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COCHAIN OPERATIONS DEFINING STEENROD $\smile_i$-PRODUCTS IN THE BAR CONSTRUCTION

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Abstract. The set of cochain operations defining Steenrod $\smile_i$-products in the bar construction $BC^*(X)$ is defined in terms of surjection operad. This structure extends a Homotopy G-algebra structure which defines only $\smile$.

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Adams's cobar construction $\Omega C_*(X)$ of the chain complex of a topological space X determined the homology $H_*(\Omega X)$ of the loop space just additively.

Later Baues [1] constructed the *geometric* diagonal

$$\nabla_0 : \Omega C_*(X) \to \Omega C_*(X) \otimes \Omega C_*(X)$$

which turns the cobar construction into a DG-Hopf algebra. This diagonal allows one to produce the next cobar construction $\Omega\Omega C_*(X)$ which models the double loop space.

Our aim here is to define on $\Omega C_*(X)$ geometric cooperations (dual to Steenrod $\smile_i$-products)

$$\{\nabla_i : \Omega(C_*(X)) \to \Omega(C_*(X)) \otimes \Omega(C_*(X)),\ i = 0, 1, \dots\}$$

satisfying the standard conditions

$$\deg \nabla_i = i,\ \nabla_i d + (d \otimes 1 + 1 \otimes d)\nabla_i = \nabla_{i-1} + T\nabla_{i-1} \tag{1}$$

(since we work here over Z_2, the signs are ignored). These cooperations are necessary (but of course not sufficient) for a further iteration of the cobar construction. Additionally, we require a certain compatibility between ∇_i-s and the standard multiplication of $\Omega C_*(X)$. This allows us to define ∇_i-s by restrictions on the generators $E_i : C \to \Omega C_*(X) \otimes \Omega C_*(X)$.

In fact, we present particular elements $\{E^i_{p,q},\ i = 0, 1, \dots;\ p, q = 1, 2, \dots\}$ in the surjection operad χ ([11]) such that the corresponding chain multicooperations

$$\{E_i^{p,q} : C_*(X) \to (C_*(X)^{\otimes p}) \otimes (C_*(X)^{\otimes q}),\ p, q = 1, 2, \dots\}$$

define ∇_i-s in the cobar construction $\Omega C_*(X)$ and the corresponding cochain multioperations

$$\{E^i_{p,q} : (C^*(X)^{\otimes p}) \otimes (C^*(X)^{\otimes q}) \to C^*(X),\ p, q = 1, 2, \dots\}$$

define $\smile_i$-s in the bar construction $BC^*(X)$.

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It is known that the $\smile_i$-products in $C^*(X)$ are represented by the following elements of χ:

$$\smile = (1,2);\quad \smile_1 = (1,2,1);\quad \smile_2 = (1,2,1,2);\ \dots\ .$$

Let us consider them as the first line cochain operations.

Let the second line be presented by a *homotopy G-algebra* structure ([4]) on $C^*(X)$ which consists of the sequence of operations

$$\{E_{1,q} : C^*(X) \otimes (C^*(X)^{\otimes q}) \to C^*(X),\ q = 1, 2, \dots\}.$$

These operations in fact define the multiplication in the bar construction $BC^*(X)$. They are represented by the following elements of χ [11]:

$$E_{1,k} = (1,2,1,3,1,\dots,1,k,1,k+1,1). \tag{2}$$

Below we present the next line cochain operations. We introduce the notion of an *extended homotopy G-algebra*. This is a DG-algebra with a certain additional structure which defines $\smile_i$-s on the bar construction. The main example of such an object is again $C^*(X)$. This structure consists of multioperations

$$\{E^i_{p,q} : (C^*(X)^{\otimes p}) \otimes (C^*(X)^{\otimes q}) \to C^*(X),\ i = 0, 1, \dots,\ p, q = 1, 2, \dots\}.$$

We present particular elements $\{E^i_{p,q} \in \chi\}$ representing these operations. In particular, $E^0_{p,q}$ coincides with the homotopy G-algebra structure;

$$\begin{aligned} E^1_{p,q} = \ & (1; p+1, 1, p+2, 1, \dots, p+q-1, 1, p+q;\\ & 1, p+q, 2, p+q, 3, \dots, p; p+q);\end{aligned}$$

and

$$\begin{aligned} E^2_{p,q} = \sum_{k=0}^{q-1}\ & (1; p+1, 1, p+2, 1, \dots, 1, p+k+1;\\ & 1, p+k+1, 2, p+k+1, 3, \dots, p+k+1, p;\\ & p+k+1, p, p+k+2, p, \dots, p+q; p).\end{aligned}$$

