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TIME-DEPENDENT BARRIER OPTIONS AND BOUNDARY CROSSING PROBABILITIES

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Abstract. The problem of pricing of time-dependent barrier options is considered in the case when interest rate and volatility are given functions in Black–Scholes framework. The calculation of the fair price reduces to the calculation of non-linear boundary crossing probabilities for a standard Brownian motion. The proposed method is based on a piecewise-linear approximation for the boundary and repeated integration. The numerical example provided draws attention to the performance of suggested method in comparison to some alternatives.

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1. Introduction

In the diffusion equation for an underlying asset S_t let us assume the coefficients $\mu(t)$ and $\sigma(t)$ to be time-dependent,

$$dS_t = \mu(t)S_t dt + \sigma(t)S_t dW_t, \quad 0 \le t \le T < \infty, \tag{1}$$

W_t is a standard Wiener process given on a probability space (Ω, F, P). We assume a bank account process B_t is driven by the equation $dB_t = r(t)B_t dt$ and hence

$$B_t = \exp\bigg(\int_0^t r(s)ds\bigg),$$

where $r(t)$ is a positive function of time, the so-called spot interest rate. The solution of equation (1) is

$$S_t = S_0 \exp\bigg\{\int_0^t \mu(s)ds - \frac{1}{2}\int_0^t \sigma^2(s)ds + \int_0^t \sigma(s)dW_s\bigg\}. \tag{2}$$

We assume here that $\mu(s)$ and $\sigma(s)$ are square-integrable and nonrandom functions. Further, we also assume that $\mu(s) = r(s)$, $0 \le s \le T$. This assumption means that we use the free-arbitrage approach to pricing of options (see details, e.g., in [1] or [2]). Then the process $\{S_t/B_t,\ t \ge 0\}$ is a martingale with respect to the information flow $F_t = \sigma\{S_s,\ 0 \le s \le t\}$ and probability measure P defined above.

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It is well known that under the free-arbitrage assumption the fair price of an option with a payoff function f_T is given by the formula

$$\mathbf{C_T} = E[f_T/B_T],$$

where $E(\cdot)$ is a symbol of expectation with respect to measure P (see details, e.g., in [1] or [2]).

A down-and-out call option is a call option that expires if the stock price falls below the prespecified "out" barrier H. "Down" here refers to an initial price of stock S_0 being above of the barrier H. A down-and-in call is a call that comes into existence if the stock price falls below the "in" barrier at any time during the option's life. Note, if we buy a down-and-out call and a down-and-in call with the same barrier price, H, strike price K, and time to expiration, T, the payoff of the portfolio is the same as for a standard call option. In the case of up-and-out option, the barrier lies above the initial stock price, and if the stock price ever rises above the barrier, then the option becomes worthless. Similarly, there exist up-and-in options. Below we consider the case of up-and-out barrier option with time-dependent upper barrier $H(t)$. In this case the payoff function is

$$f_T = (S_T - K)^+ I\{\tau > T\} = (S_T - K)\ I\{S_T > K,\ \tau > T\}$$

where we use the notation $I\{A\}$ for the indicator function of a set A and

$$\tau = \inf\{t \geq 0 : S_t \geq H(t)\}.$$

2. Pricing of Time-Dependent Barrier Options

The problem is to find a fast and accurate algorithm for the calculation of the fair price of up-and-out barrier option

$$\mathbf{C_T} = E\big[(S_T - K) I\{S_T > K,\ \tau > T\}/B_T\big].$$

This problem has been addressed, e.g., by Roberts and Shortland in [3]. For simplicity of the notation and further exposition, we assume the volatility function is a constant: $\sigma(s) = \sigma > 0$. The following proposition reduces the pricing problem to the calculation of boundary crossing probabilities by the standard Wiener process with respect to measure P.

Proposition 1. *The fair price of a up-and-out European call option with a single upper barrier $H(t)$ is given by*

$$\mathbf{C_T} = S_0 p_1 - K \exp\Big(-\int_0^T r(s)ds\Big) p_0, \tag{3}$$

where

$$p_1 = P\{\sigma W_T + \sigma^2 T > G;\ \ \sigma W_t + \sigma^2 t < g(t),\ \ t \le T\},$$
$$p_0 = P\{\sigma W_T > G;\ \ \sigma W_t < g(t),\ \ t \le T\},$$
$$G = \ln\left(\frac{K}{S_0}\right) + \frac{1}{2}\,\sigma^2 T - \int_0^T r(s)ds,$$
$$g(t) = \ln\left(\frac{H(t)}{S_0}\right) + \frac{1}{2}\,\sigma^2 t - \int_0^t r(s)ds.$$

