$. Let us consider $V'_x \cap A = V_x$.

Clearly, V_x is open in A, $V_x \subseteq O'_x$ and $\mathit{Fr}_A V_x \subseteq \mathit{Fr}_X V'_x$. Furthermore $\mathit{Fr}_A V_x$ is closed in $\mathit{Fr}_X V'_x$. Since $\mathit{Fr}_X V'_x \in \mathfrak{M}$, by c-monotonity, $\mathit{Fr}_A V_x \in \mathfrak{M}$. Thus we obtain that for every $x \in A$ and any its neighborhood O_x there is an open neighborhood V_x of x such that $x \in V_x \subseteq O_x$ and $\mathit{Fr}_A V_x \in \mathfrak{M}$. This means that $\mathit{in}\,(A, \mathfrak{M}) \le 0$. □

The following is true.

Theorem 3.3. *Let $\mathfrak{M}$ be a stable multiplicative class. Then the inequality, $\mathit{in}\,(X \times Y, \mathfrak{M}) \le 0$ takes place if and only if one of the following four conditions is fulfilled:*
1) $\mathit{ind}\,X \le 0$, $\mathit{ind}\,Y \le 0$;
2) $X \in \mathit{loc}\,\mathfrak{M}$, $Y \in \mathit{loc}\,\mathfrak{M}$;
3) $X \in \mathit{loc}\,\mathfrak{M}$, $\mathit{ind}\,X = 0$ *and* $\mathit{in}\,(Y, \mathfrak{M}) \le 0$;
4) $Y \in \mathit{loc}\,\mathfrak{M}$, $\mathit{ind}\,Y = 0$ *and* $\mathit{in}\,(X, \mathfrak{M}) \le 0$.

Proof. Necessity. Let us show that if $\mathit{in}\,(X \times Y, \mathfrak{M}) \le 0$, then one of the conditions given above is fulfilled. Let us consider four possible cases:
1. $\mathit{ind}\,X = 0$, $\mathit{ind}\,Y = 0$;
2. $\mathit{ind}\,X > 0$, $\mathit{ind}\,Y > 0$;
3. $\mathit{ind}\,X > 0$, $\mathit{ind}\,Y = 0$;

4. $ind\, X = 0$, $ind\, Y > 0$.

1. If $ind\, X = 0$ and $ind\, Y = 0$, then there is nothing to prove.

2. If $ind\, X > 0$ and $ind\, Y > 0$, then, by Theorem 3.2, $Y \in loc\, \mathfrak{M}$ and $X \in loc\, \mathfrak{M}$.

3. If $ind\, X > 0$ and $ind\, Y = 0$, then, by Theorem 3.1, $Y \in loc\, \mathfrak{M}$ and $ind\, Y = 0$.

4. The proof is similar to the proof in case 3.

Sufficiency. 1) If $ind\, X \leq 0$, $ind\, Y \leq 0$, then $ind\,(X \times Y) \leq 0$ (see [7]). Then since (by c-monotonity) $\varnothing \in \mathfrak{M}$, we have $in\,(X \times Y, \mathfrak{M}) \leq 0$.

2) If $X \in loc\, \mathfrak{M}$ and $Y \in loc\, \mathfrak{M}$, then, by Proposition 3.2, $X \times Y \in loc\, \mathfrak{M}$. Then by Proposition 3.1, we have: $in\,(X \times Y, \mathfrak{M}) \leq 0$. Suppose $X \in loc\, \mathfrak{M}$, $ind\, X = 0$ and $in\,(Y, \mathfrak{M}) \leq 0$. Let us show that $in\,(X \times Y, \mathfrak{M}) \leq 0$.

3) Suppose $z = (x, y)$ is any point of the space $X \times Y$ and U_z is any neighborhood of this point. Then there is a neighborhood O_x of x and there is a neighborhood O_y of y such that $z \in O_x \times O_y \subseteq U_z$. Since $X \in loc\, \mathfrak{M}$, there exists V_x such that $x \in V_x \subseteq [V_x] \subseteq O_x$ and $[V_x]_X \in \mathfrak{M}$. Since $ind\, X = 0$, there is W_X such that $x \in W_x \subseteq V_x \subseteq [V_x] \subseteq O_x$ and W_X is closed and open in X (i.e., $Fr_X W_x = \varnothing$). Thus W_x is closed and open in $[V_x]_X$ too. Since $\mathfrak{M}$ is c-monotone, $W_x \in \mathfrak{M}$. So W_x is closed and open and $W_x \in \mathfrak{M}$. Since $in\,(Y, \mathfrak{M}) \leq 0$, there exists an open set W_y such that $y \in W_y \subseteq [W_y]_Y \subseteq O_y$ and $Fr_Y W_y \in \mathfrak{M}$.