In Section 1 we recall the notion of a homotopy G-algebra and show that this structure defines a product in the bar construction. In Section 2 the notion of an extended homotopy G-algebra is introduced and it is shown that this structure defines $\smile_i$-products in the bar construction. Section 3 is dedicated to the construction of an extended homotopy G-algebra structure on the cochain complex $C^*(X)$.

1. Homotopy G-Algebras

In this section we recall the notion of a homotopy G-algebra from [4] in order to extend it in the next section.

1.1. The notion of homotopy G-algebra.

Definition 1. A homotopy G-algebra is a differential graded algebra (DG-algebra) $(A, d, \cdot)$ together with a given sequence of multioperations

$$E_{1,k} : A \otimes A^{\otimes k} \to A, \quad k = 1, 2, 3, \dots\ ,$$

satisfying the following conditions:
$\deg E_{1,k} = -k,\ E_{1,0} = id;$

$$
\begin{aligned}
&dE_{1,k}(a;b_1,\dots,b_k)+E_{1,k}(da;b_1,\dots,b_k)+\textstyle\sum_i E_{1k}(a;b_1,\dots,db_i,\dots,b_k)\\
&=b_1E_{1k}(a;b_2,\dots,b_k)+\textstyle\sum_i E_{1k}(a;b_1,\dots,b_ib_{i+1},\dots,b_k)\\
&+E_{1k}(a;b_1,\dots,b_{k-1})b_k;
\end{aligned}
\tag{3}
$$

$$
\begin{aligned}
&a_1E_{1,k}(a_2;b_1,\dots,b_k)+E_{1,k}(a_1\cdot a_2;b_1,\dots,b_k)+E_{1,k}(a_1;b_1,\dots,b_k)a_2\\
&=\textstyle\sum_{p=1,\dots,k-1}E_{1,p}(a_1;b_1,\dots,b_p)\cdot E_{1,m-p}(a_2;b_{p+1},\dots,b_k);
\end{aligned}
\tag{4}
$$

$$
\begin{aligned}
&E_{1,n}(E_{1,m}(a;b_1,\dots,b_m);c_1,\dots,c_n)\\
&=\textstyle\sum E_{1,n-\sum l_i+m}(a;c_1,\dots,c_{k_1},E_{1,l_1}(b_1;c_{k_1+1},\dots,c_{k_1+l_1}),c_{k_1+l_1+1},\dots,c_{k_m},\\
&\quad E_{1,l_m}(b_m;c_{k_m+1},\dots,c_{k_m+l_m}),c_{k_m+l_m+1},\dots,c_n).
\end{aligned}
\tag{5}
$$

Let us analyze these conditions in low dimensions.

For the operation $E_{1,1}$ the condition (3) gives

$$dE_{1,1}(a;b)+E_{1,1}(da;b)+E_{1,1}(a;db)=a\cdot b+b\cdot a, \tag{6}$$

i.e., the operation $E_{1,1}$ is sort of $\smile_1$ product, which measures the noncommutativity of A. Below we use the notation $E_{1,1}=\smile_1$.

The condition (4) gives

$$(a\cdot b)\smile_1 c+a\cdot(b\smile_1 c)+(a\smile_1 c)\cdot b=0, \tag{7}$$

i.e., our $E_{1,1}=\smile_1$ satisfies the so-called *Hirsch formula* which states that the map $f_b:A\to A$ defined as $f_b(x)=x\smile_1 b$ is a derivation.

The condition (3) gives

$$
\begin{aligned}
&a\smile_1(b\cdot c)+b\cdot(a\smile_1 c)+(a\smile_1 b)\cdot c\\
&=dE_{1,2}(a;b,c)+E_{1,2}(da;b,c)+E_{1,2}(a;db,c)+E_{1,2}(a;b,dc),
\end{aligned}
\tag{8}
$$

so the "left Hirsch formula" is satisfied just up to a chain homotopy and a suitable homotopy is the operation $E_{1,2}$, so this operation measures the lack of "left Hirsch formula".