Proof. Using (2) with $\sigma(s) = \sigma$ we have

$$\begin{aligned}\mathbf{C_T} &= E\Big[I\big\{S_T > K,\ \tau > T\big\}\frac{S_T}{B_T}\Big] - E\Big[I\big\{S_T > K,\ \tau > T\big\}\frac{K}{B_T}\Big] \\ &= S_0 E\Big[I\big\{S_T > K,\ \tau > T\big\}\exp\Big\{\sigma W_T - \frac{1}{2}\sigma^2 T\Big\}\Big] \\ &\quad - K\exp\Big\{-\int_0^T r(s)ds\Big\}P\big\{S_T > K,\ \tau > T\big\}.\end{aligned}$$

To see that $P\{S_T > K, \tau > T\} = p_0$ one needs just express S_t and τ in terms of W_t.

Denote the Girsanov exponent

$$Z_T(f) = \exp\Big\{\int_0^T f(s)dW_s - \frac{1}{2}\int_0^T f^2(s)ds\Big\}.$$

By the Girsanov theorem (see, e.g., [2]) for any square-integrable nonrandom function $f(s)$ and an event $A \in F_T$

$$E[I\{A\}Z_T(f)] = \widetilde{P}\{A\} \tag{4}$$

where probability measure $\widetilde{P}$ is such that the process

$$\Big\{\widetilde{W}_t := W_t - \int_0^t f(s)ds,\ \ t \ge 0\Big\}$$

is a standard Wiener process with respect to $(F_t, \widetilde{P})$. Applying this fact with $f(s) = \sigma$ we have

$$\begin{aligned}p_1 &= \widetilde{P}\{\sigma W_T > G;\ \sigma W_t < g(t),\ t \le T\} \\ &= \widetilde{P}\{\sigma \widetilde{W}_T + \sigma^2 T > G;\ \sigma \widetilde{W}_t + \sigma^2 t < g(t),\ t \le T\} \\ &= P\{\sigma W_T + \sigma^2 T > G;\ \sigma W_t + \sigma^2 t < g(t),\ t \le T\}\ . \qquad \square\end{aligned}$$

Remark 1. For other types of barrier options, such as double barrier options or partial barrier options, the equation (3) still holds with modified values of p_1 and p_0.

We now need tools for the calculation of probabilities p_1 and p_0 in Proposition 1. In fact, the calculation of boundary crossing probabilities has other important applications besides the pricing of barrier options. This problem arises in various fields such as psychology (see [4]), clinical trials (see [5]) and many other areas as physics, insurance, and nonparametric statistics. While the time of calculation for the purpose of evaluating the fair price of barrier options is very important, in other applications like clinical trials or physics a high degree of accuracy becomes more important than the time of calculation. For calculation other methods could be used, such as partial differential equations (PDE), see [6], integral equations [7] and Monte Carlo simulation approaches. We now introduce a method based on numerical integration, proposed by Wang and Pötzelberger [8] and then developed by Novikov et al. [9] which led to an another work by Pötzelberger and Wang [10]. This method may in fact have certain advantages over the other methods. One of the advantages of this approach is that it can be used in the case of boundaries which may even be discontinuous. Another important advantage is that we can control the accuracy of the approximation as it will be shown below.

Let $\widehat{g}(s)$ be the boundary on the interval $[0,T]$ which is considered as an approximation for function $g(s)$ defined in Proposition 1. For example, one may consider $\widehat{g}(s)$ as piecewise-linear continuous functions with nodes t_i, $t_0 = 0 < t_1 < \cdots < t_n = T$ (in general, this function might be discontinuous or nonlinear). Denote

$$p(i, \widehat{g} \mid x_i, x_i+1) = P\{W_s \le \widehat{g}(s),\ \ s \in (t_i, t_{i+1}) \mid W_{t_i} = x_i,\ \ W_{t_{i+1}} = x_{i+1}\}$$

When $\widehat{g}(t)$ is a linear function on the interval (t_i, t_{i+1}) the last probability is given by (see, e.g., [8], [9])

$$\begin{aligned} &p\big(i, \widehat{g} \mid x_i, x_i+1\big) \\ &= I\big\{\widehat{g}(t_i) > x_i, \widehat{g}(t_{i+1}) > x_{i+1}\big\}\left[1 - \exp\left\{-\frac{2(\widehat{g}(t_i) - x_i)(\widehat{g}(t_{i+1}) - x_{i+1})}{t_{i+1} - t_i}\right\}\right]. \end{aligned}$$