Let us consider $W_z = W_x \times W_y$. Then $z \in W_z = W_x \times W_y \subseteq O_x \times O_y \subseteq U_z$ and $Fr_Z W_z = ([W_x]_X \times Fr_Y W_y) \cap (Fr_X W_x \times [W_y]_Y)$. Since $Fr_X W_x = \varnothing$, $(Fr_X W_x \times [W_y]_Y) = \varnothing$, i.e., $Fr_Z W_z = [W_x]_X \times Fr_Y W_y = W_x \times Fr_Y W_y$ because $[W_x]_X = W_x$ (since it is closed and open). But $W_x \in \mathfrak{M}$ and $Fr_Y W_y \in \mathfrak{M}$. Since $\mathfrak{M}$ is multiplicative, we have $W_x \times Fr_Y W_y$. So $Fr_Z W_z \in \mathfrak{M}$. Since the point x and its open neighborhood O_z were chosen arbitrarily, $in\,(X \times Y, \mathfrak{M}) \leq 0$.

The fourth case is similar to the third one. □

Corollary 3.7. *$in\,(X \times Y, \mathfrak{M}_c) \leq 0$ if and only if one of the following four conditions is fulfilled:*

1) *$ind\, X = 0$, $ind\, Y = 0$;*

2) *$X \in loc\, \mathfrak{M}_c X \in loc\, \mathfrak{M}_c$;*

3) *$X \in loc\, \mathfrak{M}_c$, $ind\, X = 0$ and $in\,(Y, \mathfrak{M}_c) \leq 0$;*

4) *$Y \in loc\, \mathfrak{M}_c$, $ind\, Y = 0$ and $in\,(X, \mathfrak{M}_c) \leq 0$.*

Corollary 3.8. *$in\,(X \times Y, \mathfrak{M}_{mc}) \leq 0$ if and only if one of the following four conditions is fulfilled:*

1) *$ind\, X = 0$, $ind\, Y = 0$;*

2) *$X \in loc\, \mathfrak{M}_{mc}$, $X \in loc\, \mathfrak{M}_{mc}$;*

3) *$X \in loc\, \mathfrak{M}_{mc}$, $ind\, X = 0$ and $in\,(Y, \mathfrak{M}_{mc}) \leq 0$;*

4) *$Y \in loc\, \mathfrak{M}_{mc}$, $ind\, Y = 0$ and $in\,(X, \mathfrak{M}_{mc}) \leq 0$.*

Corollary 3.9. *$in\,(X \times Y, \mathfrak{M}_{dc}) \leq 0$ if and only if one of the following four conditions is fulfilled:*

1) *$ind\, X = 0$, $ind\, Y = 0$;*

2) *$X \in loc\, \mathfrak{M}_{dc}$, $X \in loc\, \mathfrak{M}_{dc}$;*

3) $X \in loc\,\mathfrak{M}_{dc}$, $ind\,X = 0$ *and* $in\,(Y, \mathfrak{M}_{dc}) \leq 0$;
4) $Y \in loc\,\mathfrak{M}_{dc}$, $ind\,Y = 0$ *and* $in\,(X, \mathfrak{M}_{dc}) \leq 0$.

Corollary 3.10. *$in\,(X \times Y, \mathfrak{M}_{cor}) \leq 0$ if and only if one of the following four conditions is fulfilled:*
1) $ind\,X = 0$, $ind\,Y = 0$;
2) $X \in loc\,\mathfrak{M}_{cor}$, $X \in loc\,\mathfrak{M}_{cor}$;
3) $X \in loc\,\mathfrak{M}_{cor}$, $ind\,X = 0$ *and* $in\,(Y, \mathfrak{M}_{cor}) \leq 0$;
4) $Y \in loc\,\mathfrak{M}_{cor}$, $ind\,Y = 0$ *and* $in\,(X, \mathfrak{M}_{cor}) \leq 0$.

Corollary 3.11. *$in\,(X \times Y, \mathfrak{M}_{\aleph_0}) \leq 0$ if and only if one of the following four conditions is fulfilled:*
1) $ind\,X = 0$, $ind\,Y = 0$;
2) $X \in loc\,\mathfrak{M}_{\aleph_0}$, $X \in loc\,\mathfrak{M}_{\aleph_0}$;
3) $X \in loc\,\mathfrak{M}_{\aleph_0}$, $ind\,X = 0$ *and* $in\,(Y, \mathfrak{M}_{\aleph_0}) \leq 0$;
4) $Y \in loc\,\mathfrak{M}_{\aleph_0}$, $ind\,Y = 0$ *and* $in\,(X, \mathfrak{M}_{\aleph_0}) \leq 0$.