Besides, the condition (5) gives

$$(a\smile_1 b)\smile_1 c-a\smile_1(b\smile_1 c)=E_{1,2}(a;b,c)+E_{1,2}(a;c,b), \tag{9}$$

so this $\smile_1$ is not strictly associative, but the operation $E_{1,2}$ somehow measures the lack of associativity too.

1.2. **Homotopy G-algebra structure and a multiplication in the bar construction.** For a homotopy G-algebra $(A,d,\cdot,\{E_{1,k}\})$ the sequence $\{E_{1,k}\}$ defines, in the bar construction BA of a DG-algebra $(A,d,\cdot)$, the multiplication turning BA into a DG-Hopf algebra. In fact, this means that a homotopy G-algebra is a $B(\infty)$-algebra in the sense of [5].

The sequence of operations $\{E_{1,k}\}$ defines a homomorphism $E:BA\otimes BA\to A$ by $E([]\otimes[a])=E([a]\otimes[])=a$, $E([a]\otimes[b_1|\cdots|b_n])=E_{1,n}(a;b_1,\dots,b_n)$ and $E([a_1|\cdots|a_m]\otimes[b_1|\cdots|b_n])=0$ if $m>1$.

Since the bar construction BA is a cofree coalgebra, a homomorphism E induces a graded coalgebra map $\mu_E : BA \otimes BA \to BA$.

Then the conditions (3) and (4) are equivalent to the condition

$$dE + E(d_{BA} \otimes id + id \otimes d_{BA}) + E \smile E = 0,$$

i.e., E is a twisting cochain, and this is equivalent to μ_E being a chain map. Besides, the condition (5) is equivalent to μ_E being associative. Finally we have

Proposition 1. *For a homotopy G-algebra $(A, d, \cdot, \{E_{1,k}\})$ the bar construction BA is a DG-Hopf algebra with respect to the standard coproduct $\nabla_B : BA \to BA \otimes BA$ and the multiplication $\mu_E : BA \otimes BA \to BA$.*

2. Extended Homotopy G-Algebras

In this section we introduce the notion of an *extended homotopy G-algebra*. This is a DG-algebra with a certain additional structure which defines $\smile_i$-s in the bar construction.

2.1. The notion of an extended homotopy G-algebra.

Definition 2. We define an extended homotopy G-algebra as an object

$$(A, d, \cdot, \{E^k_{p,q} : A^{\otimes p} \otimes A^{\otimes q} \to A,\ k = 0, 1, \dots ;\ p, q = 1, 2, \dots\})$$

such that

$E^0_{p>1,q} = 0$ and $(A, d, \cdot, \{E^0_{1,q}\})$ is a homotopy G-algebra

and

$$\begin{aligned}
dE^k_{m,n}(a_1, \dots, a_m; b_1, \dots, b_n) &+ \textstyle\sum_i E^k_{m,n}(a_1, \dots, da_i, \dots, a_m; b_1, \dots, b_n)\\
&+ \textstyle\sum_i E^k_{m,n}(a_1, \dots, a_m; b_1, \dots, db_i, \dots, b_n)\\
&+ \textstyle\sum_i E^k_{m-1,n}(a_1, \dots, a_i \cdot a_{i+1}, \dots, a_m; b_1, \dots, b_n)\\
&+ \textstyle\sum_i E^k_{m,n-1}(a_1, \dots, a_m; b_1, \dots, b_i \cdot b_{i+1}, \dots, b_n)\\
+a_1 E^k_{m-1,n}(a_2, \dots, a_m; b_1, \dots, b_n) &+ E^k_{m-1,n}(a_1, \dots, a_{m-1}; b_1, \dots, b_n) a_m\\
+b_1 E^k_{m,n-1}(a_1, \dots, a_m; b_2, \dots, b_n) &+ E^k_{m,n-1}(a_1, \dots, a_m; b_1, \dots, b_{n-1}) b_n\\
+ \textstyle\sum_{i=0}^k \sum_{p,q} T^i E^{k-i}_{p,q}(a_1, \dots, a_p; b_1, \dots, b_q) &\cdot E^i_{m-p,n-q}(a_{p+1}, \dots, a_m; b_{q+1}, \dots, b_n)\\
= E^{k-1}_{m,n}(a_1, \dots, a_m; b_1, \dots, b_n) &+ E^{k-1}_{n,m}(b_1, \dots, b_n; a_1, \dots, a_m).
\end{aligned} \tag{10}$$

Here $TE^i_{p,q}(x_1, \dots, x_p; y_1, \dots, y_q) = E^i_{q,p}(y_1, \dots, y_q; x_1, \dots, x_p)$.

