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INTERNAL CROSSED MODULES

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Dedicated to Professor H. Inassaridze
on the occasion of his seventieth birthday

Abstract. We introduce the notion of (pre)crossed module in a semiabelian category, and establish equivalences between internal reflexive graphs and precrossed modules, and between internal categories and crossed modules.

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Introduction

Every split epimorphism of groups is, up to an isomorphism, a semidirect product projection. This old observation is a crucial step in establishing the well-known category equivalences

$$\begin{array}{ccc} \text{Internal categories in } \mathbf{Groups} & \sim & \text{Crossed modules} \\ \cap & & \cap \\ \text{Reflexive graphs in } \mathbf{Groups} & \sim & \text{Precrossed modules} \end{array} \qquad (*)$$

although the top equivalence and the left-hand full embedding also use another old idea that leads to what is now called "commutator theory" (see [12], [13] and the references there). Having the categorical notion of semidirect product [5], and knowing (from [5]) that the equivalence between split epimorphisms and semidirect product projections holds in any semiabelian category (in the sense of [11]), we extend the equivalences (*) from the category of groups and of rings, and of some other concrete categories, for which it was known to semiabelian categories. Thus the passage from internal categories to crossed modules becomes a purely categorical procedure rather than an algebraic translation of a categorical notion.

Some historical and technical remarks:

(i) As F. Borceux told me, the top equivalence in (*) was first discovered by R. Lavendhomme. I would describe a possible reason for this reference being missed by many authors (including myself) as follows:

- In the eyes of "pure-category-theorists", the internal categories in the category of groups are obviously the same as internal groups in the category of categories, which is a very particular case of a general fact;

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however there was no use of these particular structures in pure category theory.

- In the eyes of "homotopy-theorists" an internal group (="group object") in an arbitrary category is a simple useful notion, but an internal category a "mystery"; and as motivated by considering the first and the second homotopy group of a connected space, there is a simple connection between the internal group structure in the category of groupoids and the structure of a crossed module and it is hard to say now who had noticed this connection first. R. Brown, who is certainly a leading expert in homotopical algebra of groupoids and crossed modules, should be the right person to be asked about this.
- . . . And so it was only a very special kind of "categorical geometry", where one would try to describe internal categories in the category of groups directly, and to show that every such an internal category is a groupoid.

(ii) Unlike the situation in past, the internal categories and many other internal categorical structures in Groups are now of interest in pure category theory as they are simpler than the "external" ones and therefore can be used to classify those, or at least to show that some of definitions of higher-dimensional categorical structures are not equivalent to each other.

(iii) Semiabelian categories introduced in [11] have nothing to do with semiabelian categories in the sense of D. Raikov introduced long time ago just like Barr exact categories have nothing to do with the old notion of exact category used in homological algebra. A semiabelian category (in the sense of [11]) is a pointed Barr exact and Bourn protomodular category with finite coproducts which in particular assumes the existence of finite limits and implies the existence of finite colimits. It was D. Bourn who first studied exact protomodular categories and gave a number of elegant theorems and proofs in [2], but there was an important reason for introducing a new notion by requiring just pointedness and the existence of finite coproducts in addition: it turned out that there is equivalence with various old systems of axioms used by many authors (see the references in [11]), bringing "new life" to their investigations in several branches of categorical algebra. In fact the semiabelian categories seem to be exactly what had to be found after S. Mac Lanes famous "Duality for groups" [15].

(iv) The categorical notion of semidirect product [5], which we use here, is a natural generalization of the classical one. However this is not at all a straightforward generalization, and its full explanation for the readers not familiar with theory of monads would also require to repeat a chapter from [16] and most of [1].

(v) Every semiabelian category is a Maltsev category in the sense of [7] and [6], but the converse is not true, and I see no reasonable way to extend the results of this paper to the context of Maltsev categories (as the reader could conclude from a remark of P. T. Johnstone [14]). However the Maltsev categories are exactly those for which the internal categories coincide with internal groupoids and form a full subcategory in the category of internal reflexive graphs [8].

(vi) After describing internal reflexive graphs as (internal) precrossed modules in Section 2, and internal categories as crossed modules in Section 3, we also describe an intermediate notion that we call a star-multiplicative graph in Section 4. This last description involves a simplified definition of a crossed module, and in the case of groups every star-multiplicative graph is a category.