Corollary 3.12. *$in\,(X \times Y, \mathfrak{M}_{cn}) \leq 0$ if and only if one of the following four conditions is fulfilled:*
1) $ind\,X = 0$, $ind\,Y = 0$;
2) $X \in loc\,\mathfrak{M}_{cn}$, $X \in loc\,\mathfrak{M}_{cn}$;
3) $X \in loc\,\mathfrak{M}_{cn}$, $ind\,X = 0$ *and* $in\,(Y, \mathfrak{M}_{cn}) \leq 0$;
4) $Y \in loc\,\mathfrak{M}_{cn}$, $ind\,Y = 0$ *and* $in\,(X, \mathfrak{M}_{cn}) \leq 0$.

Based on the obtained results, we determine the structure of free topological semigroups (in the sense of Marxen) and free topological groups (in the sense of Markov [11] and Graev [9]) in the case where these objects are zero-dimensional modulo a certain class.

Theorem 3.4. *Let $\mathfrak{M}$ be a stable multiplicative class. Then the space $S(X)$ of free topological semigroups of the space X has $in\,(S(X), \mathfrak{M}) \leq 0$ if and only if either $ind\,X = 0$ or $X \in loc\,\mathfrak{M}$.*

Proof. Marxen showed in [12] that the space of the topological semi-group $S(X)$ of any space X is of the form $S(X) = \bigvee\limits_{n=1}^{\infty} X^n$, where $\bigvee$ is a symbol for a topological sum and X^n is the n^{th} power of X.

As is known, if $ind\,X = 0$, then for every $n \geq 1$ we have $ind\,X^n = 0$, but the topological sum of zero-dimensional (in sense of ind), spaces is still zero-dimensional (in sense of ind). Thus $ind\,S(X) = 0$. Since $\mathfrak{M}$ is a stable class, we have $in\,(S(X), \mathfrak{M}) \leq 0$. If $X \in loc\,\mathfrak{M}$, then for every $n \geq 1$ (by multiplicity of $\mathfrak{M}$) we have $X^n \in loc\,\mathfrak{M}$. Now let us show that $S(X) \in loc\,\mathfrak{M}$. Suppose $x \in S(X)$ is a point chosen arbitrarily and O_x is any its neighborhood in $S(X)$. Then there is unique n such that $x \in X^n$. Let us consider $O'_x = O_x \cap X^n$. Since $X^n \in loc\,\mathfrak{M}$, there is a neighborhood V'_x such that $x \in V'_x \subseteq [V'_x]_{X^n} \subseteq O'_x$ and $[V'_x]_X \in \mathfrak{M}$. V'_x is open in X^n and X^n is open in $S(X)$. So V'_x is open in $S(X)$. Thus we have found an open neighborhood V_x in $S(X)$ such that $x \in V'_x \subseteq [V'_x]_{X^n} = [V'_x]_{S(X)} \subseteq O'_x \subseteq O_x$ and $[V'_x]_{S(X)} \in \mathfrak{M}$. Consequently, $S(X) \in loc\,\mathfrak{M}$. Therefore, by Proposition 3.1, $in\,(S(X), \mathfrak{M}) \leq 0$.

And vice versa suppose $in\,(S(X),\mathfrak{M}) \le 0$. We have to show that either $ind\,X = 0$ or $X \in loc\,\mathfrak{M}$. Since $S(X) = \bigvee_{n=1}^{\infty} X^n$, $X \times X$ is the closed subset of $S(X)$. So, by Proposition 3.3, $in\,(X \times X, \mathfrak{M}) \le in\,(S(X),\mathfrak{M}) \le 0$. Thus $in\,(X \times X, \mathfrak{M}) \le 0$. Since $\mathfrak{M}$ is a stable multiplicative class, by Theorem 3.2, either $ind\,X = 0$ or $X \in loc\,\mathfrak{M}$, which was to be proved. □

Definition 3.1. Let $\mathfrak{M}$ be a topologically closed class. The uncountable space X is called $\mathfrak{M}$-dispersed if for every uncountable and closed subset B we have $AM(B) \notin loc\,\mathfrak{M}$, where $AM(B)$ is a free abelian topological group of B in the sense of Markov.

Remark 3.1. One can show that every uncountable compact space is $\mathfrak{M}_c$, $\mathfrak{M}_{dc}$-, $\mathfrak{M}_{mc}$-, $\mathfrak{M}_{\aleph_0}$-dispersed (see, e.g., [6]).

Theorem 3.5. *Let X be an uncountable compact and $\mathfrak{M}$-dispersed space, where $\mathfrak{M}$ is a stable and multiplicative class. Then the following conditions are equivalent:* 1) $ind\,X \le 0$, 2) $ind\,AG(X) \le 0$, 3) $ind\,AM(X) \le 0$, 4) $in\,(AG(X),\mathfrak{M}) \le 0$, 5) $in\,(AM(X),\mathfrak{M}) \le 0$, 6) $Ind\,AM(X) \le 0$, 7) $Ind\,AG(X) \le 0$.

