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ON SEPARATION PROPERTIES FOR FAMILIES OF PROBABILITY MEASURES

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Abstract. We consider the problem of transition from a weakly separated family of probability measures to a strictly separated family.

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Let (E, S) be a measurable space. A family of probability measures $(\mu_i)_{i\in I}$ defined on this space is called weakly separated if there exists a family $(X_i)_{i\in I}$ of measurable subsets of E such that

$$(\forall i)(i \in I \ \&\ j \in I \to \mu_i(X_j) = \delta(i, j)),$$

where $\delta(i, j)$denotes Kronecker's function on the Cartesian square I^2 of the set I.

A family of probability measures $(\mu_i)_{i\in I}$ defined on the measurable space (E, S) is called strictly separated if there exists a disjoint family $(X_i)_{i\in I}$ of measurable subsets of E such that

$$(\forall i)(i \in I \to \mu_i(X_i) = 1).$$

It is clear that an arbitrary strictly separated family $(\mu_i)_{i\in I}$ of probability measures is weakly separated.

In connection with the definitions above, see [6] where the structure of weakly separated and strictly separated families of probability measures is investigated.

In a general theory of statistical decisions there often arises a question of transition from a weakly separated family of probability measures to the corresponding strictly separated family. In this context, the following result is of certain interest.

Theorem 1. *In the system of axioms* (ZFC) *the following three conditions are equivalent:*

1) *The Continuum Hypothesis* ($\mathbf{c} = 2^{\aleph_0} = \aleph_1$)*;*

2) *for an arbitrary probability space* (E, S, μ)*, the* μ*-measure of the union of any family* $(E_i)_{i\in I}$ *of* μ*-measure zero subsets, such that* $\mathrm{card}(I) < c$*, is equal to zero;*

3) *an arbitrary weakly separated family of probability measures, of cardinality continuum, is strictly separated.*

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Proof. 1) $\to$ 2). Let (E, S, μ) be an arbitrary probability space and let $(E_i)_{i\in I}$ be a family of μ-measure zero subsets of E such that $\operatorname{card}(I) < c$. Applying condition 1), we have $\operatorname{card}(I) \le \omega$, where ω denotes the cardinality of the set of all natural numbers. Finally, applying the semiadditivity of the measure μ, we obtain

$$\mu\Big(\bigcup_{i\in I} E_i\Big) \le \sum_{i\in I} \mu(E_i) = 0.$$

The implication 1) $\to$ 2) is thus proved.

2) $\to$ 3). Let ω_ϕ denote the first ordinal number of cardinality of the continuum, let $(\mu_\xi)_{\xi\prec\omega_\phi}$ be a family of probability measures defined on a measurable space (E, S) and suppose that there exists a family $(X_\xi)_{\xi\prec\omega_\phi}$ of measurable subsets of E such that

$$(\forall\xi)(\forall\tau)(\xi \prec \omega_\phi \ \&\ \tau \prec \omega_\phi \to \mu_\xi(X_\tau) = \delta(\xi, \tau)),$$

where $\delta(\xi, \tau)$ denotes Kronecker's function on the Cartesian square $[0; \omega_\phi[\times [0; \omega_\phi[$ of the set $[0; \omega_\phi[$.

Let

$$(\forall\xi)\Big(\xi \prec \omega_\phi \to Y_\xi = X_\xi \setminus \bigcup_{\tau\prec\xi} X_\tau\Big).$$

By the condition 2) we conclude that $(Y_\xi)_{\xi\prec\omega_\phi}$ is a disjoint family of measurable subsets of the space E such that

$$(\forall\xi)(\xi \prec \omega_\phi \to \mu_\xi(Y_\xi) = 1).$$

This means that the implication 2) $\to$ 3) is proved.

3) $\to$ 1). For arbitrary $x \in]0; 1[$, define the σ-algebra B_x of subsets of the space $\triangle_2 =]0; 1[\times]0; 1[$ by

$$B_x = \big\{Y | Y \subseteq \triangle_2 \ \&\ (\operatorname{card}(Y \cap (\{x\}\times]0; 1[)) \le \aleph_0) \vee (\operatorname{card}((\{x\}\times]0; 1[) \setminus Y) \le \aleph_0)\big\}.$$

For arbitrary $x \in]1; 2[$, denote by B_x the σ-algebra of subsets of the space $\triangle_2$ defined by

$$B_x = \big\{\ Y | Y \subseteq \triangle_2 \ \&\ (\operatorname{card}(Y \cap (]0; 1[\times\{x-1\})) \le \aleph_0) \\ \vee(\operatorname{card}((]0; 1[\times\{x-1\}) \setminus Y) \le \aleph_0)\big\}.$$

Let us put

$$S = \bigcap_{x\in]0;1[\cup]1;2[} B_x.$$

It is clear that each element of the families $(\{x\}\times]0; 1[)_{x\in]0;1[}$ and $(]0; 1[\times \{x-1\})_{x\in]1;2[}$ belongs to the σ-algebra S.