(vii) The results of this paper slightly improve those presented in my talks with the same title on PSSL 71 (Lovain-la-Neuve, Belgium) and Australian Category Seminar in 1999. Meanwhile D. Bourn and M. Gran proved that the category of connected internal groupoids in a semiabelian category is equivalent to the category of central extensions, and then extended this result to Maltsev categories (see [3], [4], [9]), and there is a further extension by M. Gran (not published yet).

Throughout this paper C denotes a semiabelian category.

1. Split Epimorphisms and Object Actions

1.1. Let us recall in detail the (adjoint) equivalence

$$\mathbf{SplitEpi}(\mathbb{C}) \sim \mathbf{Act}(\mathbb{C}) \tag{1.1}$$

between the category of split epimorphisms and the category of object actions in $\mathbb{C}$, and at the same time introduce some useful notation:

(a) The objects of $\mathbf{SplitEpi}(\mathbb{C})$ will be written as 4-tuples (A, B, α, β), where $\alpha : A \to B$ and $\beta : B \to A$ are morphisms in $\mathbb{C}$ with $\alpha\beta = 1$.

(b) A morphism $(A, B, \alpha, \beta) \to (A', B', \alpha', \beta')$ in $\mathbf{SplitEpi}(\mathbb{C})$ is a pair (f, g), where $f : A \to A'$ and $g : B \to B'$ are morphisms in $\mathbb{C}$ with $g\alpha = \alpha' f$ and $f\beta = \beta' g$.

(c) The objects of $\mathbf{Act}(\mathbb{C})$ will be written as triples (B, X, ξ), where $\xi : B\flat X \to X$ is a B-action on X in $\mathbb{C}$ defined as follows:

- the object $B\flat X$ is defined together with a morphism $k_{B,X} : B\flat X \to B + X$, as the kernel of the morphism $\pi_{B,X} : B + X \to B$, induced by the identity morphism of B and the zero morphism $X \to B$;
- the functor $B\flat(-) : \mathbb{C} \to \mathbb{C}$ has a canonical monad structure (see [5] and [1]), which we will write as $B\flat(-) = (B\flat(-), \eta^B, \mu^B)$, and (X, ξ) is defined as an algebra over that monad, i.e., $\xi : B\flat X \to X$ is required to make the following diagram commute

$$\begin{array}{ccccc} B\flat(B\flat X) & \xrightarrow{\mu_X^B} & B\flat X & \xleftarrow{\eta_X^B} & X \\ {\scriptstyle 1\flat\xi}\downarrow & & \downarrow{\scriptstyle \xi} & \swarrow\!\!\!\!\!= & \\ B\flat X & \xrightarrow[\xi]{} & X & & \end{array} \tag{1.2}$$

- we will actually need explicit descriptions of η^B and μ^B, and for that we use the following commutative diagrams respectively (which determine η_X^B and μ_X^B uniquely; ι's are the coproduct injections):

$$\begin{array}{ccccc} B\flat(B\flat X) & \xrightarrow{\kappa_{B,B\flat X}} & B+(B\flat X) & \xrightarrow{\pi_{B,B\flat X}} & B \\ {\scriptstyle \mu_X^B}\downarrow & & \downarrow{\scriptstyle [\iota_1,\kappa_{B,X}]} & & \| \\ B\flat X & \xrightarrow[\kappa_{B,X}]{} & B+X & \xrightarrow[\pi_{B.X}]{} & B \\ {\scriptstyle \eta_X^B}\uparrow & & \| & & \\ X & \xrightarrow[\iota_2]{} & B+X & & \end{array} \tag{1.3}$$

(d) A morphism $(B,X,\xi)\to(B',X',\xi')$ in $\mathbf{Act}(\mathbb{C})$ is a pair (g,h), where $g:B\to B'$ and $h:X\to X'$ are morphisms in $\mathbb{C}$ with $h\xi=\xi'(g\flat 1)$.

(e) The right adjoint $G:\mathbf{SplitEpi}(\mathbb{C})\to\mathbf{Act}(\mathbb{C})$ in the adjoint equivalence (1.1) has $G(A,B,\alpha,\beta)=(B,\mathrm{Ker}(\alpha),\xi)$, where ξ is the unique morphism making the diagram

$$\begin{array}{ccc} B\flat\,\mathrm{Ker}(\alpha) & \xrightarrow{\kappa_{B,\mathrm{Ker}(\alpha)}} & B+\ker(\alpha) \\ {\scriptstyle \xi}\downarrow & & \downarrow{\scriptstyle [\beta,k_\alpha]} \\ \mathrm{Ker}(\alpha) & \xrightarrow[k_\alpha]{} & A \end{array} \tag{1.4}$$

commute; here k_α is the canonical morphism defining the kernel of α.