Proof. The proof will follow the scheme: 1)⇒2)⇒3)⇒4)⇒ 5)⇒6)⇒7)⇒1).

First let us show that 1)⇒2). Since X is a compact space, $AG(X)$ is a K_ω-space (see [9]), i.e., $AG(X) = \bigcup_{n=1}^{\infty} A_i(X)$, where $A_i(X)$ symbolizes a subspace of the group $AG(X)$, which consists of words the length of which is $\le i$. It is known (see [9]) that for every $n = 1, 2, \dots$, $A_n(X)$ is a compact subspace of $AG(X)$. Let us show by induction that $dim\,A_n(X) = 0$. Indeed, when $n = 1$ we have $A_1(X) = X \bigvee (-X) \bigvee \{0\}$. Since $ind\,X = 0$, we have $dim\,X = 0$. Since $(-X)$ is homeomorphic to X, we have $dim\,(-X) = 0$, i.e., $dim\,A_1(X) = dim\,X = ind\,X = 0$. Suppose for every $p \le n-1$ it is known that $dim\,A_p(X) = 0$. Let us discuss $A_n(X)$. It is obvious that $A_n(X) = A_{n-1}(X) \cup (A_n(X) \setminus A_{n-1}(X))$. Since $A_{n-1}(X)$ is closed in $AG(X)$ (see [9]), it is closed in $A_n(X)$. Therefore $A_n(X) \setminus A_{n-1}(X)$ is open in $A_n(X)$. Now let us show that $ind\,(A_n(X) \setminus A_{n-1}(X)) \le 0$. Suppose $\alpha \in A_n(X) \setminus A_{n-1}(X)$. This means that α is a non-cancellable word of length n, i.e., $\alpha = \varepsilon_1 x_1 + \dots + \varepsilon_n x_n$, where ε_i is equal to 1 or -1 for every $i = 1, 2, \dots, n$.

Let us consider any open neighborhood O_α of the point α in $A_n(X)$. According to Arkhangelskii–Joiner's lemma (see, e.g., [3]) there exist closed and open subsets $U_1, U_2, \dots, U_n$ which satisfy the following:

1. $x_i \in U_i$ for every $i = 1, 2, \dots, n$;
2. $U_i \cap U_j = \varnothing$ as soon as $i \ne j$ $(i, j = 1, 2, \dots, n)$;
3. the set $\varepsilon_1 U_1 + \varepsilon_2 U_2 + \dots + \varepsilon_n U_n$ is an open neighborhood of the point α in $A_n(X)$;
4. $\varepsilon_1 U_1 + \varepsilon_2 U_2 + \dots + \varepsilon_n U_n \subseteq O_\alpha$ because for every i, U_i is compact. Therefore $\varepsilon_1 U_1 + \varepsilon_2 U_2 + \dots + \varepsilon_n U_n$ is an open compact neighborhood of the point $\alpha \in A_n(X) \setminus A_{n-1}(X)$.

Since the point α and its neighborhood O_α were chosen arbitrarily, then $ind\,(A_n(X)\backslash A_{n-1}(X)) \leq 0$. Furthermore, for any closed subset F of A_n, where $A_n(X) \subseteq A_n(X)\backslash A_{n-1}(X)$, we have $dim\,F \leq ind\,F \leq ind\,(A_n(X)\backslash A_{n-1}(X)) \leq 0$. Then, by the inductive assumption and according to Dowker's theorem [7], we obtain $dim\,A_n(X) \leq 0$. Since $AG(X) = \bigcup\limits_{n=1}^{\infty} A_i(X)$, we have $dim\,AG(X) \leq 0$. Thus $ind\,AG(X) \leq 0$.

1)⇒2) is proved.

2)⇒3). According to F. Ward's theorem $AM(X) \approx AG(X) \times Z$, where Z is the set of integers with discrete topology. Therefore $AM(X) = \bigvee\limits_{n=1}^{\infty} AG(X)_i$ and thus 2)⇒3).

3)⇒4). Since $\mathfrak{M}$ is a stable class, $\varnothing \in \mathfrak{M}$ and according to Nishiura's Theorem 1.8 from [14] we have $in\,(AM(X), \mathfrak{M}) \leq ind\,AM(X) = ind\,AG(X) = 0$. 3)⇒4) is proved.

4)⇒5) follows also from [16]. Indeed, $in\,(AG(X), \mathfrak{M}) = 0 \Rightarrow in\,(AM(X), \mathfrak{M}) = in\,(AG(X) \times Z, \mathfrak{M}) = in\,(\bigvee\limits_{n=1}^{\infty} AG(X)_i, \mathfrak{M}) = 0$.

