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OSCILLATION THEOREMS OF NONLINEAR DIFFERENCE EQUATIONS OF SECOND ORDER

S. H. SAKER

Abstract. Using the Riccati transformation techniques, we establish some new oscillation criteria for the second-order nonlinear difference equation

$$\Delta^2 x_n + F(n, x_n, \Delta x_n) = 0 \quad \text{for} \quad n \geq n_0.$$

Some comparison between our theorems and the previously known results in special cases are indicated. Some examples are given to illustrate the relevance of our results.

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1. Introduction

In recent years, the oscillation and asymptotic behavior of second order difference equations has been the subject of investigations by many authors. In fact, in the last few years several monographs and hundreds of research papers have been written, see, e.g., the monographs [1–8].

Following this trend in this paper, we consider the nonlinear difference equation

$$\Delta^2 x_n + F(n, x_n, \Delta x_n), \quad n \geq n_0, \tag{1.1}$$

where n_0 is a fixed nonnegative integer, Δ denotes the forward difference operator $\Delta x_n = x_{n+1} - x_n$.

Throughout, we shall assume that there exists a real sequence $\{q_n\}$ such that

$$F(n, u, v) \operatorname{sign} u \geq q_n |u|^\beta \quad \text{for} \quad n \geq n_0 \quad \text{and} \quad u, v \in R, \tag{1.2}$$

where $q_n \geq 0$ and not identically zero for large n, and $\beta > 0$ is a positive integer. We say that equation (1.1) is strictly superlinear if $\beta > 1$, strictly sublinear if $\beta < 1$ and linear if $\beta = 1$.

By a solution of (1.1) we mean a nontrivial sequence $\{x_n\}$ satisfying equation (1.1) for $n \geq n_0$. A solution $\{x_n\}$ of (1.1) is said to be oscillatory if for every $n_1 > n_0$ there exists $n \geq n_1$ such that $x_n x_{n+1} \leq 0$, otherwise it is nonoscillatory. Equation (1.1) is said to be oscillatory if all its solutions are oscillatory.

Different kinds of the dynamical behavior of solutions of second order difference equations are possible; here we shall only be concerned with the conditions which are sufficient for all solutions of (1.1) to be oscillatory.

Our concern is motivated by several papers, especially by Hooker and Patula [11], Szmanda [13], Wong and Agarwal [14] and Fu and Tsai [10].

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Our aim in this paper is to establish some new oscillation criteria for equation (1.1) by using the Riccati transformation techniques. Some comparison between our theorems and the previously known results [10, 11, 13] are indicated. Examples are given to illustrate the relevance of our results.

2. Main Results

In what follows we shall assume that equation (1.1) is strictly superlinear, strictly sublinear or linear.

First, we consider the case where (1.1) is strictly superlinear. As a variant of the Riccati transformation techniques, we shall derive new oscillation criteria which can be considered as a discrete analogue of Philos condition for the oscillation of second-order differential equations [12].

Theorem 2.1. *Assume that* (1.2) *holds, and let* $\{\rho_n\}$ *be a positive sequence. Furthermore, assume that there exists a double sequence* $\{H_{m,n} : m \geq n \geq 0\}$*such that*

(i) $H_{m,m} = 0$ *for* $m \geq 0$,

(ii) $H_{m,n} > 0$ *for* $m > n > 0$,

(iii) $\Delta_2 H_{m,n} = H_{m,n+1} - H_{m,n}$.

If

$$\lim_{m\to\infty} \sup \frac{1}{H_{m,0}} \sum_{n=0}^{m-1} \left[H_{m,n}\rho_n q_n - \frac{(\rho_{n+1})^2}{4\,\bar{\rho}_n} \left(h_{m,n} - \frac{\Delta\rho_n}{\rho_{n+1}} \sqrt{H_{m,n}} \right)^2 \right] = \infty, \quad (2.1)$$

where

$$\bar{\rho}_n = 2^{1-\beta} M^{\beta-1} \rho_n, \quad h_{m,n} = -\frac{\Delta H_{m,n}}{\sqrt{H_{m,n}}}\,. \quad (2.2)$$

for some positive constant M*, then every solution of equation* (1.1) *oscillates.*

