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ESTIMATES OF FOURIER COEFFICIENTS

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Abstract. Some well-known properties of the trigonometric system as well as of the Haar and Welsh systems are generalized to general orthonormal systems.

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1. Introduction

In the theory of functions an important place is occupied by generalization of properties of specific orthonormal series to general orthonormal systems.

Here we note only some of the authors who have significant results concerning the mentioned problems: Marcinkiewicz [1], Stechkin [2], Olevskii [3], Bochkarev [4], [5], Mitiagin [6], Kashin [7], McLaughlin [8].

It was proved that in many cases some properties of the well-known orthogonal systems are typical for general orthogonal systems (see, e.g., [2], [3], [4], [5]). However, not all properties of the well-known orthogonal systems are extending on general orthogonal systems. Therefore, in order to obtain well-known results for general orthogonal systems, we need to impose specific conditions on the given system.

2. Auxiliary Notation and Results

Let $\delta \in (0,1]$. If a function $f \in C([0,1])$, then its modulus of continuity is defined as follows

$$\omega(\delta, f) = \sup_{|h|\le\delta} \max_{0\le x\le 1-h} |f(x) - f(x+h)|,$$

where $0 < h \le 1$.

If a function $f \in L_2([0,1])$, then the integral modulus of continuity has the form

$$\omega_2(\delta, f) = \sup_{|h|\le\delta} \left(\int_0^{1-h} |f(x) - f(x+h)|^2 dx \right)^{\frac{1}{2}}.$$

We say that a function $f \in \operatorname{Lip}\alpha$ if $\omega(\delta, f) = O(\delta^\alpha)$ as $\delta \to 0$.

Let (φ_n) be an orthogonal system on $[0,1]$. Then the Fourier coefficients with respect to (φ_n) for the function $f \in L([0,1])$ are defined as follows

$$c_n(f) = \int_0^1 f(x)\varphi_n(x)\, dx, \quad n = 1, 2, \dots .$$

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The best approximation with respect to the system (φ_n) in the sense of $L_2([0,1])$ is defined by the following equality:

$$E_n^{(2)}(f) = \inf_{\{a_m\}} \left\| f(x) - \sum_{m=1}^{n} a_m \varphi_m(x) \right\|_2 .$$

If (φ_n) is a complete system on $[0,1]$, then

$$E_n^{(2)}(f) = \left(\sum_{k=n}^{\infty} c_k^2(f) \right)^{\frac{1}{2}} .$$

In the sequel we will denote by (ψ_n) either one of the orthonormal systems of Haar or Walsh (see, e.g., [9, p. 53, 54]), or the trigonometric system. For these systems the following results are valid. They are important for our purpose.

Theorem A. *If*

$$c_n(f) = \int_0^1 f(x)\psi_n(x)\, dx,$$

then the following relations are valid:
a) $c_n(f) = O\left(\omega\left(\frac{1}{n}, f\right)\right)$ *for every* $f \in C([0,1])$*;*
b) $c_n(f) = O\left(\omega_2\left(\frac{1}{n}, f\right)\right)$ *for every* $f \in L_2([0,1])$*.*

Theorem B. *For every function* $f \in L_2([0,1])$ *the relation*

$$E_n^{(2)} = O\left(\omega_2\left(\frac{1}{n}, f\right)\right)$$

holds.

The relation a) of Theorem A for the Haar system was proved by B. Golubov [10] and the relation b) was proved by P. Ul'yanov [11], for the Walsh system it was proved by Fine [12].

Theorem B for the Haar system was proved by Ul'yanov [11]. As to the trigonometric system, Theorems A and B are well-known (see, e.g., [13, p. 79]).

It is well-known that Theorems A and B are not valid for general orthogonal systems. In fact, the following theorem holds.

Theorem C. *Let* f *be a function from* $L_2([0,1])$ *and* (c_n) *be an arbitrary sequence from* ℓ_2 *satisfying the condition*

$$\sum_{n=1}^{\infty} c_n^2 = \int_0^1 f^2(x)\, dx.$$

Then there exists an orthonormal system (φ_n) *on* $[0,1]$ *such that*

$$c_n = \int_0^1 f(x)\varphi_n(x)\, dx, \quad n = 1, 2, \dots .$$

Theorem C is proved by A. Olevskii [3]. From this theorem follows that if $f \in L_2([0,1])$ and the numbers c_n satisfy the conditions

$$\text{a)}\ \sum_{n=1}^{\infty} c_n^2 = \int_0^1 f^2(x)\,dx \quad \text{and} \quad \text{b)}\ \lim_{n\to\infty} \frac{|c_n|}{\omega_2(\frac{1}{n}, f)} = +\infty,$$

then there exists an orthonormal system of functions (φ_n) such that

$$c_n = \int_0^1 f(x)\varphi_n(x)\,dx, \quad n = 1, 2, \dots .$$

Therefore there exists a function $f \in L_2([0,1])$ such that

$$\lim_{n\to\infty} \frac{|c_n(f)|}{\omega_2(\frac{1}{n}, f)} = +\infty.$$

The following propositions are valid (see [15]).

Lemma 1. *Let the function $f \in L_2([0,1])$ take finite values at every point of the interval $[0,1]$ and $\Phi \in L_2([0,1])$. Then the following equality is valid:*

$$\int_0^1 f(x)\Phi(x)\,dx = \sum_{i=1}^{n-1} \left(f\left(\frac{i}{n}\right) - f\left(\frac{i+1}{n}\right) \right) \int_0^{\frac{i}{n}} \Phi(x)\,dx + \sum_{i=1}^{n} \int_{\frac{i-1}{n}}^{\frac{i}{n}} \left(f(x) - f\left(\frac{i}{n}\right) \right) \Phi(x)\,dx + f(1) \int_0^1 \Phi(x)\,dx. \quad (1)$$

Lemma 2. *Let (φ_n) be an orthonormal system of functions on $[0,1]$. Let E_n denote the set of the natural numbers i $(i = 1, 2, \dots, n)$ for which there exists a point $t \in \left(\frac{i-1}{n}, \frac{i}{n}\right)$ such that*

$$\operatorname{sign} \int_0^t \varphi_n(x)\,dx \neq \operatorname{sign} \int_0^{\frac{i-1}{n}} \varphi_n(x)\,dx.$$

Then the inequality

$$\sum_{i \in E_n} \left| \int_0^{\frac{i}{n}} \varphi_n(x)\,dx \right| \leq \left(\int_0^1 \varphi_n^2(x)\,dx \right)^{\frac{1}{2}}$$

holds.