5)⇒6). We have to prove that $in\,(AM(X), \mathfrak{M}) \leq 0 \Rightarrow Ind\,AM(X) \leq 0$. Suppose, on the contrary, that $Ind\,AM(X) > 0$. Then $ind\,X > 0$ because if $ind\,X \leq 0$, then $ind\,AG(X) \leq 0$ and $ind\,AM(X) \leq 0$. Since $AM(X)$ is σ-compact and thus, finally compact, we have $dim\,AM(X) \leq Ind\,AM(X) = 0$. This contradicts our assumption. Therefore $ind\,X > 0$. Since X is compact and $ind\,X > 0$, we have $dim\,X > 0$. By Proposition 2.6.3 from [17], there are compact subspaces F_1 and F_2 of the compact X such that $F_1 \cap F_2 = \varnothing$, $dim\,F_1 > 0$, $dim\,F_2 > 0$. Let us consider the compact space $F = F_1 \cup F_2$. Since $F_1 \cap F_2 = \varnothing$, the compact F is homeomorphic to the topological sum of F_1 and F_2, i.e., $F = F_1 \bigvee F_2$.

According to Ward's theorem from [16], the free topological group $AM(X)$ is topologically isomorphic to $AG(X) \times Z$. Since $in\,(AM(X), \mathfrak{M}) \leq 0$ we have $in\,(AG(X), \mathfrak{M}) \leq 0$. Denote by $G(F)$ the subgroup of $AG(X)$ which is generated algebraically by the compact subspace $F \subseteq X$. It is well-known that [9] $G(F)$ is closed in $AG(X)$ and $G(F)$ is topologically isomorphic to $AG(F)$.

Thus $in\,(G(F), \mathfrak{M}) = in\,(AG(F), \mathfrak{M}) \leq 0$.

By Ward's theorem, $AM(F) = AG(F) \times Z$, i.e., $AM(F) = \bigvee\limits_{n=1}^{\infty} AG(F)_i$, where for every $i = 1, 2, \dots,$ $AG(F)_i$ is homeorphic to $AG(F)$. It is easy to show that we have $in\,(AM(F), \mathfrak{M}) \leq 0$. But $F = F_1 \bigvee F_2$. It is known that $AM(F_1 \bigvee F_2) \cong AM(F_1) \times AM(F_2)$. Then $in\,(AM(F_1) \times AM(F_2), \mathfrak{M}) \leq 0$. Consequently, by Theorem 3.3, one of the following four cases will take place:

1. $ind\,AM(F_1) \leq 0$, $ind\,AM(F_2) \leq 0$,
2. $AM(F_1) \in loc\,\mathfrak{M}$, $AM(F_2) \in loc\,\mathfrak{M}$,
3. $AM(F_1) \in loc\,\mathfrak{M}$, $ind\,AM(F_1) = 0$ and $in\,AM(F_2), \mathfrak{M}) \leq 0$,
4. $AM(F_2) \in loc\,\mathfrak{M}$, $ind\,AM(F_2) = 0$ and $in\,AM(F_1), \mathfrak{M}) \leq 0$.

But in accordance with the condition it is impossible for $AM(F_1)$ or $AM(F_2)$ to belong to $loc\,\mathfrak{M}$. Therefore only case 1) remains.

Since $F_1 \subseteq AM(F_1)$ and $F_2 \subseteq AM(F_2)$, we have $ind\,F_1 \leq 0$ and $ind\,F_2 \leq 0$. Then $dim\,F_1 \leq 0$ and $dim\,F_2 \leq 0$. But this contradicts $dim\,F_1 > 0$, $dim\,F_2 > 0$.

Thus our assumption that $Ind\,AM(X) > 0$ is false. So 5)⇒6) is proved.

6)⇒7) follows from Ward's theorem.

7)⇒1). Since $Ind\,AG(X) \leq 0$, we have $Ind\,X \leq 0$ because X is the closed subspace of $AG(X)$. Hence $ind\,X \leq 0$. □

Theorem 3.6. *Let X be an uncountable K_ω-space* [13] *and an $\mathfrak{M}$-dispersed space, where $\mathfrak{M}$ is a stable multiplicative class. Then the following conditions are equivalent:* 1) $ind\,X \leq 0$, 2) $ind\,FG(X) \leq 0$, 3) $ind\,FM(X) \leq 0$, 4) $in\,(FG(X), \mathfrak{M}) \leq 0$, 5) $in\,(FM(X), \mathfrak{M}) \leq 0$.

Proof. The proof will follow the scheme 1)⇒2)⇒3)⇒4)⇒5) ⇒1).