Define the family $(\mu_t)_{t\in]0;1[\cup]1;2[}$ of probability measures by

$$(\forall t)\Bigg(t \in]0; 1[\to (\forall Z)\Bigg(Z \in S \to \mu_t(Z) \\ = \begin{cases} 1, & \text{if } \operatorname{card}((\{t\}\times]0; 1[) \setminus Z) \le \aleph_0, \\ 0, & \text{if } \operatorname{card}((\{t\}\times]0; 1[) \cap Z) \le \aleph_0 \end{cases}\Bigg)\Bigg),$$

$$(\forall t)\Big(t \in]1;2[\to (\forall Z)\Big(Z \in S \to \mu_t(Z)$$
$$= \left\{ \begin{array}{ll} 1, & \text{if } \mathrm{card}((]0;1[\times\{t-1\}) \setminus Z) \leq \aleph_0, \\ 0, & \text{if } \mathrm{card}((]0;1[\times\{t-1\}) \cap Z) \leq \aleph_0 \end{array} \right. \Big)\Big).$$

Let us consider the family $(X_t)_{t\in]0;1[\cup]1;2[}$ of measurable subsets of the space $\triangle_2$, where

$$(\forall t)\left(t \in]0;1[\cup]1;2[\to X_t = \left\{ \begin{array}{ll} \{t\}\times]0;1[& \text{if } t<1 \\ ]0;1[\times\{t-1\} & \text{if } t>1 \end{array} \right)\right..$$

It is clear that the family $(\mu_t)_{t\in]0;1[\cup]1;2[}$ of probability measures is weakly separated because of

$$(\forall t_1)(\forall t_2)((t_1,t_2) \in \big(]0;1[\cup]1;2[\big)^2 \to \mu_{t_1}(X_{t_2}) = \delta(t_1,t_2)),$$

where $\delta(.,.)$ denotes Kronecker's function defined on the Cartesian square $(]0;1[\cup]1;2[)^2$ of the set $]0;1[\cup]1;2[$.

From the condition 3) we have that the family $(\mu_t)_{t\in]0;1[\cup]1;2[}$ of probability measures is strictly separated. This means that there exists a family of disjoint measurable subsets $(Y_t)_{t\in]0;1[\cup]1;2[}$ such that

$$(\forall t)\big(t \in]0;1[\cup]1;2[\to \mu_t(Y_t) = 1\big).$$

We may assume without loss of generality that $Y_t \subseteq X_t$ for all $t \in]0;1[\cup]1;2[$. Let us consider the sets $A = \bigcup_{t\in]0;1[} Y_t$ and $B = \bigcup_{t\in]1;2[} Y_t$. It is clear that A and B do not have common points. On the other hand, we can write

$$(\forall x)\big(x \in]0;1[\to \mathrm{card}((\{x\}\times]0;1[) \cap B) \leq \aleph_0$$
$$\& \ \mathrm{card}((]0;1[\times\{x\}) \cap A) \leq \aleph_0\big).$$

Denote by $(C_\xi)_{\xi\prec\omega_1}$ some injective transfinite sequence of horizontal segments of the space $\triangle_2$. It is clear that

$$\mathrm{card}\Big(A \cap \Big(\bigcup_{\xi\prec\omega_1} C_\xi\Big)\Big) \leq \aleph_0 \times \aleph_1 = \aleph_1.$$

We have to prove that the orthogonal projection of the set $A \cap \big(\bigcup_{\xi\prec\omega_1} C_\xi\big)$ on the interval $]0;1[\times\{0\}$ coincides with this interval. Indeed, let a be an arbitrary vertical segment of the space $\triangle_2$. Since

$$\mathrm{card}(B \cap a) \leq \aleph_0,$$

there exists an ordinal index $\xi_0 \prec \omega_1$ such that the point of the intersection of C_{ξ_0} and a belongs to the set A. This means that the set $A \cap \big(\bigcup_{\xi\prec\omega_1} C_\xi\big)$ is projected on the whole interval $]0;1[\times\{0\}$ and therefore

$$2^{\aleph_0} \leq \aleph_1. \qquad \square$$

Remark 1. Note that the implication 1) $\to$ 3) was obtained in [6]. The validity of the implication 3) $\to$ 1) was established in [15].