(f) The left adjoint $F:\mathbf{Act}(\mathbb{C})\to\mathbf{SplitEpi}(\mathbb{C})$ has $F(B,X,\xi)=(B\ltimes(X,\xi),B,\pi_\xi,\iota_\xi)$, where the object $B\ltimes(X,\xi)$ is defined together with a morphism $\sigma_\xi:B+X\to B\ltimes(X,\xi)$ via the coequalizer diagram

$$B+(B\flat X)\ \underset{1+\xi}{\overset{[\iota_1,\kappa_{B,X}]}{\rightrightarrows}}\ B+X\xrightarrow{\sigma_\xi}B\ltimes(X,\xi) \tag{1.5}$$

with $\iota_\xi=\sigma_\xi\iota_1$ and π_ξ uniquely determined by $\pi_\xi\sigma_\xi=\pi_{B,X}$. Note also that the following diagram is a pushout:

$$\begin{array}{ccc} B\flat X & \xrightarrow{\kappa_{B,X}} & B+X \\ {\scriptstyle \xi}\downarrow & & \downarrow{\scriptstyle \sigma_\xi=[\iota_\xi,\sigma_\xi\iota_2]} \\ X & \xrightarrow[\sigma_\xi\iota_2]{} & B\ltimes(X,\xi) \end{array} \tag{1.6}$$

(and also a pullback, since the horizontal arrows are normal monomorphisms with isomorphic cokernels).

1.2. Consider a diagram of the form

$$X\xrightarrow{k}A\ \underset{\beta}{\overset{\alpha}{\rightleftarrows}}\ B\ , \tag{1.7}$$

where $\alpha\beta=1$ and (X,k) is a (the) kernel of α. Since diagram (1.6) is a pushout, the equivalence (1.1) tells us that A in (1.7) can also be presented as a certain corresponding pushout. In fact it is easy to check that that pushout is

$$\begin{array}{ccc} B\flat X & \xrightarrow{\kappa_{B,X}} & B+X \\ {\scriptstyle \xi}\downarrow & & \downarrow{\scriptstyle [\beta,k]} \\ X & \xrightarrow[k]{} & A \end{array} \tag{1.8}$$

where ξ is the unique morphism making this diagram commute (as follows from 1.1(e)). This gives

1.3. **Theorem.** *Let $A, B, X, \alpha, \beta, k, \xi$ be as in 1.2 above, and C an arbitrary object in $\mathbb{C}$. Then for every two morphisms $x : X \to C$ and $b : B \to C$ there exists at most one morphism $a : A \to C$ making the diagram*

$$\begin{array}{ccccc} X & \xrightarrow{k} & A & \xleftarrow{\beta} & B \\ & {\scriptstyle x}\searrow & \downarrow{\scriptstyle a} & \swarrow{\scriptstyle b} & \\ & & C & & \end{array} \tag{1.9}$$

commute; such an a does exist if and only if the following diagram commutes

$$\begin{array}{ccc} B\flat X & \xrightarrow{\kappa_{B,X}} & B+X \\ {\scriptstyle \xi}\downarrow & & \downarrow{\scriptstyle [b,x]} \\ X & \xrightarrow[x]{} & C \end{array} \tag{1.10}$$

Proof. The first assertion is well known from D. Bourn [2]. The existence of $a : A \to C$ making (1.10) commute follows from the fact that (1.8) is a pushout. What remains is to prove that if

$$\begin{array}{ccc} B\flat X & \xrightarrow{\kappa_{B,X}} & B+X \\ {\scriptstyle \xi}\downarrow & & \downarrow{\scriptstyle [b,x']} \\ X & \xrightarrow[x]{} & C \end{array} \tag{1.11}$$

commutes, then $x' = x$. This follows from the previous argument, but a direct calculation is also easy: $x' = [b, x']\iota_2 = [b, x']\kappa_{B,X}\eta_X^B = x\xi\eta_X^B = x$, where the second equality follows from the definition of η^B, the third from the commutativity of (1.11), and the fourth from the commutativity of (1.2).

2. Internal Reflexive Graphs and Precrossed Modules

2.1. We are going to use the equivalence (1.1) to describe internal reflexive graphs in $\mathbb{C}$ as the actions above equipped with an additional structure. This is very easy since

(a) an internal reflexive graph in $\mathbb{C}$ is nothing but an object (A, B, α, β) in **SplitEpi**($\mathbb{C}$), equipped with an additional morphism $\gamma : A \to B$ with $\gamma\beta = 1$;

(b) according to Theorem 1.3, $1(f)$, to give such γ is to give $u : B \to B$ and $f : X \to B$ with

$$f\xi = [u, f]k_{B,X} \quad \text{and} \quad [u, f]\iota_1 = 1,$$

i.e., to give just $f : X \to B$ with $f\xi = [1, f]k_{B,X}$.