1)⇒2). Since X is a K_ω-space, there exists a K_ω-decomposition $\{X_i\}$ of X, i.e., there exists a system $\{X_i\}$, $i = 1, 2, \ldots$, of subspaces of the space X such that:

1. $X = \bigcup\limits_{i=1}^{\infty} X_i$;
2. X_i is compact for every $i = 1, 2, \ldots$;
3. $X_i \subseteq X_j$, for every $i < j$ $(i, j = 1, 2, \ldots)$;
4. the set A is closed in X if and only if $A \cap X_i$ is compact for every $i = 1, 2, \ldots$

(If $\{X_i\}$ is a K_ω-decomposition of X, we write $X \overset{\omega}{=} \bigcup\limits_{i=1}^{\infty} X_i$).

As we know (see [13]), if $X \overset{\omega}{=} \bigcup\limits_{i=1}^{\infty} X_i$, then $FG(X) \overset{\omega}{=} \bigcup\limits_{j=1}^{\infty} \bigcup\limits_{i=1}^{\infty} F_j(X_i)$, where $F_j(X_i)$ denotes the set of words generated by elements of X_i, the length of which is $\leq j$. It is clear that $F_j(X_i) \subseteq FG(X_i)$ for every $j = 1, 2, \ldots$. Let us show that $ind\,FG(X_i) = 0$ for every $i = 1, 2, \ldots$. We know that $F_n(X_i)$ is compact for every n. Let us show by induction that $dim\,F_n(X_i) = 0$ for every i . For $n = 1$ we have $dim\,X_i = 0$. Then $dim\,F_1(X_i) = 0$, where $F_1(X_i) = X_i \bigvee X_i^{-1} \bigvee \{1\}$. Suppose for every $p \leq n-1$ the assertion is already proved and that $dim\,F_p(X_i) = 0$. Let us consider $F_n(X_i) = F_{n-1}(X_i) \cup (F_n(X_i) \backslash F_{n-1}(X_i))$. Since $F_{n-1}(X_i)$ is closed in $FG(X_i)$, then it is also closed in $F_n(X_i)$. Thus $F_n(X_i) \backslash F_{n-1}(X_i)$ is open in $F_n(X_i)$. Let us show that $ind\,(F_n(X_i) \backslash F_{n-1}(X_i)) \leq 0$. Indeed, suppose $\alpha \in F_n(X_i) \backslash F_{n-1}(X_i)$. Thus $\alpha = x_1^{\varepsilon_1} x_2^{\varepsilon_2} \ldots x_n^{\varepsilon_n}$, where ε_i is equal to 1 or -1. Let us consider any open neighborhood O_α of the point α *in* $F_n(X_i)$. Then according to Arkhangelskii–Joiner's lemma (which is also true for the group $FG(X_i)$), there exist open compact sets $U_1, U_2, \ldots, U_n$ which satisfy the following conditions:

1. $x_i \in U_i$, $i = 1, 2, \ldots$;
2. $U_i \cap U_j = \varnothing$, $i \neq j$ $(i, j = 1, 2, \ldots)$;
3. the set $U_1^{\varepsilon_1} U_2^{\varepsilon_2} \ldots U_n^{\varepsilon_n}$ is an open neighborhood of the point α in $F_n(X_i)$;
4. $U_1^{\varepsilon_1} U_2^{\varepsilon_2} \ldots U_n^{\varepsilon_n} \subseteq O_a$.

Since U_i is compact for every $i = 1, 2, \ldots, n$, $U_1^{\varepsilon_1} U_2^{\varepsilon_2} \ldots U_n^{\varepsilon_n}$ is an open compact neighborhood of the point $\alpha \in F_n(X_i) \backslash F_{n-1}(X_i)$. Since the point α and its neighborhood O_α where chosen arbitrarily, we have $ind\,(F_n(X_i) \backslash F_{n-1}(X_i)) \le 0$.

For every compact $F \subseteq F_n(X_i) \backslash F_{n-1}(X_i)$ we have: $dim\,F \le 0$. Since according to the inductive assumption, $dim\,F_{n-1}(X_i) \le 0$, by Dowker's theorem (see[7]) we conclude that $dim\,F_n(X_i) \le 0$, i.e., for every i we have shown that $dim\,Fn(X_i) \le 0$. Hence $FG(X_i) \le 0$. Since $FG(X)$ is compact and $FG(X) = \bigcup\limits_{n=1}^{\infty} \bigcup\limits_{i=1}^{\infty} F_n(X_i)$, we have $dim\,FG(X) \le 0$. Thus $ind\,FG(X) \le 0$.

2)⇒3). Suppose $ind\,FG(X) = 0$. Then $ind\,X = 0$ because $X \subseteq FG(X)$. Let us consider the space $Y = X \bigvee \{p\}$, where $p \notin X$, which is a topological sum of the space X and the singleton $\{p\}$. It is clear that since X is the K_ω-space, we have $ind\,Y = 0$. So by means of 1)⇒2) we get $ind\,FG(Y) = 0$. But it is known (see [9]) that $FM(X)$ is topologically isomorphic to $FG(Y)$. Thus $ind\,FM(X) = 0$.