Proof. Suppose the contrary that $\{x_n\}$ is an eventually positive solution of (1.1), say, $x_n > 0$ for all $n \geq n_1 \geq n_0$. We shall consider only this case, because the proof when $x_n < 0$ is similar. From equations (1.1) and (1.2) we have

$$\Delta^2 x_n \leq -q_n x_n^\beta \leq 0 \quad \text{for} \quad n \geq n_1 \quad (2.3)$$

and so $\{\Delta x_n\}$ is a nonincreasing sequence. We first have to show that $\Delta x_n \geq 0$ for $n \geq n_1$. Indeed, if there exists an integer $n_2 \geq n_1$ such that $\Delta x_{n_2} = c < 0$, then $\Delta x_n \leq c$ for $n \geq n_2$, that is

$$x_n \leq x_{n_2} + c(n - n_0) \to -\infty \quad \text{as} \quad n \to \infty, \quad (2.4)$$

which contradicts the fact that $x_n > 0$ for $n \geq n_1$. Therefore we have

$$\Delta x_n \geq 0 \quad \text{and} \quad \Delta^2 x_n \leq 0 \quad \text{for} \quad n \geq n_1. \quad (2.5)$$

Define the sequence

$$w_n = \rho_n \frac{\Delta x_n}{x_n^\beta}. \quad (2.6)$$

Then $w_n > 0$ and

$$\Delta w_n = \Delta x_{n+1} \Delta \left[\frac{\rho_n}{x_n^\beta} \right] + \frac{\rho_n \Delta^2 x_n}{x_n^\beta};$$

this and (2.4) imply

$$\Delta w_n \leq -\rho_n q_n + \frac{\Delta \rho_n}{\rho_{n+1}} w_{n+1} - \frac{\rho_n \Delta x_{n+1} \Delta (x_n^\beta)}{x_n^\beta x_{n+1}^\beta} . \tag{2.7}$$

But (2.5) implies that $x_{n+1} \geq x_n$; then from (2.7) we have

$$\Delta w_n \leq -\rho_n q_n + \frac{\Delta \rho_n}{\rho_{n+1}} w_{n+1} - \frac{\rho_n \Delta x_{n+1} \Delta (x_n^\beta)}{(x_{n+1}^\beta)^2} . \tag{2.8}$$

Now, by using the inequality

$$x^\beta - y^\beta \geqslant 2^{1-\beta} (x-y)^\beta \quad \text{for all} \quad x \geq y > 0 \quad \text{and} \quad \beta \geqslant 1$$

we find that

$$\Delta (x_n^\beta) = x_{n+1}^\beta - x_n^\beta \geqslant 2^{1-\beta} (x_{n+1} - x_n)^\beta = 2^{1-\beta} \left(\Delta x_n \right)^\beta, \quad \beta \geqslant 1. \tag{2.9}$$

Substituting (2.9) in (2.8), we have

$$\Delta w_n \leq -\rho_n q_n + \frac{\Delta \rho_n}{\rho_{n+1}} w_{n+1} - \rho_n \frac{2^{1-\beta} \left(\Delta x_n \right)^\beta \Delta x_{n+1}}{\left(x_{n+1}^\beta \right)^2} . \tag{2.10}$$

From (2.5) and (2.10) we obtain

$$\Delta w_n \leq -\rho_n q_n + \frac{\Delta \rho_n}{\rho_{n+1}} w_{n+1} - \rho_n \frac{2^{1-\beta} \left(\Delta x_{n+1} \right)^{2\beta}}{\left(x_{n+1}^\beta \right)^2 \left(\Delta x_{n+1} \right)^{\beta - 1}} ;$$

this and (2.6) imply that

$$\Delta w_n \leq -\rho_n q_n + \frac{\Delta \rho_n}{\rho_{n+1}} w_{n+1} - \frac{2^{1-\beta} \rho_n}{\left(\rho_{n+1} \right)^2} w_{n+1}^2 \frac{1}{\left(\Delta x_{n+1} \right)^{\beta - 1}} . \tag{2.11}$$