The next formula gives the representation for a boundary crossing probability of the form

$$P(\widehat{g}, K, T) := P\big\{W_t \le \widehat{g}(t),\ \ t \le T;\ \ W_T > K\big\}$$

as an n-fold repeated integral of $p(i, \widehat{g}|x_i, x_i+1)$ and the transition probability of the Wiener process:

$$P(\widehat{g}, K, T) = E\big[I\{W_T > K\}\big] \prod_{i=0}^{n-1} p(i, \widehat{g} \mid W_{t_i}, W_{t_{i+1}}). \tag{5}$$

This formula seems to be firstly noted by Wang and Pötzelberger [8] for the case of piecewise one-sided continuous linear boundaries. Its generalization to the case of nonlinear two-sided boundaries was presented by Novikov et al. [9], see also [10].

Denote the density of the standard Wiener process W_t by $\varphi(x,t) = \frac{1}{\sqrt{2\pi t}} \times \exp(-\frac{x^2}{2t})$. The following recurrent algorithm indicated in [9] for the general case of two-sided boundaries can be used to evaluate the boundary crossing probability $P(\widehat{g}, K, T)$ in (5).

Let $z_0(x) = p(0, \widehat{g}_n \mid 0, x)\varphi(x, \Delta t_1)$,

$$z_{k+1}(x) = \int_{-\infty}^{\widehat{g}_n(t_k)} z_k(u) p(k+1, \widehat{g}_n \mid u, x)\varphi(x-u, \Delta t_k)du, \quad k = 0, \dots, n-1.$$

Then

$$P(\widehat{g}, K, T) = \int_K^{\widehat{g}_n(T)} z_{n-1}(u)\varphi(u, T)du. \tag{6}$$

We should note that formula (3′) in [9] contains a misprint which is corrected in (6). An example which illustrates calculation using this algorithm will be presented in Section 3. Here we discuss an accuracy of the approximation.

In [9] it is demonstrated how the measure transformation method based on the Girsanov theorem can be used for estimating difference between probabilities $P(g, K, T)$ and $P(\widehat{g}, K, T)$ in terms of the distance

$$\Delta_T(\widehat{g}, g) := \int_0^T \Big(\frac{d}{ds}\big(\widehat{g}(s) - g(s)\big)\Big)^2 ds.$$

Here the same technique is used to get an estimate for an accuracy of approximation for option prices.

We will always assume that the boundaries $g(t)$ and $\widehat{g}(t)$ which define the price $\mathbf{C_T}$ and its approximation $\widehat{\mathbf{C}}_\mathbf{T}$, respectively (see the notation in Proposition 1) have the following properties:

$$g(s) - \widehat{g}(s) \ \text{ is a continuous function,}$$
$$g(0) - \widehat{g}(0) = g(T) - \widehat{g}(T) = 0, \quad \Delta_T(\widehat{g}, g) < \infty.$$

Theorem 1.

$$|\mathbf{C_T} - \widehat{\mathbf{C}}_\mathbf{T}| \le C\sqrt{\Delta_T(\widehat{g}, g)},$$

where

$$C = \sqrt{\frac{2}{\pi\sigma^2}}\, E\big[(S_T - K)^+ / B_T\big]. \tag{7}$$

Proof. We will use the following representations for $\mathbf{C_T}$ and $\widehat{\mathbf{C}}_\mathbf{T}$ in terms of the original Wiener process:

$$\mathbf{C_T} = E\left[\left(S_0 B_T \exp\left\{\sigma W_T - \frac{\sigma^2 T}{2}\right\} - K\right)^+ I\left\{\sigma W_t < g(t),\ \ t \le T\right\}/B_T\right],$$
$$\widehat{\mathbf{C}}_\mathbf{T} = E\left[\left(S_0 B_T \exp\left\{\sigma W_T - \frac{\sigma^2 T}{2}\right\} - K\right)^+ I\left\{\sigma W_t < \widehat{g}(t),\ \ t \le T\right\}/B_T\right].$$