Thus we arrive at

Definition and Theorem. An *internal precrossed module* in $\mathbb{C}$ is a 4-tuple (B, X, ξ, f), in which (B, X, ξ) is an object in $\mathbf{Act}(\mathbb{C})$, and $f : X \to B$ a morphism in $\mathbb{C}$ making the diagram

$$\begin{array}{ccc} B\flat X & \xrightarrow{\kappa_{B,X}} & B + X \\ {\scriptstyle \xi}\downarrow & & \downarrow{\scriptstyle [1,f]} \\ X & \xrightarrow[x]{} & B \end{array} \tag{2.1}$$

commute. The procedure above determines an equivalence

$$\mathbf{ReflGraph}(\mathbb{C}) \sim \mathbf{PreCrossMod}(\mathbb{C}) \tag{2.2}$$

between the category of internal reflexive graphs and the category of internal precrossed modules in $\mathbb{C}$.

3. Internal Categories and Crossed Modules

3.1. As shown in [10] in the context of Maltsev varieties, and extended to Maltsev categories by A. Carboni, M. C. Pedicchio, and N. Pirovano [8], an internal category in $\mathbb{C}$ can be described as a diagram in $\mathbb{C}$ of the form

$$A \times_B A \xrightarrow{m} A \underset{\gamma}{\overset{\alpha}{\underset{\longleftarrow \beta \longleftarrow}{\rightrightarrows}}} B\ , \tag{3.1}$$

where:

- the objects A and B with the three arrows between them form an internal reflexive graph;
- the object $A \times_B A = A \times_{(\alpha,\beta)} A$ is defined as the pullback of α and β;
- the morphism m makes the following diagram commute:

$$\begin{array}{ccccc} X & \xrightarrow{\langle 1,\beta\alpha\rangle} & A \times_B A & \xleftarrow{\langle \beta\gamma,1\rangle} & A \\ & \searrow^{=} & \vdots\,{\scriptstyle m} & {}^{=}\swarrow & \\ & & A & & \end{array} \tag{3.2}$$

– with no further conditions on m. Moreover, for a given internal reflexive graph $(A, B, \alpha, \beta, \gamma)$, such m is uniquely determined whenever it exists; and if this is the case, the graph $(A, B, \alpha, \beta, \gamma)$ is called *multiplicative*.

3.2. Let $(A, B, \alpha, \beta, \gamma)$ be an internal reflexive graph, and $k : X \to A$ the (fixed) kernel of α. Applying Theorem 1.3 to

$$X \xrightarrow{\langle k,0\rangle} A \times_B A \underset{\langle \beta\gamma,1\rangle}{\overset{\mathrm{proj}_2}{\rightleftarrows}} A \tag{3.3}$$

(instead of (1.7)), we obtain

Lemma. *In the notation above, let* $\xi' : A\flat X \to X$ *be the* A*-action on* X *corresponding to* (3.3), *i.e. the unique morphism making the diagram*

$$\begin{array}{ccc} A\flat X & \xrightarrow{\kappa_{A,X}} & A+X \\ {\scriptstyle \xi'}\downarrow & & \downarrow{\scriptstyle [\langle \beta\gamma,1\rangle,\langle k,0\rangle]} \\ X & \xrightarrow[\langle k,0\rangle]{} & A \times_B A \end{array} \tag{3.4}$$

commute. Then the following conditions are equivalent:

(*a*) *there exists a* (*unique*) *morphism* $m : A \times_B A \to A$ *making the diagram*

$$\begin{array}{ccccc} X & \xrightarrow{\langle k,0\rangle} & A \times_B A & \xleftarrow{\langle \beta\gamma,1\rangle} & A \\ & {\scriptstyle k}\searrow & \downarrow{\scriptstyle m} & \swarrow\!\!= & \\ & & A & & \end{array} \tag{3.5}$$

commute;

(*b*) *the diagram*

$$\begin{array}{ccc} A\flat X & \xrightarrow{\kappa_{A,X}} & A+X \\ {\scriptstyle \xi'}\downarrow & & \downarrow{\scriptstyle [1,k]} \\ X & \xrightarrow[k]{} & A \end{array} \tag{3.6}$$

commutes.