3. Main Results

Let (a_n) be a sequence of real numbers from ℓ_2, Introduce the following notation:

$$P_N^{(m)}(x) \equiv \sum_{k=N}^{N+m} a_k \varphi_k(x), \quad m = 1, 2, \dots ,$$
$$e_N \equiv \left(\sum_{k=N}^{\infty} c_k^2(f) \right)^{\frac{1}{2}},$$
$$H_N \equiv \sup_m \sum_{k=1}^{N-1} \left| \int_0^{\frac{k}{N}} P_N^{(m)}(x)\, dx \right|,$$
$$h_N \equiv \left(\sum_{k=N}^{\infty} a_k^2 \right)^{\frac{1}{2}}.$$

Theorem 1. *Let $f \in C([0,1])$ and (φ_n) be an orthonormal system of functions on $[0,1]$ satisfying $\int\limits_0^1 \varphi_n(x)\, dx = 0$, $n = 1, 2, \dots$.*

Then the relation

$$e_N = O\left(\omega\left(\frac{1}{N}, f \right) \right)$$

holds if and only if the condition

$$H_N = O(h_N) \quad \textit{for every sequence } (a_k) \in \ell_2$$

is fulfilled.

Proof. Sufficiency. If $c_n(f) = \int\limits_0^1 f(x)\varphi_n(x) dx$, then

$$\sum_{k=N}^{N+m} c_k^2(f) = \sum_{k=N}^{N+m} c_k(f) \int_0^1 f(x)\varphi_k(x)\, dx$$
$$= \int_0^1 f(x) \sum_{k=N}^{N+m} c_k(f)\varphi_k(x)\, dx = \int_0^1 f(x) P_N^{(m)}(x)\, dx. \tag{2}$$

Using the assertion of Lemma 1, we obtain

$$\int_0^1 f(x) P_N^{(m)}(x)\, dx = \sum_{i=1}^{N-1} \left(f\left(\frac{i}{N}\right) - f\left(\frac{i+1}{N}\right) \right) \int_0^{\frac{i}{N}} P_N^{(m)}(x)\, dx$$
$$+ \sum_{i=1}^{N} \int_{\frac{i-1}{N}}^{\frac{i}{N}} \left(f(x) - f\left(\frac{i}{N}\right) \right) P_N^{(m)}(x)\, dx + f(1) \int_0^1 P_N^{(m)}(x)\, dx. \tag{3}$$

Estimate the right-hand side of equality (3). We have

$$\begin{aligned}
&\left|\sum_{i=1}^{N-1}\left(f\left(\frac{i}{N}\right)-f\left(\frac{i+1}{N}\right)\right)\int_0^{\frac{i}{N}} P_N^{(m)}(x)\,dx\right| \\
&\quad\le \sum_{i=1}^{N-1}\left|f\left(\frac{i}{N}\right)-f\left(\frac{i+1}{N}\right)\right|\left|\int_0^{\frac{i}{N}} P_N^{(m)}(x)\,dx\right| \\
&\quad\le \omega\left(\frac{1}{N},f\right)\sum_{i=1}^{N-1}\left|\int_0^{\frac{i}{N}} P_N^{(m)}(x)\,dx\right| \le O(1)\,\omega\left(\frac{1}{N},f\right)h_N. \qquad (4)
\end{aligned}$$

Applying the Hölder inequality, we obtain

$$\begin{aligned}
&\left|\sum_{i=1}^{N-1}\int_{\frac{i-1}{N}}^{\frac{i}{N}}\left(f(x)-f\left(\frac{i}{N}\right)\right)P_N^{(m)}(x)\,dx\right| \\
&\quad\le \omega\left(\frac{1}{N},f\right)\sum_{i=1}^{N-1}\int_{\frac{i-1}{N}}^{\frac{i}{N}}\left|P_N^{(m)}(x)\right|\,dx \le \omega\left(\frac{1}{N},f\right)\left(\int_0^1\left(P_N^{(m)}(x)\right)^2dx\right)^{\frac12} \\
&\quad= O(1)\omega\left(\frac{1}{N},f\right)\left(\sum_{k=N}^{N+m}c_k^2(f)\right)^{\frac12} \le O(1)\omega\left(\frac{1}{N},f\right)e_N. \qquad (5)
\end{aligned}$$

Finally, since $\int\limits_0^1 \varphi_n(x)\,dx = 0$, $n=1,2,\dots$, from (3), (4) and (5) we have

$$\begin{aligned}
\left|\int_0^1 f(x)P_N^{(m)}(x)\,dx\right| &\le O\left(\omega\left(\frac{1}{N},f\right)e_N\right)+O\left(\omega\left(\frac{1}{N},f\right)e_N\right) \\
&= O\left(\omega\left(\frac{1}{N},f\right)e_N\right).
\end{aligned}$$

Hence, applying (2), we get

$$\sum_{k=N}^{N+m}c_k^2(f) = O\left(\omega\left(\frac{1}{N},f\right)e_N\right),$$

i.e.,

$$\sum_{k=N}^{\infty}c_k^2(f) = O\left(\omega\left(\frac{1}{N},f\right)e_N\right), \qquad (6)$$

From (6) we have

$$\left(\sum_{k=N}^{\infty}c_k^2(f)\right)^{\frac12} = O\left(\omega\left(\frac{1}{N},f\right)\right).$$

Sufficiency of the theorem is proved.