Let probability measure $\widetilde{P}$ be defined by formula (4) with

$$f(s) = \frac{d}{ds}\big(g(s) - \widehat{g}(s)\big)/\sigma.$$

Then by the Girsanov theorem the process

$$\big\{\widetilde{W}_t = W_t + (\widehat{g}(t) - g(t))/\sigma,\ \ t \ge 0\big\} \tag{8}$$

is a standard Wiener process with respect to $(F_t, \widetilde{P})$. Note that due to the assumption $\widehat{g}(T) - g(T) = 0$ we have the equality $\widetilde{W}_T = W_T$. Besides, expressing W_t via $\widetilde{W}_t$ from (8) and substituting it into the representation for $Z_T(f)$ we also have

$$(Z_T(f))^{-1} = \exp\left\{-\int_0^T \frac{d}{ds}\big(\widehat{g}(s) - g(s)\big)/\sigma d\widetilde{W}_s - \frac{\Delta(\widehat{g}(s), g(s))}{2\sigma^2}\right\}.$$

As $E(\ .\ ) = \widetilde{E}[(Z_T(f))^{-1}(\ .\ )]$, we have

$$\begin{aligned}
\mathbf{C_T} &= \widetilde{E}\Big[\big(Z_T(f)\big)^{-1}\Big(S_0 B_T \exp\Big\{\sigma W_T - \frac{\sigma^2 T}{2}\Big\} - K\Big)^+ \\
&\qquad \times I\big\{\sigma W_t < g(t),\ t \le T\big\}/B_T\Big] \\
&= \widetilde{E}\Big[\big(Z_T(f)\big)^{-1}\Big(S_0 B_T \exp\Big\{\sigma \widetilde{W}_T - \frac{\sigma^2 T}{2}\Big\} - K\Big)^+ \\
&\qquad \times I\big\{\sigma \widetilde{W}_t < \widehat{g}(t),\ t \le T\big\}/B_T\Big] \\
&= E\Big[\big(Z_T(-f)\big)\Big(S_0 B_T \exp\Big\{\sigma W_T - \frac{\sigma^2 T}{2}\Big\} - K\Big)^+ \\
&\qquad \times I\big\{\sigma W_t < \widehat{g}(t),\ t \le T\big\}/B_T\Big].
\end{aligned}$$

Using this representation we get

$$\begin{aligned}
|\mathbf{C_T} - \widehat{\mathbf{C}}_\mathbf{T}| &= \big|E\big[(Z_T(-f) - 1)(S_T - K)^+ I\{\sigma W_t < \widehat{g}(t), t \le T\}/B_T\big]\big| \\
&\le E\big[|Z_T(-f) - 1|(S_T - K)^+/B_T\big].
\end{aligned}$$

Here the random variables $Z_T(-f)$ and S_T are independent as they are functions of Gaussian random variables $\int_0^T f(s)dW(s)$ and W_T which are independent.

Indeed, due to the properties of stochastic integrals and the choice of function $f(s)$ the covariance of those random variables is

$$E\left[W_T \int_0^T f(s)dW(s)\right] = \int_0^T f(s)ds = \big(g(T) - \widehat{g}(T) + \widehat{g}(0) - g(0)\big)/\sigma = 0$$

Hence

$$E\big[|Z_T(f) - 1|\,(S_T - K)^+/B_T\big] = E\big[|Z_T(f) - 1|\big]\,E\big[(S_T - K)^+/B_T\big]$$

To complete the proof we note that a random variable $\log(Z_T(f)$ is normally distributed with mean $-\Delta_T(\widehat{g}, g)/(2\sigma^2)$ and variance $\Delta_T(\widehat{g}, g)/(\sigma^2)$. By direct calculation we have the equality

$$E|Z_T(f) - 1| = 2\left(\Phi\big(\sqrt{\Delta_T(\widehat{g}, g)/\sigma^2}\,\big) - \frac{1}{2}\right),$$

where $\Phi(x)$ is a standard normal distribution. As $\Phi(x) - 1/2 \leq x/\sqrt{2\pi}$, $x > 0$ it follows that

$$E|Z_T(f) - 1| \leq \sqrt{\frac{2\Delta_T(\widehat{g}, g)}{\pi\sigma^2}}\,. \qquad \square$$

Remark 2. The price of the ordinary call option $E[(S_T - K)^+/B_T]$ in (7) is easy to evaluate by the famous Black–Scholes formula. If we assume that the boundary $g(t)$ is a twice continuously differentiable function and the lengths of intervals (t_i, t_{i+1}), $i = 1, \ldots, n-1$, for a piecewise-linear approximating function $\widehat{g}_t$ are equal (i.e., a uniform partition is considered), then, obviously, $\Delta_T(\widehat{g}, g) = O(\frac{1}{n^2})$ as $n \to \infty$. Hence by Theorem 1 we have

$$|\mathbf{C_T} - \widehat{\mathbf{C}}_\mathbf{T}| = O\Big(\frac{1}{n}\Big).$$

We can essentially improve this estimate by using Theorem 3 from [9] along with Proposition 1.