Since $\Delta^2 x_n \leq 0$, from (2.5) it follows that Δx_n is a nonincreasing and positive sequence and there exists sufficiently large $n_2 \geq n_1$ such that $\Delta x_n \leq M$ for some positive constant M and $n \geq n_2$, and hence $\Delta x_{n+1} \leq M$ so that

$$\frac{1}{\left(\Delta x_{n+1} \right)^{\beta - 1}} \geqslant M^{^{\beta - 1}}. \tag{2.12}$$

Now, from (2.11) and (2.12), we have

$$\rho_n q_n \leq -\Delta w_n + \frac{\Delta \rho_n}{\rho_{n+1}} w_{n+1} - \frac{\bar{\rho}_n}{\left(\rho_{n+1} \right)^2} w_{n+1}^2. \tag{2.13}$$

Therefore

$$\sum_{n=n_2}^{m-1} H_{m,n}\rho_n q_n \leq -\sum_{n=n_2}^{m-1} H_{m,n}\Delta w_n + \sum_{n=n_2}^{m-1} H_{m,n}\frac{\Delta\rho_n}{\rho_{n+1}}\, w_{n+1}$$
$$-\sum_{n=n_2}^{m-1} H_{m,n}\frac{\bar{\rho}_n}{(\rho_{n+1})^2}\, w_{n+1}^2, \tag{2.14}$$

which yields after summing by parts

$$\begin{aligned}
&\sum_{n=n_2}^{m-1} H_{m,n}\rho_n q_n \\
\leq\ & H_{m,n_2}w_{n_2} + \sum_{n=n_2}^{m-1} w_{n+1}\Delta_2 H_{m,n} + \sum_{n=n_2}^{m-1} H_{m,n}\frac{\Delta\rho_n}{\rho_{n+1}} w_{n+1} \\
& -\sum_{n=n_2}^{m-1} H_{m,n}\frac{\bar{\rho}_n}{(\rho_{n+1})^2} w_{n+1}^2 \\
=\ & H_{m,n_2}w_{n_2} - \sum_{n=n_2}^{m-1} h_{m,n}\sqrt{H_{m,n}}w_{n+1} + \sum_{n=n_2}^{m-1} H_{m,n}\frac{\Delta\rho_n}{\rho_{n+1}} w_{n+1} \\
& -\sum_{n=n_2}^{m-1} H_{m,n}\frac{\bar{\rho}_n}{(\rho_{n+1})^2} w_{n+1}^2 \\
=\ & H_{m,n_2}w_{n_2} - \sum_{n=n_2}^{m-1}\left[\frac{\sqrt{H_{m,n}\ \bar{\rho}_n}}{\rho_n} w_{n+1}\right. \\
& \left. + \frac{\rho_{n+1}}{2\sqrt{H_{m,n}\ \bar{\rho}_n}}\left(h_{m,n}\sqrt{H_{m,n}} - \frac{\Delta\rho_n}{\rho_{n+1}} H_{m,n}\right)\right]^2 \\
& + \frac{1}{4}\sum_{n=n_2}^{m-1}\frac{(\rho_{n+1})^2}{\bar{\rho}_n}\left(h_{m,n} - \frac{\Delta\rho_n}{\rho_{n+1}}\sqrt{H_{m,n}}\right)^2 \\
<\ & H_{m,n_2}w_{n_2} + \frac{1}{4}\sum_{n=n_2}^{m-1}\frac{(\rho_{n+1})^2}{\bar{\rho}_n}\left(h_{m,n} - \frac{\Delta\rho_n}{\rho_{n+1}}\sqrt{H_{m,n}}\right)^2.
\end{aligned}$$

Therefore

$$\sum_{n=n_2}^{m-1}\left[H_{m,n}\rho_n q_n - \frac{(\rho_{n+1})^2}{4\ \bar{\rho}_n}\left(h_{m,n} - \frac{\Delta\rho_n}{\rho_{n+1}}\sqrt{H_{m,n}}\right)^2\right] < H_{m,n_2}w_{n_2} \leq H_{m,0}w_{n_2}$$

which implies that

$$\sum_{n=0}^{m-1}\left[H_{m,n}\rho_n q_n - \frac{(\rho_{n+1})^2}{4\ \bar{\rho}_n}\left(h_{m,n} - \frac{\Delta\rho_n}{\rho_{n+1}}\sqrt{H_{m,n}}\right)^2\right] < H_{m,0}\left(w_{n_2} + \sum_{n=0}^{n_2-1}\rho_n q_n\right)$$