Let us analyze this condition in low dimensions.

For the operation $E^k_{1,1}$ the condition (10) gives

$$dE^k_{1,1}(a; b) + E^k_{1,1}(da; b) + E^k_{1,1}(a; db) = E^{k-1}_{1,1}(a; b) + E^{k-1}_{1,1}(b; a),$$

i.e., the operation $E^k_{1,1}$ is the homotopy which measures the lack of commutativity of $E^{k-1}_{1,1}$. Keeping in mind that $E^0_{1,1} = \smile_1$, we can say that $E^k_{1,1}$ is sort of $\smile_{k+1}$ product on A. Below we use the notation $E^k_{1,1} = \smile_{k+1}$.

Besides, the condition (10) also gives

$$\begin{aligned}(a \cdot b) \smile_k c + a \cdot (b \smile_k c) + (a \smile_k c) \cdot b + E_{2,1}^{k-2}(a, b; c) + E_{1,2}^{k-2}(c; a, b)\\ = dE_{2,1}^{k-1}(a, b; c) + E_{2,1}^{k-1}(da, b; c) + E_{2,1}^{k-1}(a, db; c) + E_{2,1}^{k-1}(a, b; dc)\end{aligned} \tag{11}$$

and

$$\begin{aligned}a \smile_k (b \cdot c) + b \cdot (a \smile_k c) + (a \smile_k b) \cdot c + E_{1,2}^{k-2}(a; b, c) + E_{2,1}^{k-2}(b, c; a)\\ = dE_{1,2}^{k-1}(a; b, c) + E_{1,2}^{k-1}(a; db, c) + E_{1,2}^{k-1}(a; b, dc),\end{aligned} \tag{12}$$

these are up to homotopy Hirsch type formulae connecting $\smile_k$ and $\cdot$. We remark here that the homotopy G-algebra structure controls the connection between $\cdot$ and $\smile_1$, while the extended homotopy G-algebra structure controls the connections between $\cdot$ and $\smile_k$-s (but not between $\smile_m$ and $\smile_n$, in general).

As we already know, the homotopy G-algebra structure defines the multiplication in the bar construction. Below we are going to show that the extended homotopy G-algebra structure defines Steenrod $\smile_i$ products in the bar construction. But before we need some preliminary notions.

2.2. **DG-Hopf algebras with Seenrod coproducts.** Let a *DG-coalgebra with Steenrod coproducts* be an object

$$(A; d; \nabla_0, \nabla_1, \nabla_2, \dots),$$

where $(A; d; \nabla_0)$ is a DG-coalgebra (with $\deg d = -1$), and $\nabla_i : A \to A \otimes A$, $i > 0$, are cooperations, dual to Steenrods $\smile_i$ products, i.e., they satisfy the conditions (1).

Suppose now that A is additionally equipped with a multiplication $\cdot : A \otimes A \to A$ which turns $(A, d, \cdot)$ into a DG-algebra. We are interested in what kind of compatibility of ∇_i-s with the multiplication $\cdot$ it is natural to require.

The following notion was introduced in [6], the dual notion was introduced by V. Smirnov in [12] and called a $\smile_\infty$-Hopf algebra.

Definition 3. We define a DG-Hopf algebra with Steenrod coproducts as an object

$$(A, d, \cdot, \nabla_0, \nabla_1, \nabla_2, \dots),$$

where $(A, d, \cdot)$ is a DG-algebra, ∇_i-s satisfy (1) and, additionally, we require the following connections between ∇_i-s and the product $\cdot$ (*decomposition rule*):

$$\nabla_n(a \cdot b) = \sum_{k=0}^{n} \nabla_k(a) \cdot T^k \nabla_{n-k}(b), \tag{13}$$

where $T : A \otimes A \to A \otimes A$ is the permutation map $T(a \otimes b) = b \otimes a$ and T^k is its iteration.