3.3. Let us compare diagrams (3.4) and (3.6). Since the commutativity of (3.4) implies (and in fact is equivalent to) $k\xi' = [\beta y, k]\kappa_{A,X}$, Lemma 3.2 gives

Corollary. *In the notation above, the following conditions are equivalent:*

(*a*) *there exists a* (*unique*) *morphism* $m : A \times_B A \to A$ *making the diagram* (3.5) *commute;*

(*b*) *the diagram*

$$\begin{array}{ccc} A\flat X & \xrightarrow{\kappa_{A,X}} & A+X \\ {\scriptstyle \kappa_{A,X}}\downarrow & & \downarrow{\scriptstyle [1,k]} \\ A+X & \xrightarrow[{[\beta\gamma,k]}]{} & A \end{array} \tag{3.7}$$

commutes.

3.4. Let us now recall what we have done and explain what are we going to do: According to Section 2, an internal reflexive graph $(A, B, \alpha, \beta, \gamma)$ in $\mathbb{C}$ can equivalently be described as a 4-tuple (B, X, ξ, f) as in Definition and Theorem 2.1. And an internal category in C is nothing but an internal reflexive graph $(A, B, \alpha, \beta, \gamma)$ in $\mathbb{C}$ equipped with a (uniquely determined if it exists) morphism $m : A \times_B A \to A$ making diagram (3.2) commute. On the other hand, Corollary 3.3 gives a necessary and sufficient condition for the existence of a (unique) morphism $m : A \times_B A \to A$ making diagram (3.5) commute. After that in order to obtain a complete description of internal categories in $\mathbb{C}$, we are going to show (in Subsection 3.5) that diagram (3.2) is commutative if and only if so is (3.5), and then translate that condition in terms of (B, X, ξ, f) (in the rest of Section 3).

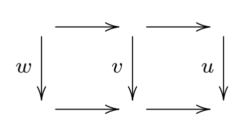
3.5. Let us compare the following equalities:
(a) $m\langle 1, \beta\alpha\rangle = 1$;
(b) $m\langle 1, \beta\alpha\rangle\beta = \beta$;
(c) $m\langle 1, \beta\alpha\rangle k = k$;
(d) $m\langle \beta\gamma, 1\rangle = 1$;
(e) $m\langle k, 0\rangle = k$.
We have:

- the commutativity of (3.2) is the same as $(a)\&(d)$;
- the commutativity of (3.5) is the same as $(d)\&(e)$;
- $(a) \Leftrightarrow (b)\&(c)$ since β and k are jointly epic (as follows from the protomodularity of $\mathbb{C}$);
- $(c) \Leftrightarrow (e)$ since $\alpha k = 0$;
- $(d) \Leftrightarrow (b)$ since $m\langle 1, \beta\alpha\rangle\beta = m\langle \beta, \beta\alpha\beta\rangle = m\langle \beta, \beta\rangle = m\langle \beta\gamma\beta, \beta\rangle = m\langle \beta\gamma, 1\rangle\beta$.

Hence (3.2) is commutative if and only if so is (3.5), as desired.

3.6. The following lemma is known (and in fact inexplicitly used in [11]) to be true in any pointed exact Maltsev category; hence we omit the proof.

Lemma. *Let*



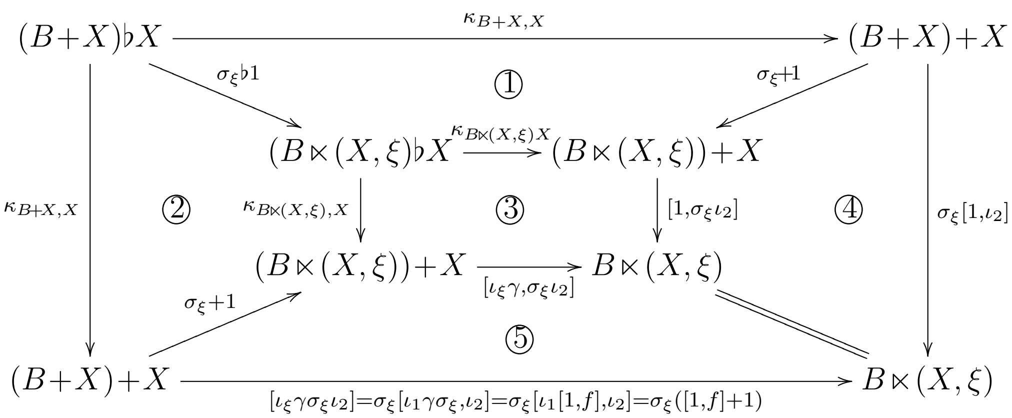
be a commutative diagram, whose rows are short exact sequences, and the right-hand square is a pushout. If u and v are regular epimorphisms, then so is w.