Hence

$$\lim_{m\to\infty}\sup \frac{1}{H_{m,0}}\sum_{n=0}^{m-1}\left[H_{m,n}\rho_n q_n - \frac{(\rho_{n+1})^2}{4\ \bar{\rho}_n}\left(h_{m,n} - \frac{\Delta\rho_n}{\rho_{n+1}}\sqrt{H_{m,n}}\right)^2\right]$$
$$< \left(w_{n_2} + \sum_{n=0}^{n_2-1}\rho_n q_n\right) < \infty,$$

which contradicts (2.1). Therefore every solution of (1.1) oscillates. □

Remark 2.1. From Theorem 2.1 we can obtain different conditions for oscillation of all solutions of equation (1.1) when (1.2) holds by different choices of $\{\rho_n\}$ and $H_{m,n}$.

Let $H_{m,n} = 1$. By Theorem 2.1 we have the following result.

Corollary 2.1. *Assume that* (1.2) *holds. Furthermore, assume that there exists a positive sequence* $\{\rho_n\}$*such that for some positive constant* M

$$\lim_{n\to\infty}\sup \sum_{l=n_0}^{n}\left[\rho_l q_l - \frac{(\Delta\rho_l)^2}{2^{3-\beta}M^{\beta-1}\rho_l}\right] = \infty. \tag{2.15}$$

Then every solution of equation (1.1) *oscillates.*

Let $\rho_n = n^\lambda$, $n \geq n_0$, $\lambda \geqslant 1$ be a constant and $H_{m,n} = 1$; then from Theorem 2.1 we have the following result.

Corollary 2.2. *Assume that all the assumptions of Theorem* 2.1 *hold except that condition* (2.1) *is replaced by*

$$\lim_{n\to\infty}\sup \sum_{s=n_0}^{n}\left[s^\lambda q_s - \frac{((s+1)^\lambda - s^\lambda)^2}{2^{3-\beta}M^{\beta-1}s^\lambda}\right] = \infty. \tag{2.16}$$

Then every solution of equation (1.1) *oscillates.*

Remark 2.2. Note that when $F(n,u,v) = q_n u$, equation (1.1) reduces to the linear difference equation

$$\Delta^2 x_n + q_n x_n = 0, \quad n = 0, 1, 2, \ldots, \tag{2.17}$$

and condition (2.15) reduces to

$$\lim_{n\to\infty}\sup \sum_{l=n_0}^{n}\left[\rho_l q_l - \frac{(\Delta\rho_l)^2}{4\rho_l}\right] = \infty. \tag{2.18}$$

Then Theorem 2.1 is an extension of Theorem 4 in [13] and improves Theorem A in [10].

Remark 2.3. If $F(n,u,v) = q_n u^\beta$, then equation (1.1) reduces to the equation

$$\Delta^2 x_n + q_n x_n^\beta = 0, \quad n = 0, 1, 2, \ldots,$$

and condition (2.1) reduces to

$$\lim_{n\to\infty}\sup\sum_{l=n_0}^{n}\left[\rho_l q_l-\frac{(\Delta\rho_l)^2}{2^{3-\beta}M^{\beta-1}\rho_l}\right]=\infty,$$

which improves Theorem 4.1 in [11].

The following example is illustrative.