Necessity. Let $H_N \ne O(h_N)$, i.e.,

$$\overline{\lim_N}\,\frac{H_N}{h_N} = +\infty.$$

Therefore there exist a sequence $(a_n) \in \ell_2$ and natural numbers $m_{_N} \uparrow \infty$ such that

$$\overline{\lim_N} \frac{H_N^{(m_N)}}{h_N} = +\infty, \tag{7}$$

where

$$H_N^{(m_N)} = \sum_{k=1}^{N-1} \left| \int_0^{\frac{k}{N}} P_N^{(m_N)}(x)\, dx \right|$$

and

$$P_N^{(m_N)}(x) = \sum_{k=N}^{N+m_{_N}} a_k \varphi_k(x).$$

Consider the sequence of functions

$$f_N(x) = \int_0^x \left(\operatorname{sign} \int_0^u P_N^{(m_N)}(t)\, dt \right) du, \quad N = 1, 2, \dots . \tag{8}$$

Taking into account the condition $\int\limits_0^1 \varphi_n(x)\, dx = 0$, $n = 1, 2, \dots$, from (1) we obtain

$$\int_0^1 f_N(x) P_N^{(m_N)}(x)\, dx = \sum_{i=1}^{N-1} \left(f_N\left(\frac{i}{N}\right) - f_N\left(\frac{i+1}{N}\right) \right) \int_0^{\frac{i}{N}} P_N^{(m_N)}(x)\, dx$$
$$+ \sum_{i=1}^{N} \int_{\frac{i-1}{N}}^{\frac{i}{N}} \left(f_N(x) - f_N\left(\frac{1}{N}\right) \right) P_N^{(m_N)}(x)\, dx. \tag{9}$$

Applying (8) we have

$$\begin{aligned} &\left| \sum_{i=1}^{N} \int_{\frac{i-1}{N}}^{\frac{i}{N}} \left(f_N(x) - f_N\left(\frac{i}{N}\right) \right) P_N^{(m_N)}(x)\, dx \right| \\ &\quad \le \frac{1}{N} \sum_{i=1}^{N} \int_{\frac{i-1}{N}}^{\frac{i}{N}} \left| P_N^{(m_N)}(x) \right| dx \\ &\quad \le \frac{1}{N} \left(\int_0^1 \left(P_N^{(m_N)}(x) \right)^2 dx \right)^{\frac{1}{2}} = \frac{1}{N} \left(\sum_{k=N}^{N+m_N} a_k^2 \right)^{\frac{1}{2}}. \end{aligned} \tag{10}$$

Taking into account the assertions of Lemma 2 and (8), we get

$$\begin{aligned} &\sum_{i=1}^{N-1} \left(f_N\left(\frac{i}{N}\right) - f_N\left(\frac{i+1}{N}\right) \right) \int_0^{\frac{i}{N}} P_N^{(m_N)}(x)\, dx \\ &\quad = \sum_{i \in F_N \setminus E_N} \left(f_N\left(\frac{i}{N}\right) - f_N\left(\frac{i+1}{N}\right) \right) \int_0^{\frac{i}{N}} P_N^{(m_N)}(x)\, dx \end{aligned}$$

$$+\sum_{i\in E_N}\left(f_N\left(\frac{i}{N}\right)-f_N\left(\frac{i+1}{N}\right)\right)\int_0^{\frac{i}{N}} P_N^{(m_N)}(x)\,dx, \tag{11}$$

where $F_N=\{1,2,\ldots,N\}$.

If $i\in F_N\setminus E_N$, then

$$f_N\left(\frac{i}{N}\right)-f_N\left(\frac{i+1}{N}\right)=-\int_{\frac{i}{N}}^{\frac{i+1}{N}}\operatorname{sign}\int_0^u P_N^{(m_N)}(t)\,dt\,du$$
$$=-\frac{1}{N}\operatorname{sign}\int_0^{\frac{i}{N}} P_N^{(m_N)}(t)dt.$$

Therefore

$$\left|\sum_{i\in F_N\setminus E_N}\left(f_N\left(\frac{i}{N}\right)-f_N\left(\frac{i+1}{N}\right)\right)\int_0^{\frac{i}{N}} P_N^{(m_N)}(x)\,dx\right|$$
$$=\frac{1}{N}\sum_{i\in F_N\setminus E_N}\left|\int_0^{\frac{i}{N}} P_N^{(m_N)}(x)\,dx\right|. \tag{12}$$

At last, as far as

$$\left|f_N\left(\frac{i}{N}\right)-f_N\left(\frac{i+1}{N}\right)\right|<\frac{1}{N} \tag{13}$$

and, by virtue of Lemma 2, we have

$$\sum_{i\in E_N}\left|\int_0^{\frac{i}{N}} P_N^{(m_N)}(x)\,dx\right|\le\left(\int_0^1\left(P_N^{(m_N)}(x)\right)^2dx\right)^{\frac12}$$
$$=\left(\sum_{k=N}^{N+m_N}a_k^2\right)^{\frac12}\le\left(\sum_{k=N}^{\infty}a_k^2\right)^{\frac12}\equiv h_N. \tag{14}$$

Then, taking into account (13) and (14), we obtain

$$\left|\sum_{i\in E_N}\left(f_N\left(\frac{i}{N}\right)-f_N\left(\frac{i+1}{N}\right)\right)\int_0^{\frac{i}{N}} P_N^{(m_N)}(x)\,dx\right|\le\frac{1}{N}\,h_N, \tag{15}$$
$$N=1,2,\ldots\,.$$