Remark 2. M. Coldstern [4] offers a different proof of the equivalence of the conditions 1) and 2). His proof is based on the following fact:

Fact *A*: There is a measure space and a family of $\aleph_1$-many measure zero sets whose union is not measure zero, and not even measurable.

Notice that Fact A is true in the usual axiomatic set theory (e.g., in ZFC).

One proof of Fact A reads as follows:

Take any uncountable set X. Consider the σ-algebra of those subsets of X which are either at most countable or whose complement is at most countable. Define the measure μ by letting $\mu(C) = 0$ and $\mu(X \setminus C) = 1$ whenever C is countable. This is a complete measure and serves as an example for Fact A.

Here is the second example (proposed by the same author) with an incomplete measure.

Consider the σ-algebra of Borel sets equipped with the Lebesgue measure.

Then there is a family of $\aleph_1$-many measure zero sets whose union is not measurable. This example can be found in [3](see Volume 5, Exercise $511Xj$).

Remark 3. In the system of axioms $(ZFC)\&(\neg CH)\&(MA)$ the family of probability measures $(\mu_t)_{t\in]0;1[\cup]1;2[}$ considered in Theorem 1 is an example of a weakly separated family of probability measures which is not strictly separated.

Remark 4. It is reasonable to note that the pair $\{A, B\}$ constructed in Theorem 1 is similar to the Sierpiński partition of the unit square $]0;1[^2$ (see, e.g., [16]).

Remark 5. Applying the well-known results of Cohen and Gödel (see [1] and [5]), we conclude that each of the following statements:

– "for an arbitrary probability space (E, S, μ) the μ-measure of the union of every family $(E_i)_{i\in I}$ of μ-measure zero subsets, such that $\text{card}(I) < c$, is equal to zero";

– "an arbitrary weakly separated family of probability measures is strictly separated whenever its cardinality is not greater than $2^{\aleph_0}$",

is independent of the theory ZFC.

Let us consider the question of transition from a weakly separated family of probability measures to a strictly separated one when the family of probability measures is defined on the so-called Radon metric space (about the notion of a Radon metric space, see, e.g., [9], [17]). The next auxiliary proposition plays the key role in our further consideration.

Lemma 1. *Let (E, ρ) be a Radon metric space. Let μ be an arbitrary σ-finite Borel measure defined on E. Then there exists a closed separable subspace $E(\mu)$ of E such that*

$$\mu(E \setminus E(\mu)) = 0.$$

Remark 6. We remind the reader that a cardinal number α is real-valued measurable if there exists a continuous probability measure defined on the class of all subsets of some set of cardinality α. In connection with Lemma 1, we must also recall that an arbitrary complete metric space (E, ρ) whose topological weight is not a real-valued measurable cardinal, is a Radon metric space (see, e.g., [9], p. 48, Theorem 7).

The following important result is essentially due to Martin and Solovay (see, e.g., [2] and [7]).

Lemma 2. *Let (F,ρ) be a separable metric space equipped with some probability Borel measure μ. If $(E_i)_{i\in I}$ is a family of μ-measure zero subsets of F, such that* $\mathrm{card}(I) < c$, *then (in the system of axioms (ZFC) & (MA)) the outer measure μ^* of the set $E = \bigcup_{i\in I} E_i$ is equal to zero.*

The proof of Lemma 4 can be found, e.g., in [7]. The following theorem is valid.

Theorem 2. *Let (F,ρ) be a Radon metric space. Let $(\mu_i)_{i\in I}$ be a weakly separated family of Borel probability measures with* $\mathrm{card}(I) \leq \mathbf{c}$ *defined on (F,ρ). Then, in the system of axioms (ZFC) & (MA), the family $(\mu_i)_{i\in I}$ is strictly separated.*

Proof. Note that an arbitrary Borel probability measure μ defined on the space (F,ρ) has the property

$$(\forall J)(\forall (X_i)_{i\in J})\Big(\mathrm{card}(J) < 2^{\aleph_0} \ \&\ (\forall i)(i\in J \to \mu(X_i)=0) \to \mu^*\Big(\bigcup_{i\in J} X_i\Big) = 0\Big).$$

Indeed, by Lemma 3 applied to μ, there exists a separable closed support $F(\mu)$ in (F,ρ). Let us consider the set

$$\bigcup_{i\in J} X_i = \Big[\Big(\bigcup_{i\in J} X_i\Big)\cap F(\mu)\Big] \cup \Big[(F\setminus F(\mu))\cap\Big(\bigcup_{i\in J} X_i\Big)\Big].$$

Using Lemma 4, we conclude that the set $\big(\bigcup_{i\in J} X_i\big)\cap F(\mu)$ is a μ^*-measure zero subset of $F(\mu)$. Note that the outer measure of the set $(F\setminus F(\mu))\cap \big(\bigcup_{i\in J} X_i\big)$ is equal to zero because $\mu(F\setminus F(\mu)) = 0$.