In particular, (13) gives that ∇_0 is a multiplicative map, i.e., $(A, d, \cdot, \nabla_0)$ is a DG-Hopf algebra; ∇_1 is a $(\nabla_0, T\nabla_0)$-derivation, etc.

The decomposition rule (13) has the following sense: if $(A, \cdot)$ is a free (i.e tensor) algebra, i.e., $A = T(V)$ (for example, the cobar construction), then (13) allows us to construct the cooperations ∇_i on the generating vector space V

and after that to extend them onto whole A by a suitable *extension rule* which follows from the above decomposition rule (13).

Let (C, d, Δ) be a DG-coalgebra and ΩC be its cobar construction. By definition, ΩC is the tensor algebra $T(s^{-1}\bar{C})$ generated by the desuspension $s^{-1}\bar{C}$ of the coaugmentation coideal $\bar{C}$.

A sequence of coproducts $\nabla_i : \Omega C \to \Omega C \otimes \Omega C$ satisfying (13) is determined by the restrictions $E_i : C \to \Omega C \otimes \Omega C$ which are homomorphisms of degree $i - 1$.

For ∇_i to satisfy (1) E^i must satisfy the condition

$$dE_i + E_i d + \textstyle\sum_{k=0}^{i} E_k \smile T^k E_{i-k} = E_{i-1} + TE_{i-1} \tag{14}$$

which is the restriction of (1) on C.

So if we want to construct, on ΩC, a sequence ∇_i forming the structure of a DG-Hopf algebra with Steenrod coproducts we have to construct a sequence of *higher twisting cochains* – homomorphisms $\{E_i,\ i = 0, 1, \ldots;\ \deg E_i = i - 1\}$ satisfying (14). Note that E_0 is an ordinary twisting cochain

$$dE_0 + E_0 d + E_0 \smile E_0 = 0.$$

2.3. **DG-Hopf algebras with Steenrod products.** Here we dialyze the previous section.

Let us define a *DG-algebra with Steenrod products* as an object

$$(A; d; \smile_0, \smile_1, \smile_2, \ldots\ ),$$

where $(A; d; \smile_0)$ is a DG-algebra (with $\deg d = +1$), and $\smile_i$: $A \otimes A \to A,\ i > 0$, are Steenrods $\smile_i$ products, i.e., they satisfy the conditions

$$d(a \smile_i b) = da \smile_i b + a \smile_i db + a \smile_{i-1} b + b \smile_{i-1} b. \tag{15}$$

Suppose now that A is additionally equipped with the diagonal $\nabla : A \to A \otimes A$ which turns (A, d, ∇) into a DG-coalgebra. We are interested in what kind of compatibility of $\smile_i$-s with the diagonal ∇ must be required.

Definition 4. We define a DG-Hopf algebra with Steenrod products as an object $(A, d, \nabla, \smile_0, \smile_1, \smile_2, \ldots)$ where (A, d, ∇) is a DG-coalgebra, the products $\smile_i$: $A \otimes A \to A$ satisfy (15) and additionally we require the following connections between $\smile_i$-s and the diagonal ∇:

$$\nabla \cdot \smile_n = \sum_{k=0}^{n} (\smile_k \otimes \smile_{n-k} \cdot T^k) \nabla_{A \otimes A}. \tag{16}$$

In particular, $\smile_0$ is a coalgebra map, i.e., $(A, d, \nabla, \smile_0)$ is a DG-Hopf algebra.

Let $(C, d, \cdot)$ be a DG-algebra and BC be its bar construction. By definition, BC is the tensor coalgebra $T^c(s^{-1}\bar{C})$ generated by the desuspension $s^{-1}\bar{C}$ of the augmentation ideal $\bar{C}$.

Since T^c is cofree, the sequence of products $\smile_i$: $BC \otimes BC \to BC$ satisfying (16) is determined by the projections $E^i : BC \otimes BC \to BC \to C$ which are homomorphisms of degree $1 - i$.

For $\smile_i$ to satisfy (15) E^i must satisfy the condition

$$dE^i + E^i(d_{BC} \otimes id + id \otimes d_{BC}) + \sum_{k=0}^{i} E^k \smile E^{i-k}T^k$$
$$= E^{i-1} + E^{i-1}T, \tag{17}$$

which is the projection of (15) on C.