Using (12) and (15), we get

$$\left|\sum_{i=1}^{N-1}\left(f_N\left(\frac{i}{N}\right)-f_N\left(\frac{i+1}{N}\right)\right)\int_0^{\frac{i}{N}} P_N^{(m_N)}(x)\,dx\right|$$
$$\ge\frac{1}{N}\sum_{i\in F_N\setminus E_N}\left|\int_0^{\frac{i}{N}} P_N^{(m_N)}(x)\,dx\right|-\frac{1}{N}\sum_{i\in E_N}\left|\int_0^{\frac{i}{N}} P_N^{(m_N)}(x)\,dx\right|$$
$$\ge\frac{1}{N}\,H_N^{(m_N)}-\frac{h_N}{N}\,.$$

Hence due to (7) we have

$$\frac{N\left|\int\limits_0^1 f_N(x)P_N^{(m_N)}(x)\,dx\right|}{h_N} \geq \frac{H_N^{(m_N)}}{h_N} - \frac{h_N}{h_N}\,,$$

and consequently

$$\overline{\lim_{n\to\infty}}\ \frac{N\left|\int\limits_0^1 f_N(x)P_N^{(m_N)}(x)\,dx\right|}{h_N} = +\infty. \tag{16}$$

Further, since

$$\|f_N\|_{\mathrm{Lip}\,1} = \|f_N\|_C + \sup_{x,y}\frac{|f_N(x)-f_N(y)|}{|x-y|}\,,$$

then from (8) it follows that

$$\|f_N\|_{\mathrm{Lip}\,1} \leq 2. \tag{17}$$

Finally, by virtue of the Banach–Steinhaus theorem (see (16) and (17)) there exists a function $f_0 \in \mathrm{Lip}\,1$ such that

$$\overline{\lim_{n\to\infty}}\ \frac{N\left|\int\limits_0^1 f_N(x)P_N^{(m_N)}(x)\,dx\right|}{h_N} = +\infty. \tag{18}$$

As far as

$$\begin{aligned}\left|\int_0^1 f_0(x)P_N^{(m_N)}(x)\,dx\right| &= \left|\int_0^1 f_0(x)\sum_{k=N}^{N+m_N} a_k\varphi_k(x)\,dx\right| \\ &= \left|\sum_{k=N}^{N+m_N} a_k\int_0^1 f_0(x)\varphi_k(x)\,dx\right| = \left|\sum_{k=N}^{N+m_N} a_kc_k(f_0)\right| \\ &\leq \left(\sum_{k=N}^{N+m_N} a_k^2\right)^{\frac12}\left(\sum_{k=N}^{N+m_N} c_k^2(f_0)\right)^{\frac12} < h_N\cdot e_N,\end{aligned}$$

then taking into account that $\omega\left(\frac1N\,,f_0\right) = O\left(\frac1N\right)$, from (18) we have

$$\lim_{n\to\infty}\frac{h_N\cdot e_N}{\omega\left(\frac1N\,,f_0\right)\cdot h_N} = +\infty,$$

i.e.,

$$\lim_{n\to\infty}\frac{e_N}{\omega\left(\frac1N\,,f_0\right)} = +\infty. \qquad \square$$

Theorem 2. *Let the functions f and Φ be from $L_2([0,1])$. Then*

$$\left|\int_0^1 f(x)\Phi(x)\,dx\right| \le \omega_2\left(\frac{1}{n}\,, f\right)(V_n + 2\|\Phi\|_2) + n\int_{1-\frac{1}{n}}^1 |f(x)|\,dx\left|\int_0^1 \Phi(x)\,dx\right|, \tag{19}$$

where $V_n = \sum\limits_{k=1}^{n-1}\left|\int_0^{k/n}\Phi(x)\,dx\right|$.

Proof. Applying the Abel transformation, we get

$$\begin{aligned} n\sum_{k=1}^{n}&\int_{\frac{k-1}{n}}^{\frac{k}{n}} f(x)\,dx\int_{\frac{k-1}{n}}^{\frac{k}{n}}\Phi(x)\,dx \\ &= n\sum_{k=1}^{n-1}\left(\int_{\frac{k-1}{n}}^{\frac{k}{n}} f(x)\,dx - \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(x)\,dx\right)\int_0^{\frac{k}{n}}\Phi(x)\,dx \\ &\quad + n\int_{1-\frac{1}{n}}^1 f(x)\,dx\int_0^1\Phi(x)\,dx \\ &= n\sum_{k=1}^{n-1}\int_{\frac{k-1}{n}}^{\frac{k}{n}}\left(f(x) - f\left(x+\frac{1}{n}\right)\right)\,dx\int_0^{\frac{k}{n}}\Phi(x)\,dx \\ &\quad + n\int_{1-\frac{1}{n}}^1 f(x)\,dx\int_0^1\Phi(x)\,dx. \end{aligned} \tag{20}$$

Since

$$\begin{aligned} \int_{\frac{k-1}{n}}^{\frac{k}{n}}& f(x)\Phi(x)\,dx - n\int_{\frac{k-1}{n}}^{\frac{k}{n}} f(t)\,dt\int_{\frac{k-1}{n}}^{\frac{k}{n}}\Phi(x)\,dx \\ &= n\int_{\frac{k-1}{n}}^{\frac{k}{n}} dt\int_{\frac{k-1}{n}}^{\frac{k}{n}} f(x)\Phi(x)\,dx - n\int_{\frac{k-1}{n}}^{\frac{k}{n}} f(t)\,dt\int_{\frac{k-1}{n}}^{\frac{k}{n}}\Phi(x)\,dx \\ &= n\int_{\frac{k-1}{n}}^{\frac{k}{n}}\left(\int_{\frac{k-1}{n}}^{\frac{k}{n}} f(x)\,dt - \int_{\frac{k-1}{n}}^{\frac{k}{n}} f(t)\,dt\right)\Phi(x)\,dx \\ &= n\int_{\frac{k-1}{n}}^{\frac{k}{n}}\int_{\frac{k-1}{n}}^{\frac{k}{n}}(f(x) - f(t))\,\Phi(x)\,dx\,dt \end{aligned} \tag{21}$$

and

$$\int_0^1 f(x)\Phi(x)\,dx = \sum_{k=1}^{n}\int_{\frac{k-1}{n}}^{\frac{k}{n}} f(x)\Phi(x)\,dx, \tag{22}$$