3.7. **Corollary**. *For every object X in $\mathbb{C}$, the functor $(-)\flat X : \mathbb{C} \to \mathbb{C}$ preserves regular epimorphisms.*

Proof. Given a regular epimorphism $u : U \to V$ in $\mathbb{C}$, just apply Lemma 3.6 to the diagram

$$\begin{array}{ccccc} U\flat X & \xrightarrow{\kappa_{U,X}} & U+X & \xrightarrow{\pi_{U,X}} & U \\ {\scriptstyle u\flat 1}\downarrow & & \downarrow{\scriptstyle u+1} & & {\scriptstyle u}\downarrow \\ V\flat X & \xrightarrow[\kappa_{V,X}]{} & V+X & \xrightarrow[\pi_{V,X}]{} & V \end{array}$$

3.8. For a given internal precrossed module (B, X, ξ, f) in $\mathbb{C}$, consider the diagram



(3.8)

in which

- the parts ①, ②, ④, obviously commute;
- the same is true for ⑤, which uses the equality $\gamma\sigma_\xi = [1, f]$ that defines γ in terms of f (or can be used to define f in terms of γ see 2.1);
- ③ is nothing but (3.7) translated in terms of (B, X, ξ, f) with γ defined via $\gamma\sigma_\xi = [1, f]$.

Corollary 3.7 tells us that $\sigma_\xi\flat 1$ is a regular epimorphism, and therefore the commutativity of ③ is equivalent to the commutativity of the enveloping square, i.e. to $\sigma_\xi([1, f] + 1)\kappa_{B+X,X} = \sigma_\xi[1, \iota_2]\kappa_{B+X,X}$, which can also be expressed as the equality of the following two composites:

$$(B + X)\flat X \xrightarrow{\kappa_{B+X,X}} (B + X) + X \underset{[1,f]+1}{\overset{[1,\iota_2]}{\rightrightarrows}} B + X \xrightarrow{\sigma_\xi} B \ltimes (X, \xi). \tag{3.9}$$

Thus, from the previous results we obtain

Lemma. *The internal reflexive graph corresponding to an internal precrossed module* (B, X, ξ, f) *under the equivalence* (2.2) *is multiplicative* (*or, equivalently, has an internal category structure*) *if and only if the two composites* (3.9) *coincide.*

3.9. Why should not we stop here and call Lemma 3.8 the description of internal categories in $\mathbb{C}$? The reason is that as the semidirect product $B \ltimes (X, \xi)$

is defined as a coequalizer (or as a pushout), there is no simple method of verifying whether two morphisms with the codomain $B \ltimes (X, \xi)$ are equal or not. Fortunately, we are able make a further simplification as follows:

(a) the commutative diagram

$$\begin{array}{ccccc}
(B+X)\flat X & \xrightarrow{[1,\iota_2]^*} & B\flat X & \xrightarrow{\xi} & X \\
{\scriptstyle \kappa_{B+X,X}}\downarrow & ① & \downarrow{\scriptstyle \kappa_{B,X}} & ② & \downarrow{\scriptstyle \sigma_\xi\iota_2} \\
(B+X)+X & \xrightarrow[{[1,\iota_2]}]{} & B+X & \xrightarrow[\sigma_\xi]{} & B \ltimes (X,\xi) \\
{\scriptstyle \pi_{B+X,X}}\downarrow & ③ & \downarrow{\scriptstyle \pi_{B,X}} & & \\
B+X & \xrightarrow[\pi_{B,X}]{} & B & &
\end{array} \tag{3.10}$$

in which $[1, \iota_2]^\#$ is defined via ① (which is possible since ③ commutes) and ② is the same as "flipped" (1.6), presents the top composite in (3.10) as

$$\sigma_\xi[1, \iota_2]\kappa_{B+X,X} = \sigma_\xi\iota_2\xi[1, \iota_2]^\#; \tag{3.11}$$

(b) the commutative diagram

$$\begin{array}{ccccc}
(B+X)\flat X & \xrightarrow{[1,f]\flat 1} & B\flat X & \xrightarrow{\xi} & X \\
{\scriptstyle \kappa_{B+X,X}}\downarrow & & \downarrow{\scriptstyle \kappa_{B,X}} & & \downarrow{\scriptstyle \sigma_\xi\iota_2} \\
(B+X)+X & \xrightarrow[{[1,f]+1}]{} & B+X & \xrightarrow[\sigma_\xi]{} & B \ltimes (X,\xi)
\end{array} \tag{3.12}$$

presents the bottom composite in (3.10) as

$$\sigma_\xi([1, f] + 1)\kappa_{B+X,X} = \sigma_\xi\iota_2([1, f]\flat 1); \tag{3.13}$$