So $ind\,FM(X) = 0$. According to Morris' theorem (see [13]), the group $FM(X)$ is topologically isomorphic to the coproduct of the group $FG(X)$ and the discrete group Z of integers i.e., $FM(X) \cong FG(X) * Z$. Hence $FG(X)$ is a closed subspace of $FM(X)$. Thus $ind\,FG(X) = 0$ is proved.

3)⇒4). Since $\mathfrak{M}$ is the stable class and $\varnothing \in \mathfrak{M}$, according to Nishiura's theorem (see [14]), $in\,(FM(X), \mathfrak{M}) \le in\,(FM(X), \varnothing) = ind\,FM(X) \le 0$.

4)⇒5). Suppose $in\,(FM(X), \mathfrak{M}) \le 0$. We have to show that $in\,(FG(X), \mathfrak{M}) \le 0$. According to Morris' theorem, $FM(X) \cong FG(X) * Z$. Since $FG(X)$ is a topological group the space of which is a K_ω-space, it is complete in the sense of A. Weil (see [9]). Moreover, $FG(X)$ is a closed subgroup of $FM(X)$. Therefore, by c-monotonity of the function $in\,(X, P)$, we get $in\,(FG(X), \mathfrak{M}) \le 0$.

Now let us prove 5)⇒1). Suppose $in\,(FG(X), \mathfrak{M}) \le 0$. We have to show that $ind\,X = 0$.

Let us assume that the opposite is true, i.e., $ind\,X > 0$. Then $dim\,X > 0$. Since X is the K_ω-space, there exists its ω decomposition $X \stackrel{\omega}{=} \bigcup\limits_{i=1}^{\infty} X_i$, where each X_i is compact.

Suppose i_0 is a natural number such that $dim\,X_{i_0} > 0$ (if there is no such i_0 and for every i $dim\,X_i$ is equal to 0, then $dim\,X$ is also equal to 0).

Since X_{i_0} is compact, then, based on the well-known result that $dim\,X_{i_0}$ is equal to the supremum of dim dimensions of the components of X_{i_0}, there exists a connected compact $\phi^{i_0} \subseteq X_{i_0}$ such that $dim\,\phi^{i_0} > 0$.

According to Zambakhidze's Theorem 2.6.3 (see [17]) there exist subsets $F^{i_0(1)}$ and $F^{i_0(2)}$ closed in ϕ^{i_0} (and therefore in X too) such that $F^{i_0(1)} \cap F^{i_0(2)} = \varnothing$, $dim\,F^{i_0(1)} > 0$, $dim\,F^{i_0(2)} > 0$. Since ϕ^{i_0} is connected and compact, there exists a point $z = \phi^{i_0} \backslash F^{i_0(1)} \bigcup F^{i_0(2)}$ (otherwise ϕ^{i_0} would be decomposed into the sum of two separate closed subsets, which contradicts the connectedness of ϕ^{i_0}). Suppose $Y_{i_0} = F^{i_0(1)} \bigcup F^{i_0(2)}$ and $Y'_{i_0} = F^{i_0(1)} \bigcup F^{i_0(2)} \bigcup \{z\}$. Since $F^{i_0(1)} \bigcap F^{i_0(2)} = \varnothing$ and $z \notin F^{i_0(1)} \bigcup F^{i_0(2)}$, we have $Y_{i_0} = F^{(i_0(1)} \bigvee F^{i_0(2)}$, $Y'_{i_0} = F^{i_0(1)} \bigvee F^{i_0(2)} \bigvee \{z\}$. Obviously, Y_{i_0} and Y'_{i_0} are compact subspaces of the space X. Then according to Graev's theorem (see [9]), $FM(Y_{i_0})$ is topologically isomorphic to the group $FG(Y'_{i_0})$. Since Y'_{i_0} is a compact subspace of the K_ω-space, the subgroup $G(Y'_{i_0})$ of the group $FG(X)$ (which is generated algebraically from the compact Y'_{i_0}) is closed in $FG(X)$ and is topologically isomorphic to the free topological group $FG(Y'_{i_0})$. By the condition, $\mathit{in}\,(FG(X),\mathfrak{M}) \leq 0$. So $\mathit{in}\,(FG(Y'_{i_0}),\mathfrak{M}) \leq 0$ and thus $\mathit{in}\,(FG(Y_{i_0}),\mathfrak{M}) \leq 0$.