then, taking into account (20), (21) and (22), one has

$$\int_0^1 f(x)\Phi(x)\,dx=\sum_{k=1}^{n}\int_{\frac{k-1}{n}}^{\frac{k}{n}} f(x)\Phi(x)\,dx-n\sum_{k=1}^{n}\int_{\frac{k-1}{n}}^{\frac{k}{n}} f(t)\,dt\int_{\frac{k-1}{n}}^{\frac{k}{n}}\Phi(x)\,dx$$

$$+n\sum_{k=1}^{n-1}\int_{\frac{k-1}{n}}^{\frac{k}{n}}\left(f(x)-f\left(x+\frac{1}{n}\right)\right)\,dx\int_0^{\frac{k}{n}}\Phi(x)\,dx+n\int_{1-\frac{1}{n}}^{1} f(x)\,dx\int_0^1\Phi(x)\,dx$$

$$=\sum_{k=1}^{n}\left(\int_{\frac{k-1}{n}}^{\frac{k}{n}} f(x)\Phi(x)\,dx-n\int_{\frac{k-1}{n}}^{\frac{k}{n}} f(t)\,dt\int_{\frac{k-1}{n}}^{\frac{k}{n}}\Phi(x)\,dx\right)$$

$$+n\sum_{k=1}^{n-1}\int_{\frac{k-1}{n}}^{\frac{k}{n}}\left(f(x)-f\left(x+\frac{1}{n}\right)\right)\,dx\int_0^{\frac{k}{n}} f(x)\Phi(x)\,dx$$

$$+n\int_{1-\frac{1}{n}}^{1} f(x)\,dx\int_0^1\Phi(x)\,dx=n\sum_{k=1}^{n}\int_{\frac{k-1}{n}}^{\frac{k}{n}}\int_{\frac{k-1}{n}}^{\frac{k}{n}}\left(f(x)-f(t)\right)\Phi(x)\,dx\,dt$$

$$+n\sum_{k=1}^{n-1}\int_{\frac{k-1}{n}}^{\frac{k}{n}}\left(f(x)-f\left(x+\frac{1}{n}\right)\right)\,dx\int_0^{\frac{k}{n}}\Phi(x)\,dx+n\int_{1-\frac{1}{n}}^{1} f(x)\,dx\int_0^1\Phi(x)\,dx.$$

Consequently,

$$\int_0^1 f(x)\Phi(x)\,dx=n\sum_{k=1}^{n-1}\int_{\frac{k-1}{n}}^{\frac{k}{n}}\left(f(x)-f\left(x+\frac{1}{n}\right)\right)\,dx\int_0^{\frac{k}{n}}\Phi(x)\,dx$$

$$+n\sum_{k=1}^{n}\int_{\frac{k-1}{n}}^{\frac{k}{n}}\int_{\frac{k-1}{n}}^{\frac{k}{n}}\left(f(x)-f(t)\right)\Phi(x)\,dx\,dt+n\int_{1-\frac{1}{n}}^{1} f(x)\,dx\int_0^1\Phi(x)\,dx. \tag{23}$$

Assuming $\xi_n^{(k)}(t)=\frac{1}{n}\,t+\frac{k-1}{n}$, we get

$$\int_{\frac{k-1}{n}}^{\frac{k}{n}}\left(f(x)-f\left(x+\frac{1}{n}\right)\right)\,dx=\frac{1}{n}\int_0^1\left(f\left(\frac{1}{n}\,t+\frac{k-1}{n}\right)-f\left(\frac{1}{n}\,t+\frac{k}{n}\right)\right)\,dt$$

$$=\frac{1}{n}\int_0^1\left(f\left(\xi_n^{(k)}(t)\right)-f\left(\xi_n^{(k)}(t)+\frac{1}{n}\right)\right)\,dt.$$

Hence we have

$$\left|\int_{\frac{k-1}{n}}^{\frac{k}{n}}\left(f(x)-f\left(x+\frac{1}{n}\right)\right)\,dx\right|\le\frac{1}{n}\int_0^1\left|f\left(\xi_n^{(k)}(t)\right)-f\left(\xi_n^{(k)}(t)+\frac{1}{n}\right)\right|\,dt$$

$$\le\frac{1}{n}\int_0^1\left|f(t)-f\left(t+\frac{1}{n}\right)\right|\,dt\le\frac{1}{n}\sup_{|h|\le\frac{1}{n}}\left(\int_0^1|f(t)-f(t+h)|^2\,dt\right)^{\frac{1}{2}}$$

$$=\frac{1}{n}\,\omega_2\left(\frac{1}{n}\,,f\right). \tag{24}$$

By virtue of the Hölder inequality we obtain

$$\begin{aligned}
&\left|\int_{\frac{k-1}{n}}^{\frac{k}{n}}\int_{\frac{k-1}{n}}^{\frac{k}{n}}(f(x)-f(t))\,\Phi(x)\,dx\,dt\right| \\
&\quad\le \int_{\frac{k-1}{n}}^{\frac{k}{n}}\left(\int_{\frac{k-1}{n}}^{\frac{k}{n}}(f(x)-f(t))^2\,dx\right)^{\frac{1}{2}}\left(\int_{\frac{k-1}{n}}^{\frac{k}{n}}\Phi^2(x)\,dx\right)^{\frac{1}{2}}dt \\
&\quad\le \frac{1}{\sqrt{n}}\left(\int_{\frac{k-1}{n}}^{\frac{k}{n}}\int_{\frac{k-1}{n}}^{\frac{k}{n}}(f(x)-f(t))^2\,dx\,dt\right)^{\frac{1}{2}}\left(\int_{\frac{k-1}{n}}^{\frac{k}{n}}\Phi^2(x)\,dx\right)^{\frac{1}{2}}. \qquad (25)
\end{aligned}$$