(c) as follows from (3.11), (3.13), and the fact that $\sigma_\xi\iota_2$ is a monomorphism, the equality of the two composites in (3.9) is equivalent to $\xi[1, \iota_2]^\# = \xi([1, f]\flat 1)$ yielding the following

Definition and Theorem. An internal crossed module in $\mathbb{C}$ is an internal precrossed module (B, X, ξ, f) in $\mathbb{C}$, for which the diagram

$$\begin{array}{ccc}
(B+X)\flat X & \xrightarrow{[1,f]\flat 1} & B\flat X \\
{\scriptstyle [1,\iota_2]^\#}\downarrow & & \downarrow{\scriptstyle \xi} \\
B\flat X & \xrightarrow[\xi]{} & X
\end{array} \tag{3.14}$$

commutes. The equivalence (2.2) induces an equivalence

$$\mathbf{Cat}(\mathbb{C}) \sim \mathbf{CrossMod}(\mathbb{C}) \tag{3.15}$$

between the category of internal categories and the category of internal crossed modules in $\mathbb{C}$.

3.10. **Example.** Let us take $\mathbb{C}$ =**Groups**, the category of groups. Then as follows from the results of [5] (see also [1]), we have:

(a) To give an action $\xi : B\flat X \to X$ is the same as to give an ordinary action $h : B \times X \to X$, i.e. a group homomorphism $B \to \mathrm{Aut}(X)$ written as $b \mapsto (x \mapsto bx = h(b,x)))$; these ξ and h determine each other via $\xi(b,x,-b) = bx$, using the additive notation for the group structure and bx for $h(b,x)$ as above.

(b) Since the group $B\flat X$ is generated by elements of the form $(b,x,-b)$, diagram (2.1) commutes if and only if $f\xi(b,x,-b) = [1,f]\kappa_{B,X}(b,x,-b)$, i.e.

$$f(bx) = b + f(x) - b \tag{3.16}$$

for all b in B and x in X. Therefore the internal precrossed modules in the category of groups are the same as the ordinary precrossed modules.

(c) The group $(B+X)\flat X$ is generated by elements of the form $((b_1,x_1,...,b_n,x_n),x,(-x_n,-b_n,\dots,-x_1,-b_1))$, where $b_1,\dots,b_n$ are in B and $x_1,\dots,x_n$ in X (with $b_2,\dots,b_n$ and $x_1,\dots,x_{n-1}$ nonzero and b_1 and/or x_n removed if one or both of them are zero); and in diagram (3.14) we have

$$\begin{aligned}
&\xi[1,\iota_2]^{\#}((b_1,x_1,\dots,b_n,x_n),x,(-x_n,-b_n,\dots,-x_1,-b_1))\\
&= \xi(b_1,x_1,\dots,x_{n-1},b_n,x_n+x-x_n,-b_n,-x_{n-1},\dots,-x_1,-b_1)\\
&= \xi(b_1,x_1,-b_1) + \xi(b_1+b_2,x_2,-b_2-b_1) + \dots + \xi(b_1+\dots+b_{n-1},x_{n-1},\\
&- b_{n-1}\dots - b_1) + \xi(b_1+\dots+b_n,x_n+x-x_n,-b_n\dots-b_1)\\
&+ \xi(b_1+\dots+b_{n-1},-x_{n-1},-b_{n-1}\dots-b_1) + \dots + \xi(b_1+b_2,-x_2,-b_2-b_1)\\
&+ \xi(b_1,-x_1,-b_1) = b_1x_1 + (b_1+b_2)x_2 + \dots + (b_1+\dots+b_{n-1})x_{n-1}\\
&+ (b_1+\ldots+b_n)(x_n+x-x_n) - (b_1+\ldots+b_{n-1})x_{n-1} - \ldots - (b_1+b_2)x_2 - b_1x_1
\end{aligned}$$

and

$$\begin{aligned}
&\xi([1,f]\flat 1)((b_1,x_1,\dots,b_n,x_n),x,(-x_n,-b_n,\dots,-x_1,-b_1))\\
&= \xi(b_1+f(x_1)+\dots+b_n+f(x_n)),x,(-f(x_n)-b_n\dots-f(x_1)-b_1))\\
&= (b_1+f(x_1)+\dots+b_n+f(x_n))x.
\end{aligned}$$