Furthermore, Y_{i_0} is homeomorphic to the topological sum of the spaces $F^{i_0(1)}$ and $F^{i_0(2)}$. According to Morris' theorem (see [13]) the group $FM(Y_{i_0})$ is topologically isomorphic to the free product $FM(F^{i_0(1)}) * FM(F^{i_0(2)})$ of topological groups $FM(F^{i_0(1)})$ and $FM(F^{i_0(2)})$. Thus $\mathit{in}\,(FM(F^{i_0(1)}) * FM(F^{i_0(2)}),\mathfrak{M}) \leq 0$. According to Ordman's theorem (see [15]), in the sequence $FM(F^{i_0(1)}) \times FM(F^{i_0(2)}) \xrightarrow{i} FM(F^{i_0(1)}) * FM(F^{i_0(2)}) \xrightarrow{\rho} FM(F^{i_0(1)}) \times FM(F^{i_0(2)})$, where i is a homeomorphic inclusion of $FM(F^{i_0(1)}) \times FM(F^{i_0(2)})$ into $FM(F^{i_0(1)}) * FM(F^{i_0(2)})$ and ρ is a continuous mapping of $FM(F^{i_0(1)}) * FM(F^{i_0(2)})$ onto $FM(F^{i_0(1)}) \times FM(F^{i_0(2)})$, the composition $i\rho$ is the identical mapping. Hence $i(FM(F^{i_0(1)}) \times FM(F^{i_0(2)}))$ is closed in $FM(F^{i_0(1)}) * FM(F^{i_0(2)})$. Therefore $\mathit{in}\,(FM(F^{i_0(1)} \times FM(F^{i_0(2)}),\mathfrak{M}) \leq 0$.

According to Theorem 3.3 $\mathit{in}\,(FM(F^{i_0(1)}) \times FM(F^{i_0(2)}),\mathfrak{M}) \leq 0$ is true if and only if one of the following conditions is fulfilled:

1. $\mathit{ind}\,FM(F^{i_0(1)}) = 0$, $\mathit{ind}\,FM(F^{i_0(2)}) = 0$;
2. $FM(F^{i_0(1)}) \in \mathit{loc}\,\mathfrak{M}$, $FM(F^{i_0(2)}) \in \mathit{loc}\,\mathfrak{M}$;
3. $FM(F^{i_0(1)}) \in \mathit{loc}\,\mathfrak{M}$, $\mathit{ind}\,FM(F^{i_0(1)}) = 0$ and $\mathit{in}\,(FM(F^{i_0(2)}),\mathfrak{M}) \leq 0$;
4. $FM(F^{i_0(2)}) \in \mathit{loc}\,\mathfrak{M}$, $\mathit{ind}\,FM(F^{i_0(2)}) = 0$ and $\mathit{in}\,(FM(F^{i_0(1)}),\mathfrak{M}) \leq 0$.

As we know (see [11]), abelian free topological groups $AM(F^{i_0(1)})$ and $AM(F^{i_0(2)})$ are topologically isomorphic to the factor groups $FM(F^{i_0(1)})/K_1$ and $FM(F^{i_0(2)})/K_2$, where K_1 and K_2 are commutants. The mappings

$$\pi_1 : FM(F^{i_0(1)}) \to FM(F^{i_0(1)})/K_1 = AM(F^{i_0(1)}),$$
$$\pi_2 : FM(F^{i_0(2)}) \to FM(F^{i_0(2)})/K_2 = AM(F^{i_0(2)})$$

are open. If $FM(F^{\alpha_0(1)}) \in \mathit{loc}\,\mathfrak{M}$ and $FM(F^{\alpha_0(2)}) \in \mathit{loc}\,\mathfrak{M}$, then $AM(F^{i_0(1)}) \in \mathit{loc}\,\mathfrak{M}$ and $AM(F^{i_0(2)}) \in \mathit{loc}\,\mathfrak{M}$, respectively. But this is impossible because the space X is $\mathfrak{M}$-dispersed. Thus we exclude cases 2), 3) and 4). So we have $\mathit{ind}\,FM(F^{i_0(1)}) = 0$ and $\mathit{ind}\,FM(F^{i_0(2)}) = 0$. On the other hand, we have $\mathit{dim}\,FM(F^{i_0(1)}) > 0$ and $\mathit{dim}\,FM(F^{i_0(2)}) > 0$ (since $\mathit{dim}\,F^{i_0(1)} > 0$ and $\mathit{dim}\,F^{i_0(2)} > 0$). So, $FM(F^{i_0(1)})$ and $FM(F^{i_0(2)})$ are σ-compacts and $\mathit{ind}\,FM(F^{i_0(1)}) > 0$, $\mathit{ind}\,FM(F^{i_0(2)}) > 0$. This is the contradiction. Thus our assumption that $\mathit{ind}\,X > 0$ is not true. Therefore $\mathit{ind}\,X \leq 0$. □

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