Applying the equality (see [11])

$$\int_a^b\int_a^b|f(x)-f(t)|^p dx\,dt = 2\int_0^{b-a}\left(\int_a^{b-\xi}|f(y+\xi)-f(y)|^p dy\right)d\xi,$$

we get

$$\int_{\frac{k-1}{n}}^{\frac{k}{n}}\int_{\frac{k-1}{n}}^{\frac{k}{n}}(f(x)-f(t))^2\,dx\,dt = 2\int_0^{\frac{1}{n}}\left(\int_{\frac{k-1}{n}}^{\frac{k}{n}-\xi}|f(y+\xi)-f(y)|^2 dy\right)d\xi. \qquad (26)$$

Therefore, by virtue of (25) and (26) we have

$$\begin{aligned}
&\left|\sum_{k=1}^{n}\int_{\frac{k-1}{n}}^{\frac{k}{n}}\int_{\frac{k-1}{n}}^{\frac{k}{n}}(f(x)-f(t))\,\Phi(x)dx\,dt\right| \\
&\quad\le \frac{2}{\sqrt{n}}\sum_{k=1}^{n}\left(\int_0^{\frac{1}{n}}\int_{\frac{k-1}{n}}^{\frac{k}{n}-\xi}(f(y+\xi)-f(y))^2\,dy\,d\xi\right)^{\frac{1}{2}}\cdot\left(\int_{\frac{k-1}{n}}^{\frac{k}{n}}\Phi^2(x)\,dx\right)^{\frac{1}{2}} \\
&\quad\le \frac{2}{\sqrt{n}}\left(\int_0^{\frac{1}{n}}\sum_{k=1}^{n}\int_{\frac{k-1}{n}}^{\frac{k}{n}}(f(y+\xi)-f(y))^2\,dy\,d\xi\right)^{\frac{1}{2}}\cdot\left(\sum_{k=1}^{n}\int_{\frac{k-1}{n}}^{\frac{k}{n}}\Phi^2(x)\,dx\right)^{\frac{1}{2}} \\
&\quad\le \frac{2}{\sqrt{n}}\left(\int_0^{\frac{1}{n}}\int_0^1(f(y+\xi)-f(y))^2\,dy\,d\xi\right)^{\frac{1}{2}}\cdot\left(\int_0^1\Phi^2(x)\,dx\right)^{\frac{1}{2}} \\
&\quad\le \frac{2}{\sqrt{n}}\cdot\frac{1}{\sqrt{n}}\,\omega_2\left(\frac{1}{n},f\right)\cdot\|\Phi\|_2 = \frac{2\|\Phi\|_2}{n}\,\omega_2\left(\frac{1}{n},f\right). \qquad (27)
\end{aligned}$$

At last, taking into account (24), we have

$$\begin{aligned}
&\left|\sum_{k=1}^{n-1}\int_{\frac{k-1}{n}}^{\frac{k}{n}}\left(f(x)-f\left(x+\frac{1}{n}\right)\right)dx\int_0^{\frac{k}{n}}\Phi(x)\,dx\right| \\
&\quad\le \frac{1}{n}\,\omega_2\left(\frac{1}{n},f\right)\cdot\sum_{k=1}^{n-1}\left|\int_0^{\frac{k}{n}}\Phi(x)\,dx\right| = \frac{\omega_2\left(\frac{1}{n},f\right)}{n}\,V_n.
\end{aligned}$$

Hence from (23) and (27) we get

$$\left|\int_0^1 f(x)\Phi(x)\,dx\right| \le n\cdot\frac{\omega_2\left(\frac{1}{n},f\right)}{n}\,V_n + n\,\frac{2\|\Phi\|_2}{n}\,\omega_2\left(\frac{1}{n},f\right)$$
$$+n\int_{1-\frac{1}{n}}^1 |f(x)|dx\left|\int_0^1 \Phi(x)\,dx\right|.$$

Theorem 2 is proved completely. □

Theorem 3. *Let (φ_n) be an orthonormal system of functions on $[0,1]$ satisfying the condition $\int\limits_0^1 \varphi_n(x)\,dx=0$, $n=1,2,\dots$. Then*

$$c_n(f)=O\left(\omega_2\left(\frac{1}{n},f\right)\right)\quad \textit{for every } f\in L_2([0,1])$$

if and only if

$$V_n=O(1),$$

where $V_n=\sum\limits_{k=1}^{n-1}\left|\int_0^{\frac{k}{n}}\varphi_n(x)\,dx\right|$.

Proof. Sufficiency. Assuming in (19) that $\Phi(x)=\varphi_n(x)$, we obtain

$$\left|\int_0^1 f(x)\varphi_n(x)\,dx\right| \le \omega_2\left(\frac{1}{n},f\right)(V_n+2\|\varphi_n\|_2)+n\int_{1-\frac{1}{n}}^1 |f(x)|\,dx\left|\int_0^1 \varphi_n(x)\,dx\right|.$$

Since $V_n=\sum\limits_{k=1}^{n-1}\left|\int\limits_0^{\frac{k}{n}}\varphi_n(x)\,dx\right|=O(1)$ and $\int\limits_0^1\varphi_n(x)\,dx=0$, $n=1,2,\dots$, we have

$$|c_n(f)|=O\left(\omega_2\left(\frac{1}{n},f\right)\right).$$

Necessity. If $V_n\ne O(1)$, then as it is known (see [15]) there exists $f_0\in\operatorname{Lip}1$ such that

$$\overline{\lim_{n\to\infty}}\, n|c_n(f_0)|=+\infty. \tag{28}$$

As far as $\omega_2\left(\frac{1}{n},f_0\right)=O\left(\frac{1}{n}\right)$, from (28) we have

$$\lim_{n\to\infty}\frac{c_n(f_0)}{\omega_2\left(\frac{1}{n},f_0\right)}=+\infty.$$ □

Theorem 4. *Let (φ_n) be an orthonormal system on $[0,1]$ satisfying*

$$\int\limits_0^1 \varphi_n(x)dx=0,\quad n=1,2,\dots.$$

Then the relation

$$e_N=O\left(\omega_2\left(\frac{1}{N},f\right)\right)\quad \textit{for every } f\in L_2([0,1])$$

holds if and only if
$$H_N = O(h_N)$$
for every sequence $(a_n) \in \ell_2$.