Example 2.1. Consider the discrete Euler equation

$$\Delta^2 x_n+\frac{\mu}{n^2}x_n=0,\quad n\geq 1. \tag{2.19}$$

Here $\beta=1$,

$$F(n,x_n,\Delta x_n)=\frac{\mu}{n^2}\,x_n,$$

where $\mu>\frac{1}{4}$. Thus $q_n=\frac{\mu}{n^2}$. Therefore if $\rho_n=n$, then (2.18) becomes

$$\begin{aligned}\lim_{n\to\infty}\sup\sum_{l=n_0}^{n}\left[\rho_l q_l-\frac{(\Delta\rho_l)^2}{4\rho_l}\right]&=\lim_{n\to\infty}\sup\sum_{s=n_0}^{n}\left[\frac{\mu}{s}-\frac{1}{4s}\right]\\&=\lim_{n\to\infty}\sup\sum_{s=n_0}^{n}\frac{4\mu-1}{s}\to\infty.\end{aligned}$$

By Corollary 2.1, every solution of the discrete Euler equation oscillates. It is known [15] that when $\mu\leq\frac{1}{4}$, the discrete Euler equation has a nonoscillatory solution. Hence Theorem 2.1 and Corollary 2.1 are sharp.

Remark 2.4. We can use a general class of double sequences $\{H_{m,n}\}$ as the parameter sequences in Theorem 2.1 to obtain different conditions for oscillation of equation (1.1). By choosing specific sequence $\{H_{m,n}\}$, we can derive several oscillation criteria for equation (1.1). Let us consider the double sequence $\{H_{m,n}\}$ defined by

$$\begin{aligned}H_{m,n}&=(m-n)^\lambda, && m\geq n\geq 0,\ \ \lambda\geq 1,\\ H_{m,n}&=\left(\log\tfrac{m+1}{n+1}\right)^\lambda, && m\geq n\geq 0,\ \ \lambda\geq 1.\end{aligned} \tag{2.20}$$

Then $H_{m,m}=0$ for $m\geq 0$, and $H_{m,n}>0$ and $\Delta_2 H_{m,n}\leq 0$ for $m>n\geq 0$. Hence we have the following results.

Corollary 2.3. *Assume that all the assumptions of Theorem* 2.2 *hold except that condition* (2.1) *is replaced by*

$$\begin{aligned}\lim_{m\to\infty}\sup\frac{1}{m^\lambda}\sum_{n=0}^{m}\Bigg[&(m-n)^\lambda\rho_n q_n\\&-\frac{\rho_{n+1}^2}{4\,\bar{\rho}_n}\left(\lambda(m-n)^{\frac{\lambda-2}{2}}-\frac{\Delta\rho_n}{\rho_{n+1}}(m-n)^{\frac{\lambda}{2}}\right)^2\Bigg]=\infty.\end{aligned} \tag{2.21}$$

Then every solution of equation (1.1) *oscillates.*

Corollary 2.4. *Assume that all the assumptions of Theorem 2.2 hold except that condition (2.1) is replaced by*

$$\lim_{m\to\infty}\sup \frac{1}{(\log(m+1))^{\lambda}}\sum_{n=0}^{m}\left[\left(\log\frac{m+1}{n+1}\right)^{\lambda}\rho_n q_n-\frac{\rho_{n+1}^2 A_{m,n}}{4\,\bar{\rho}_n}\right]=\infty, \quad (2.22)$$

where

$$A_{m,n}=\left(\frac{\lambda}{n+1}\left(\log\frac{m+1}{n+1}\right)^{\frac{\lambda-2}{2}}-\frac{\Delta\rho_n}{\rho_{n+1}}\left(\log\frac{m+1}{n+1}\right)^{\frac{\lambda}{2}}\right)^2.$$

Then every solution of equation (1.1) oscillates.

Another $H_{m,n}$ may be chosen as

$$H_{m,n}=\phi(m-n), \quad m\geq n\geq 0,$$

or

$$H_{m,n}=(m-n)^{(\lambda)}, \quad \lambda>2, \quad m\geq n\geq 0,$$

where $\phi:[0,\infty)\to[0,\infty)$ is a continuously differentiable function which satisfies $\phi(0)=0$ and $\phi(u)>0$, $\phi'(u)\geq 0$ for $u>0$, and $(m-n)^{(\lambda)}=(m-n)(m-n+1)\cdots(m-n+\lambda-1)$ and

$$\Delta_2(m-n)^{(\lambda)}=(m-n-1)^{(\lambda)}-(m-n)^{(\lambda)}=-\lambda(m-n)^{(\lambda-1)}.$$

The corresponding corollaries can also be stated.