Proposition 2. *Let $g(t)$ be a twice continuously differentiable function and $\widehat{g}(t)$ be a piecewise-linear continuous function such*

$$\widehat{g}(t_i) = g(t_i), \quad t_i = \frac{iT}{n}, \quad i = 0, \ldots, n.$$

Then as $n \to \infty$

$$|\mathbf{C_T} - \widehat{\mathbf{C}}_\mathbf{T}| = O\left(\frac{\log n}{n^{3/2}}\right).$$

Theoretically, we can improve this estimate for the rate of convergence if we allow the use of a non-uniform partition. In the context of boundary crossing problems it has recently been shown by Pötzelberger and Wang [10] that under some conditions on boundaries with the use of a specifically designed non-uniform partition

$$\big|P\{W_t < g(t),\ t \leq T\} - P\{W_t < \widehat{g}(t),\ t \leq T\}\big| = O\left(\frac{1}{n^2}\right).$$

Applying this fact along with Proposition 1 we get

$$|\mathbf{C_T} - \widehat{\mathbf{C}}_\mathbf{T}| = O\left(\frac{1}{n^2}\right).$$

Note that a search for an optimal non-uniform partition could be a rather time-consuming procedure especially for large n.

3. Numerical Example

This section contains a numerical example of the calculation of the fair price of a barrier option which was considered by Roberts and Shortland in [3]. In this paper the Vasicek model is used for the risk-free interest rate I_t:

$$I_t - r = a + \int_0^t (r - I_s)ds + \sigma \widehat{W}_t,$$

where $\widehat{W}_t$ is a standard Wiener process independent of W_t. Then $r(t) = EI_t = r + ae^{-t}$ and $\int_0^t r(s)ds = rt + a(1 - e^{-t})$. Note that the interest rate is now considered to be stochastic rather than deterministic as in Section 2.

Roberts and Shortland considered in [5] the example with $S_0 = 10$, $\sigma = 0.1$, $r = 0.1$, and $a = 0.05$. The style of option was the up-and-in European call option with boundary $H = 12$, strike price $K = 11$, and maturity at $T = 1$. To price this option we use that the sum of prices of "up-and-down" and "up-and-in" options equals to the price of "standard call" and hence the assertion of Theorem 1 is true for "up-and-in" options also.

The boundary function $g(t)$ for this example is

$$g(t) = \ln(H/S_0) + \sigma^2 t/2 - \int_0^t r(s)ds = 0.18232 - 0.095t - 0.05(1 - e^{-t}).$$

By using an analytic approximation Roberts and Shortland obtained the following bounds for the fair price:

$$0.51675 \leq \mathbf{C_T} \leq 0.51796.$$

They also used the Monte-Carlo method to evaluate the fair price of the option. By simulating 1 million sample paths of the stock price with step size 0.001 they obtained

$$\mathbf{C_T} = 0.513903$$

with standard error 0.016. This value of the fair price is less than the lower bound, although a 95% confidence interval for $\mathbf{C_T}$ does include these bounds. In order for a 95% confidence interval to have comparable width to the analytic bounds, we would require about 700 million sample paths with step size 0.001. The computational time required to do this would clearly prevent the direct Monte Carlo method from being useful. However, the use of the variance reduction technique might dramatically reduce the required sample size.

Using the suggested numerical integration method with piecewise-linear approximation for 50 and 400 uniformly spaced nodes, we obtained for both cases the following value for the approximation of the fair price:

$$\widehat{\mathbf{C}}_{\mathbf{T}} = 0.51683\ . \tag{9}$$

This value is within the analytic bounds obtained by Roberts and Shortland. Note that by Theorem 1 the upper bound for errors of these estimates are $9 \cdot 10^{-4}$ and $1.1 \cdot 10^{-4}$, respectively for $n = 50$ and $n = 400$. The stability of numerical integration is verified by using the Gaussian quadrature method with 32 and 64 nodes, the reported numbers are the same as in (9).

For the calculation of boundary probabilities in Proposition 1 we also used the integral equation method from [7]. Solving the integral equation iteratively, for three iterations only we obtained the fair price as $\mathbf{C}_{\mathbf{T}} = 0.51695$. This is also within the bounds given by Roberts and Shortland.

By using the PDE approach we obtained $\mathbf{C}_{\mathbf{T}} = 0.51671$ as the fair price. It is noteworthy that this is slightly less than the lower bound obtained by Roberts and Shortland, although the difference is only in the fifth digit. However this is an acceptable accuracy for the bank practice.

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