Proof. Sufficiency. Let $f \in L_2([0,1])$. As it was shown above (see (2)), the equality
$$\sum_{m=N}^{N+k} c_m^2(f) = \int_0^1 f(x) \sum_{m=N}^{N+k} c_m \varphi_m(x)\, dx = \int_0^1 f(x) P_N^{(k)}(x)\, dx \tag{29}$$
holds. By virtue of Theorem 2, we get
$$\left| \int_0^1 f(x) P_N^{(k)}(x)\, dx \right| \le \omega_2\left(\frac{1}{N}, f\right) \left(H_N + 2\|P_N^{(k)}\|_2\right). \tag{30}$$
Since
$$\|P_N^{(k)}\|_2 = \left(\int_0^1 \left(P_N^{(k)}(x)\right)^2\right)^{\frac{1}{2}} = \left(\int_0^1 \left(\sum_{m=N}^{N+k} c_m(f)\varphi_m(x)\right)^2 dx\right)^{\frac{1}{2}}$$
$$= \left(\sum_{m=N}^{N+k} c_m^2(f)\right)^{\frac{1}{2}} \le e_N,$$
and by virtue of the condition of Theorem 4 $H_N = O(e_N)$, from (30) we have
$$\left| \int_0^1 f(x) P_N^{(k)}(x)\, dx \right| \le c \cdot \omega_2\left(\frac{1}{N}, f\right) e_N,$$
where c does not depend on N.

Finally, from (29) we get
$$\sum_{m=N}^{N+k} c_m^2(f) \le c \cdot \omega_2\left(\frac{1}{N}, f\right) e_N, \quad k = 1, 2, \dots,$$
hence
$$\sum_{m=N}^{\infty} c_m^2(f) \le c \cdot \omega_2\left(\frac{1}{N}, f\right) e_N. \tag{31}$$
From (31) we have
$$e_N = O\left(\omega_2\left(\frac{1}{N}, f\right)\right).$$
Thus the sufficiency is proved.

Necessity. Let $H_N \ne O(h_N)$. Then by virtue of Theorem 1 there exists a function $f_0 \in \operatorname{Lip} 1$ such that
$$\varlimsup_{n\to\infty} N e_N = +\infty.$$

As far as for the function f_0 the relation $\omega_2\left(\frac{1}{N}, f_0\right) = O\left(\frac{1}{N}\right)$ holds, from (31) it follows
$$\lim_{n\to\infty} \frac{e_N}{\omega_2\left(\frac{1}{n}, f_0\right)} = +\infty.$$

Theorem is proved completely. $\square$

Theorem 5. *From every orthonormal system φ_n on $[0,1]$ satisfying the condition $\int\limits_0^1 \varphi_n(x)\,dx = 0$, $n = 1, 2, \dots$, one can choose a subsystem $\psi_k = \varphi_{n_k}$ for which the following conditions are fulfilled:*

1) $c_n(f) = O\left(\omega_2\left(\frac{1}{n}, f\right)\right)$,
2) $e_N = O\left(\omega\left(\frac{1}{N}, f\right)\right)$,
3) $e_N = O\left(\omega_2\left(\frac{1}{N}, f\right)\right)$.

Proof. Let

$$\varepsilon_i(k) = \sum_{s=1}^{\infty} \left(\int_0^{\frac{i}{k}} \varphi_s(x)\,dx \right)^2, \quad i = 1, 2, \dots, k.$$

By virtue of the Bessel inequality, $\varepsilon_i(k) \le 1$ for every $i = 1, 2, \dots, k$. Therefore for each fixed k we can choose a number $s_i(k)$ such that the inequality

$$\sum_{s=s_i(k)}^{\infty} \left(\int_0^{\frac{i}{k}} \varphi_s(x)\,dx \right)^2 < \frac{1}{k^3}$$

holds. Now if $s(k) = \max\limits_{1\le i\le k} s_i(k)$, then

$$\sum_{s=s(k)}^{\infty} \left(\int_0^{\frac{i}{k}} \varphi_s(x)\,dx \right)^2 < \frac{1}{k^3}.$$

Hence for every $i = 1, 2, \dots, k$ we have

$$\left| \int_0^{\frac{i}{k}} \varphi_s(x)\,dx \right| < \frac{1}{k^{3/2}}$$

if $s \ge s(k)$. Assuming $\varphi_{s(k)} = \psi_k$, we get

$$\left| \int_0^{\frac{i}{k}} \psi_k(x)\,dx \right| < \frac{1}{k^{3/2}}.$$

This inequality implies the inequality

$$V_k = \sum_{i=1}^{k-1} \left| \int_0^{\frac{i}{k}} \psi_k(x)\,dx \right| < 1, \quad n = 1, 2, \dots, \tag{32}$$

and also the estimate

$$\sum_{i=1}^{k-1} \left| \int_0^{\frac{i}{k}} \sum_{m=k}^{k+l} a_m \psi_m(x)\,dx \right| \le \left(\sum_{m=k}^{k+l} a_m^2 \right)^{\frac{1}{2}} \sum_{i=1}^{k-1} \left(\sum_{m=k}^{k+l} \left(\int_0^{\frac{i}{k}} \psi_m(x)\,dx \right)^2 \right)^{\frac{1}{2}}$$

$$\le \left(\sum_{m=k}^{k+l} a_m^2 \right)^{\frac{1}{2}} \cdot \sum_{i=1}^{k-1} \frac{1}{k} < \left(\sum_{m=k}^{k+l} a_m^2 \right)^{\frac{1}{2}}.$$

is valid. Therefore
$$H_k = O(h_k). \tag{33}$$
Hence the validity of Theorem 5 follows from Theorems 1,2,3,4 and from the relations (32) and (33). □

Remark 1. It follows from the proof of Theorem 1 that the relation
$$e_N = O\left(\omega_2\left(\frac{1}{N}, f\right)\right) \tag{34}$$
holds for every $f \in C([0,1])$ if and only if relation (34) holds for every $f \in \mathrm{Lip}\,1$.