So if we want to construct, on BC, a sequence of $\smile_i$-s forming the structure of a DG-Hopf algebra with Steenrod products, we have to construct a sequence of *higher twisting cochains* – homomorphisms $\{E^i,\ i = 0, 1, \ldots; \deg E^i = 1 - i\}$ satisfying (17). Note that E^0 is an ordinary twisting cochain

$$dE^0 + E^0 d + E^0 \smile E^0 = 0.$$

2.4. **Structure of an extended homotopy G-algebra and Steenrod products in the bar construction.** As we already know, the part of extended homotopy G-algebra – the sequence of operations $\{E^0_{p,q}\}$ (which in fact is a homotopy G-algebra structure) defines, in the bar construction BA, the multiplication turning BA into a DG-Hopf algebra. Here we show that for an extended homotopy G-algebra $(A, d, \cdot, \{E^k_{p,q}\})$ the sequence $\{E^{k>0}_{p,q}\}$ defines, in the bar construction BA of a DG-algebra $(A, d, \cdot)$, the $\smile_i$-products turning BA into a DG-Hopf algebra with Steenrod products.

Indeed, sequences of operations $\{E^k_{p,q}\}$ define homomorphisms

$$\{E^k : BA \otimes BA \to A,\ k = 0, 1, \ldots\}$$

by $E^k([a_1|\cdots|a_m] \otimes [b_1|\cdots|b_n]) = E^k_{m,n}(a_1, \ldots, a_m; b_1, \ldots, b_n)$.

The condition (10), which verifies our $\{E^k_{p,q}\}$-s, is equivalent to the condition (17) for the sequence $\{E^k\}$ and thus $\{E^k_{p,q}\}$ define the correct $\smile_k$-s on BC.

Finally we have

Proposition 2. *For an extended homotopy G-algebra $(A, d, \cdot, \{E^k_{p,q}\})$ the bar construction BA is a DG-Hopf algebra with Steenrod products.*

3. Cochain Complex $C^*(X)$ as an Extended Homotopy G-Algebra

The main example of an extended homotopy G-algebra is given by the following

Theorem 1. *The cochain complex of a topological space $C^*(X)$ carries the structure of an extended homotopy G-algebra.*

Proof. In [10] it is shown that the bar construction $BC^*(X)$ is actually the cochain complex of a certain cubical set Q, and in [6] the explicit formulae for the Steenrods $\smile_i$ products are constructed in the cochains of a cubical set. So we have $\smile_i$-s on $BC^*(X) = C^*(Q)$. Moreover, these $\smile_i$-s are well connected with the standard comultiplication of $BC^*(X)$ in the sense of (16) and thus these $\smile_i$-s are determined by the compositions

$$E^i_{p,q} : C^*(X)^{\otimes p} \otimes C^*(X)^{\otimes q} \to BC^*(X) \otimes BC^*(X) \stackrel{\smile_i}{\to} BC^*(X) \to C^*(X)$$

which form the needed structure. $\square$

The rest of the paper will be dedicated to the description of these operations in terms of the surjection operad.

3.1. **Operations $E^k_{p,q}$ in the surjection operad.** A surjection operad χ [11] is defined as a sequence of chain complexes $\chi(r)$ where $\chi(r)_d$ is spanned by nondegenerate surjections $u:(1,2,\dots,r+d)\to(1,2,\dots,r)$, $u(i)\neq u(i+1)$. A surjection u is written as a string $(u(1),u(2),\dots,u(r+d))$.

For the structure maps of this operad and the filtration $F_1\chi\subset\cdots\subset F_n\chi\subset\cdots\subset\chi$, with $F_n\chi$ equivalent to a little n-cub operad, we refer to [2].

Here we briefly recall the definition of the action of χ on $C_*(X)$ (on $C^*(X)$). Let us take an interval $(0,\dots,n)$ and cut it into $r+d$ subintervals

$$0=n_0\le n_1\le\cdots\le n_{r+d-1}\le n_{r+d}=n.$$

We label the i-th interval $(n_{i-1},\dots,n_i)$ by the integer $u(i)$. Let $C_{(k)}$ be the concatenation of all intervals labelled by k. Then the operation $AW(u):C_*(X)\to C_*(X)^{\otimes r}$ determined by the surjection u is defined as

$$AW(u)(\sigma(0,\dots,n))=\sum\sigma(C_{(1)})\otimes\cdots\otimes\sigma(C_{(r)}),$$

where the summation is taken with respect to all cuttings of $(0,\dots,n)$.