Now, we consider the case where (1.1) is strictly sublinear.

Theorem 2.2. *Assume that (1.2) holds, and let $\{\rho_n\}$ be a positive sequence. Furthermore, assume that there exists a double sequence $\{H_{m,n}: m\geq n\geq 0\}$ such that*

(i) $H_{m,m}=0$ *for* $m\geq 0$,
(ii) $H_{m,n}>0$ *for* $m>n>0$,
(iii) $\Delta_2 H_{m,n}=H_{m,n+1}-H_{m,n}$.
If

$$\lim_{m\to\infty}\sup\frac{1}{H_{m,0}}\sum_{n=0}^{m-1}\left[H_{m,n}\rho_n q_n-\frac{\rho_{n+1}^2}{4P_n}\left(h_{m,n}-\frac{\Delta\rho_n}{\rho_{n+1}}\sqrt{H_{m,n}}\right)^2\right]=\infty, \quad (2.23)$$

where $P_n=\frac{\beta\rho_n}{b^{1-\beta}(n+1)^{1-\beta}}$, *then every solution of equation (1.1) oscillates.*

Proof. Proceeding as in the proof of Theorem 2.1, we assume that equation (1.1) has a nonoscillatory solution $x_n>0$ for all n$\geq n_0$. Defining again w_n by (2.6), we obtain (2.8). Now, using the inequality (cf. [9, p. 39]),

$$x^{\beta}-y^{\beta}\geqslant\beta x^{\beta-1}(x-y) \quad \text{for all} \quad x\neq y>0 \quad \text{and} \quad 0<\beta\leq 1$$

we find that

$$\Delta(x_n^{\beta})=x_{n+1}^{\beta}-x_n^{\beta}\geqslant\beta\left(x_{n+1}\right)^{\beta-1}\left(x_{n+1}-x_n\right)=\beta\left(x_{n+1}\right)^{\beta-1}\left(\Delta x_n\right). \quad (2.24)$$

Substituting (2.24) in (2.8), we have

$$\Delta w_n \leq -\rho_n q_n + \frac{\Delta\rho_n}{\rho_{n+1}} w_{n+1} - \rho_n a_n \frac{\beta\,(x_{n+1})^{\beta-1}\,(\Delta x_n)\,(\Delta x_{n+1})}{\left(x_{n+1}^{\beta}\right)^2}\,.$$

From (2.5) and the last inequality we obtain

$$\Delta w_n \leq -\rho_n q_n + \frac{\Delta\rho_n}{\rho_{n+1}} w_{n+1} - \frac{\beta\rho_n}{(\rho_{n+1})^2\,(x_{n+1})^{1-\beta}} \frac{(\rho_{n+1})^2\,(\Delta x_{n+1})^2}{\left(x_{n+1}^{\beta}\right)^2}\,.$$

Hence

$$\Delta w_n \leq -\rho_n q_n + \frac{\Delta\rho_n}{\rho_{n+1}}\, w_{n+1} - \frac{\beta\rho_n}{(\rho_{n+1})^2\,(x_{n+1})^{1-\beta}}\, w_{n+1}^2\,. \tag{2.25}$$

From (2.5) we conclude that

$$x_n \leq x_{n_0} + \Delta x_{n_0}(n-n_0), \quad n \geq n_0,$$

and consequently there exists $n_1 \geq n_0$ and an appropriate constant $b \geq 1$ such that

$$x_n \leq bn \quad \text{for} \quad n \geq n_1.$$

This implies that

$$x_{n+1} \leq b(n+1) \quad \text{for} \quad n \geq n_2 = n_1 - 1$$

and hence

$$\frac{1}{(x_{n+1})^{1-\beta}} \geq \frac{1}{b^{1-\beta}(n+1)^{1-\beta}}\,. \tag{2.26}$$

From (2.25) and (2.26) we obtain

$$\rho_n q_n \leq -\Delta w_n + \frac{\Delta\rho_n}{\rho_{n+1}}\, w_{n+1} - \frac{P_n}{(\rho_{n+1})^2}\, w_{n+1}^2. \tag{2.27}$$

The remainder of the proof is similar to that of the proof of Theorem 2.1 and hence is omitted. □

From Theorem 2.2 we can obtain different conditions for oscillation of all solutions of equation (1.1) when (1.2) holds by different choices of $\{\rho_n\}$ and $H_{m,n}$. Let $H_{m,n} = 1$. By Theorem 2.2 we have the following result.