Hence it follows

Corollary 1. *Relation* (34) *holds for every* $f \in L_2([0,1])$ *if and only if the condition*
$$e_N = O\left(\frac{1}{N}\right)$$
holds for every function $f \in \mathrm{Lip}\,1$.

Corollary 2. *The relation*
$$c_n(f) = O\left(\omega_2\left(\frac{1}{n}, f\right)\right)$$
holds for every $f \in L_2([0,1])$ *if and only if the condition*
$$c_n(f) = O\left(\frac{1}{n}\right)$$
holds for every $f \in \mathrm{Lip}\,1$ (*see* [15]).

Remark 2. a) If (φ_n) is a trigonometric system, then the inequality
$$V_n = \sum_{k=1}^{n-1}\left|\int_0^{\frac{k}{n}} \sin \pi n x\, dx\right| \le \sum_{k=1}^{n-1}\frac{1}{\pi n} < \frac{1}{\pi}$$
is valid.

b) Let (χ_n) be the Haar system (see [11]). Then if $n = 2^m + l$ $(1 \le l \le 2^m)$, we have
$$\left|\int_0^{\frac{k}{n}} \chi_n(x)\, dx\right| \le 2^{-\frac{m}{2}} < \frac{2}{\sqrt{n}}\,.$$

On the other hand, note that only one integral $\int\limits_0^{\frac{k}{n}} \chi_n(x)\, dx$ is different from zero when $k = 1, 2, \ldots, n$. Thus
$$V_n = \sum_{k=1}^{n-1}\left|\int_0^{\frac{k}{n}} \chi_n(x)\, dx\right| \le \frac{2}{\sqrt{n}}\,.$$

Further on, it is easy to verify that if $n = 2^s + l$, in case $l \le 2^s$ the following estimate is valid:

$$\sum_{k=1}^{n-1} \left| \int_0^{\frac{k}{n}} P_n^{(q)}(x)\, dx \right| \equiv \sum_{k=1}^{n-1} \left| \int_0^{\frac{k}{n}} \sum_{m=n}^{n+q} a_m \chi_m(x)\, dx \right|$$
$$\le \sum_{k=1}^{n} \left| \int_0^{\frac{k}{2^{s+1}}} \sum_{m=n}^{n+q} a_m \chi_m(x)\, dx \right| + \sum_{k=1}^{n} \left| \int_{\frac{k}{2^{s+1}}}^{\frac{k}{n}} P_n^{(q)}(x)\, dx \right|. \tag{36}$$

Applying the Hölder inequality, we get

$$\sum_{k=1}^{n} \left| \int_{\frac{k}{2^{s+1}}}^{\frac{k}{n}} P_n^{(q)}(x)\, dx \right| \le \int_0^1 |P_n^{(q)}(x)|\, dx \le \left(\sum_{m=n}^{n+q} a_m^2 \right)^{\frac{1}{2}} \le h_n. \tag{37}$$

Then, if $n + q > 2^{s+1}$, then

$$\sum_{k=1}^{n} \left| \int_0^{\frac{k}{2^{s+1}}} \sum_{m=n}^{n+q} a_m \chi_m(x)\, dx \right| \le \sum_{k=1}^{n} \left| \int_0^{\frac{k}{2^{s+1}}} \sum_{m=n}^{2^{s+1}} a_m \chi_m(x)\, dx \right|$$
$$+ \sum_{k=1}^{n} \left| \int_0^{\frac{k}{2^{s+1}}} \sum_{m=2^{s+1}+1}^{n+q} a_m \chi_m(x)\, dx \right|. \tag{38}$$

Since $m \ge 2^{s+1}$, we have

$$\int_0^{\frac{k}{2^{s+1}}} \chi_m(x)\, dx = 0$$

and the second summand in (38) is equal to zero.

On the other hand, as far as $2^{s+1} - n \le 2^s$, we get

$$\sum_{k=1}^{n} \left| \int_0^{\frac{k}{2^{s+1}}} \sum_{m=n}^{2^{s+1}} a_m \chi_m(x)\, dx \right| \le \sum_{m=n}^{2^{s+1}} |a_m| \sum_{k=1}^{n} \left| \int_0^{\frac{k}{2^{s+1}}} \chi_m(x)\, dx \right|$$
$$\le \sum_{m=n}^{2^{s+1}} |a_m| \cdot \frac{1}{\sqrt{m}} \le \left(\sum_{m=n}^{2^{s+1}} a_m^2 \right)^{\frac{1}{2}} \left(\sum_{m=n}^{2^{s+1}} \frac{1}{m} \right)^{\frac{1}{2}} \le h_n. \tag{39}$$

Finally, taking into account in (36) the inequalities (37), (38) and (39), we obtain

$$\sum_{k=1}^{n} \left| \int_0^{\frac{k}{n}} P_n^{(q)}(x)\, dx \right| = O(h_n).$$

Consequently,

$$H_n = O(h_n).$$

c) Let now (φ_n) be the Walsh system (see [12]). Then since

$$\left| \int_0^{\frac{k}{n}} \varphi_n(x)\, dx \right| < \frac{1}{n},$$

we get

$$V_n = \sum_{k=1}^{n-1} \left| \int_0^{\frac{k}{n}} \varphi_n(x)\, dx \right| < 1.$$

In this case the inequality

$$H_n = O(h_n)$$

is analogously proved as in case of the Haar system.

In conclusion, we can say that the efficiency of the conditions of Theorems 1, 2, and 3 is evident.

It should be noted that the above results were partially announced in [14].

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