Let $(\mu_i)_{i\in J}$ be a weakly separated family of Borel probability measures with $\mathrm{card}(J) \leq \mathbf{c}$. Let us represent this family as an injective sequence $(\mu_\xi)_{\xi\prec\omega_\alpha}$, where the first ordinal number of cardinality J is denoted by ω_α. Since the family $(\mu_\xi)_{\xi\prec\omega_\alpha}$ is weakly separated, there exists a family $(X_\xi)_{\xi\prec\omega_\alpha}$ of Borel subsets of the space F such that

$$(\forall\xi)(\forall\tau)(\xi\in[0;\omega_\alpha[\ \&\ \tau\in[0;\omega_\alpha[\to \mu_\xi(X_\tau) = \delta(\xi,\tau)),$$

where $\delta(.,.)$ denotes Kronecker's function on the Cartesian square $[0;\omega_\alpha[^2$ of the set $[0;\omega_\alpha[$. Let us define an ω_α-sequence of subsets $(B_\xi)_{\xi\prec\omega_\alpha}$ of the metric space F so that:

1)$(\forall\xi)(\xi\prec\omega_\alpha \to B_\xi$ is a Borel subset in $F)$;
2)$(\forall\xi)(\xi\prec\omega_\alpha \to B_\xi\subseteq X_\xi)$;
3)$(\forall\tau_1)(\forall\tau_2)(\tau_1\prec\omega_\alpha \ \&\ \tau_2\prec\omega_\alpha \ \&\ \tau_1\neq\tau_2 \to B_{\tau_1}\cap B_{\tau_2} = \emptyset)$;
4)$(\forall\tau)(\tau\prec\omega_\alpha \to \mu_\tau(B_\tau) = 1)$.

Take $B_0 = X_0$. Let, for $\xi \prec \omega_\alpha$, the partial sequence $(B_\tau)_{\tau\prec\xi}$ be already constructed. It is clear that

$$\mu_\xi^*\Big(\bigcup_{\tau\prec\xi} B_\tau\Big) = 0.$$

This means that there exists a Borel subset Y_ξ of the space F such that

$$\bigcup_{\tau\prec\xi} B_\tau \subseteq Y_\xi, \mu_\xi(Y_\xi) = 0.$$

We put $B_\xi = X_\xi \setminus Y_\xi$. Now it can easily be verified that the ω_α-sequence $(B_\xi)_{\xi\prec\omega_\alpha}$ of disjoint measurable subsets of the space F is constructed so that

$$(\forall\xi)\big(\xi \prec \omega_\alpha \to \mu_\xi(B_\xi) = 1\big). \qquad \square$$

Remark 7. Theorem 2 generalizes the main results obtained in [15] and [18]. Similar results are also discussed in [8], [10], [11], [13] and [14].

The next remark shows that all complete metric spaces can be assumed to be Radon (under some additional set-theoretic hypothesis.

Remark 8. The following conditions are equivalent:

a) an arbitrary complete metric space is a Radon space;

b) there does not exist a real-valued measurable cardinal.

Proof. *a*) $\to$ *b*). Assume the contrary and let J be a real-valued measurable cardinal. Let μ be a continuous probability measure defined on the class of all subsets of J.

Let us define a metric space (V, ρ) by

1) $V = J$;

2) $(\forall x)(\forall y)(x \in V \& y \in V \to \rho(x, y) = 1$ if $x \neq y$, and $\rho(x, y) = 0$ if $x = y)$.

It is clear that (V, ρ) is a complete metric space whose topological weight is equal to J. The measure μ is not concentrated on a separable closed subset, because such a subset is at most countable and hence has μ-measure zero.

b) $\to$ *a*). Let (V, ρ) be an arbitrary complete metric space and W be its topological weight. By using the validity of the condition *b*), we have that W is not a real-valued measurable cardinal. In view of Remark 6 we conclude that (V, ρ) is a Radon metric space. $\square$

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