Corollary 2.5. *Assume that* (1.2) *holds. Furthermore, assume that there exists a positive sequence* $\{\rho_n\}$ *such that for every* $b \geq 1$

$$\limsup_{n\to\infty} \sum_{l=0}^{n} \left[\rho_l q_l - \frac{b^{1-\beta}(l+1)^{1-\beta}(\Delta\rho_n)^2}{4\beta\rho_l}\right] = \infty. \tag{2.28}$$

Then every solution of equation (1.1) *oscillates.*

Remark 2.5. Corollary 2.5 improves Theorem 4.3 in [11].

Let $H_{m,n} = 1$ and $\rho_n = n^{\lambda}$, $n \geq n_0$ and $\lambda \geqslant 1$ be a constant; then we have the following result.

Corollary 2.6. *Assume that all the assumptions of Theorem* 2.3 *hold except that condition* (2.28) *is replaced by*

$$\lim_{n\to\infty}\sup\sum_{s=n_0}^{n}\left[s^{\lambda}q_s-\frac{b^{1-\beta}(s+1)^{1-\beta}((s+1)^{\lambda}-s^{\lambda})^2}{4\beta s^{\lambda}}\right]=\infty. \tag{2.29}$$

Then every solution of equation (1.1) *oscillates.*

Example 2.2. Consider the difference equation

$$\Delta^2 x_n+\frac{2n+1}{[n(n+1)^2]^{\frac{1}{3}}}(x_n)^{\frac{1}{3}}=0,\quad n\geq 1.$$

Since $\beta=\frac{1}{3}$, we have

$$q_n=\frac{1+2n}{[n(n+1)^2]^{\frac{1}{3}}}\ .$$

By choosing $\rho_n=n+1$ and $b=1$, we have

$$\begin{aligned}\sum_{s=1}^{n}\left[\rho_s q_s-\frac{(s+1)^{1-\beta}(\Delta\rho_l)^2}{4\beta\rho_l}\right]&=\sum_{s=1}^{n}\left[(1+2s)-\frac{3(s+1)^{\frac{2}{3}}}{4(s+1)}\right]\\&\geq\sum_{s=1}^{n}\left[(1+2s)-\frac{3(s+1)^2}{4(s+1)}\right]\to\infty\end{aligned}$$

as $n\to\infty$. So according to Corollary 2.6 every solution of this equation oscillates.

The following corollaries follow immediately from Theorem 2.2.

Corollary 2.7. *Assume that all the assumptions of Theorem* 2.2 *hold except that condition* (2.23) *is replaced by*

$$\lim_{m\to\infty}\sup\frac{1}{m^{\lambda}}\sum_{n=0}^{m}\left[(m-n)^{\lambda}\rho_n q_n-\frac{\rho_{n+1}^2}{4P_n}\left(\lambda(m-n)^{\frac{\lambda-2}{2}}-\frac{\Delta\rho_n}{\rho_{n+1}}(m-n)^{\frac{\lambda}{2}}\right)^2\right]=\infty.$$

Then every solution of equation (1.1) *oscillates.*

Corollary 2.8. *Assume that all the assumptions of Theorem* 2.2 *hold except that condition* (2.23) *is replaced by*

$$\lim_{m\to\infty}\sup\frac{1}{(\log(m+1))^{\lambda}}\sum_{n=0}^{m}\left[\left(\log\frac{m+1}{n+1}\right)^{\lambda}\rho_n q_n-\frac{\rho_{n+1}^2 A_{m,n}}{4P_n}\right]=\infty,$$

where $A_{m,n}$ is as defined in Corollary 2.4. *Then, every solution of equation* (1.1) *oscillates.*

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Author's address:

Mathematics Department
Faculty of Science
Mansoura University
Mansoura, 35516
Egypt
E-mail: shsaker@mum.mans.eun.eg