For example, the Alexander–Whitney diagonal

$$\nabla\sigma(0,\dots,n)=\sum_k\sigma(0,\dots,k)\otimes\sigma(k,\dots,n)$$

is represented by the surjection $(1,2)\in F_1\chi(2)_0$; the diagonal

$$\nabla_1\sigma(0,\dots,n)=\sum_{k<l}\sigma(0,\dots,k,l,l+1,\dots,n)\otimes\sigma(k,\dots,l)$$

is represented by $(1,2,1)\in F_2\chi(2)_1$, the diagonal

$$\nabla_2\sigma(0,\dots,n)=\sum_{k<l<t}\sigma(0,\dots,k,l,l+1,\dots,t)\otimes\sigma(k,\dots,l,t,t+1,\dots,n)$$

is represented by $(1,2,1,2)\in F_2\chi(2)_1$, etc.

Here we present particular elements of χ representing operations $E^k_{p,q}$. They are obtained from *admissible tables* which we define now.

The first row of an admissible table consists of the only number 1. The second is $p+1,1,p+2,1,\dots,1,p+k$. As we see, it consists of a *stable part* (1-s at even places) and of an *increasing part* ($p+1,p+2,\dots,p+k$ at odd places).

Each next row starts with the stable number of the previous row which gives rise to an increasing part at odd places. At even places the maximal number from the previous row serves as a stable part. For example, if the previous row ends with $\dots,i-2,s,i-1,s,i$, then the next row is $s,i,s+1,i,s+2,i,\dots,i,s+t$. Each row consists of an odd number of terms. An important remark: the stabile part of a row may be *empty*, in this case under the *stabile part* of this row we mean the maximal number of the previous row.

As we see, the table consists of two increasing sequences of integers $1,2,\dots$ and $p+1,p+2,\dots$ (of course, with repetitions and permutations). The main restriction is that the first sequence should necessarily end with p. The table always ends with a one term row.

Next, we put all rows of an admissible table in one string and obtain an *admissible string* in χ. We say that this string belongs to $E^k_{p,q}$ if: its table consists of $k+3$ rows; the first element of the second row is $p+1$ and the maximal number which occurs in the string is $p+q$.

Here are some examples (by ';' we indicate the ends of rows in admissible tables). The admissible strings

$$(1;5,1,6,1,7;1,7,2,7,3,7,4;7,4,8,4,9;4)$$

and

$$1;5,1,6,1,7,1,8;1,8,2,8,3,8,4;8,4,9;4$$

both belong to $E^2_{4,5}$, while

$$(1;4,1,5,1,6;1,6,2;6;2,6,3;6;3)$$

belongs to $E^4_{3,3}$.

We define an element $E^k_{p,q} \in \chi$ as the sum of all admissible strings belonging to it.

In particular, $E^{2k}_{1,1} = (1;2;1;\dots;1;2)$ and $E^{2k-1}_{1,1} = (1;2;1;\dots;1;2;1)$. They correspond to $\smile_{2k+1}$ and $\smile_{2k}$, respectively.

Moreover, $E^0_{1,q} = (1;2,1,3,\dots 1,q+1;1)$. These elements generate $F_2\chi$ [11] and they determine on $C^*(X)$ a structure of homotopy G-algebra..

Here are the examples of higher operations:

$$\begin{aligned} E^1_{p,q} =& (1;p+1,1,p+2,1,\dots,p+q-1,1,p+q;\\ & 1,p+q,2,p+q,3,\dots,p;p+q);\\ E^2_{p,q} =& \textstyle\sum_{k=0}^{q-1}(1;p+1,1,p+2,1,\dots,1,p+k+1;\\ & 1,p+k+1,2,p+k+1,3,\dots,p+k+1,p;\\ & p+k+1,p,p+k+2,p,\dots,p+q;p). \end{aligned}$$

Generally, $E^k_{p,q}$ belong to the filtration $F_{k+2}\chi$.

Remark 1. The elements $E^0_{1,q}$-s satisfy the defining conditions of a homotopy G-algebra already in χ. So do $E^{k>0}_{p,q}$-s: they satisfy (10) already in χ. For example, the condition (7) is a result of

$$(1,2,1)\circ_1(1,2)+(1,2)\circ_2(1,2,1)+(id\times T)(1,2)\circ_1(1,2,1)=0 \tag{18}$$

(which is not the case in, say, of the Barrat–Eccles operad: a suitable combination is just *homological to zero.* Note also that the Barrat–Eccles operad acts on $C(X)$ via χ, see [2]). The condition (8) is a result of

$$(1,2,1)\circ_2(1,2)+(T\times id)(1,2)\circ_2(1,2,1)+(1,2)\circ_1(1,2,1)=d(1,2,1,3,1).$$

Remark 2. The extended homotopy G-algebra structure, consisting of the operations $E^k_{p,q}$, establishes connections just between $\smile_k$ and $\smile$ (equivalently, between ∇_k and ∇ or between $E^k_{1,1}$ and $1,2$ in χ), but not connections between

$\smile_m$ and $\smile_n$ generally. Here are two operations establishing connections between $\smile_2$ and $\smile_1$: $G_{1,2}=(1,2,1,3,1,3,2),\ G_{2,1}=(1,2,3,2,3,1,3)\in F_3\chi(3)_4$ satisfy the conditions

$$\begin{aligned}dG_{2,1}(a,b;c)+G_{2,1}(da,b;c)+G_{2,1}(a,db;c)+G_{2,1}(a,b;dc)\\ =(a\smile_1 b)\smile_2 c+a\smile_1(b\smile_2 c)+(a\smile_2 c)\smile_2 b\\ +E^1_{2,1}(a,b;c)+E^1_{2,1}(b,a;c)\end{aligned}$$

and

$$\begin{aligned}dG_{1,2}(a;b,c)+G_{1,2}(da;b,c)+G_{1,2}(a;db,c)+G_{1,2}(a;b,dc)\\ =a\smile_2(b\smile_1 c)+b\smile_1(a\smile_2 c)+(a\smile_2 c)\smile_1 b\\ +E^1_{1,2}(a;b,c)+E^1_{1,2}(a;c,b)\end{aligned}$$

already in the operad χ. Note that the element $(1,2)\in F_1\chi$ generates the operad $F_1\chi$. Furthermore, $(1,2)$ and $E^1_{1,k}=(1,2,1,3,1,\dots,1,k,1,k+1,1)\in F_2\chi$ generate the operad $F_2\chi$ [11] (but not freely: for example (18) is a relation). The elements $E^1_{p,q}$ belong to the suboperad $F_3\chi$ but they, together with $(1,2)$ and $E^0_{1,k}$, do not generate $F_3\chi$: it is possible to check that the elements $G_{1,2}=(1,2,1,3,1,3,2),\ G_{2,1}=(1,2,3,2,3,1,3)\in F_3\chi(3)_4$ cannot be obtained.

Remark 3. Besides $C^*(X)$, the elements $E^0_{1,q}\in F_2\chi$ act also on the Hochschild cochain complex of an associative algebra $C^*(U,U)$ (they determine the operations described in [7] and [5]). This action answers the so-called Deligne conjecture about the action of the operad $F_2\chi$ on $C^*(U,U)$ since it is generated by $(1,2)\in F_1\chi$ and $E^0_{1,q}\in F_2\chi$ [11]. It follows from (9) that for a homotopy G-algebra $(A,d,\cdot,\{E^0_{1,k}\})$ the commutator $[a,b]=a\smile_1 b+b\smile_1 a$ satisfies the Jacobi identity. Together with (6) it implies on the $H(A)$ a Lie bracket of degree -1. Besides, (7) and (8) imply that $[a,-]:H(A)\to H(A)$ is a derivation so that $H(A)$ becomes a Gerstenhaber algebra [3]. This structure is generally nontrivial for $A=C^*(U,U)$ but the trivial for $A=C^*(X)$ because of the existence of $\smile_2$ there. The nontriviality of the Gerstenhaber bracket on the Hochschild cohomology also implies that $C^*(U,U)$ is not an extended homotopy G-algebra in general: here $E^1_{1,1}=\smile_2$ cannot act. One more example of a homotopy G-algebra is the cobar construction of a DG-Hopf algebra [8].

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