That is, the direct translation of the commutativity of (3.14) is the equality

$$\begin{aligned}
&b_1x_1 + (b_1+b_2)x_2 + \dots + (b_1+\ldots+b_{n-1})x_{n-1} + (b_1+\dots+b_n)(x_n+x-x_n)\\
&- (b_1+\dots+b_{n-1})x_{n-1} - \dots - (b_1+b_2)x_2 - b_1x_1\\
&= (b_1+f(x_1)+\dots+b_n+f(x_n))x. \qquad (3.17)
\end{aligned}$$

(d) However there is an easy way to simplify this equality: putting $b_1 = \dots = b_n = 0$ in B, and $x_1 = \dots = x_{n-1} = 0$ and $x_n = y$ in X yields

$$y + x - y = f(y)x, \tag{3.18}$$

and, conversely, using (3.18) we obtain

$$\begin{aligned}
&b_1x_1+(b_1+b_2)x_2+\dots+(b_1+\dots+b_{n-1})x_{n-1}+(b_1+\dots+b_n)(x_n+x-x_n)\\
&-(b_1+\dots+b_{n-1})x_{n-1}-\dots-(b_1+b_2)x_2-b_1x_1\\
&=f(b_1x_1)((b_1+b_2)x_2+\dots+(b_1+\dots+b_{n-1})x_{n-1}\\
&+(b_1+\dots+b_n)(x_n+x-x_n)-(b_1+\dots+b_{n-1})x_{n-1}-\dots-(b_1+b_2)x_2)\\
&=\dots=f(b_1x_1)(f((b_1+b_2)x_2)(\dots(f((b_1+\dots+b_{n-1})x_{n-1})(b_1+\dots\\
&+b_n)(x_n+x-x_n)))\dots))=(f(b_1x_1)+f((b_1+b_2)x_2)+\dots\\
&+f((b_1+\dots+b_{n-1})x_{n-1})+(b_1+\dots+b_n))(x_n+x-x_n)\\
&=(b_1+f(x_1)-b_1+b_1+b_2+f(x_2)-b_2-b_1+\dots+b_1+\dots+b_{n-1}\\
&+f(x_{n-1})-b_{n-1}-\dots-b_1+b_1+\dots+b_n)(x_n+x-x_n)\\
&=(b_1+f(x_1)+b_2+f((x_2)+\dots+b_{n-1}+f(x_{n-1})\\
&+b_n)(x_n+x-x_n)=((b_1+f(x_1)+\dots+b_n+f(x_n))x.
\end{aligned}$$

That is, (3.17) is equivalent to (3.18). Since (3.16) and (3.18) define the ordinary notion of a crossed module, we conclude that the internal crossed modules in the category of groups are the same as the ordinary crossed modules.

4. Star-Multiplication

4.1. Let C be an internal category, say, in **Groups**, displayed as (3.1); one usually writes $A=C_1$ and $A\times_B A=C_2$, and we will also write

$$C_{*1}=\operatorname{Ker}(\alpha)=\big\{a\in A|\alpha(a)=0\big\},$$
$$C_{*2}=\big\{(a',a)\in A\times_B A|\alpha(a)=0\big\},$$

calling C_{*1} and C_{*2} the star of morphisms and the star of composable pairs (of morphisms) respectively. The multiplication (=composition) map $m:A\times_B A\to A$ induces a map $m*:C_{*2}\to C_{*1}$, which we will call the star-multiplication in C. It is straightforward to repeat this for an internal category in an abstract semiabelian category $\mathbb{C}$, and according to the notation of Section 3, we will have $C_{*1}=X, C_{*2}$ defined as the pullback

$$\begin{array}{ccc}
C_{*2}=A\times_B X & \xrightarrow{\text{proj}_2} & X\\
{\scriptstyle\text{proj}_1}\downarrow & & \downarrow{\scriptstyle\gamma k}\\
A & \xrightarrow[\alpha]{} & B
\end{array}\tag{4.1}$$

and diagram (3.2) "restricted to stars" becoming

$$\begin{array}{ccccc}
X & \xrightarrow{\langle k,0\rangle} & A\times_B X & \xleftarrow{\langle\beta\gamma k,1\rangle} & X\\
& \searrow^{=} & \vdots\,{\scriptstyle m*} & \swarrow_{=} & \\
& & X & &
\end{array}\tag{4.2}$$

4.2. Just like the multiplication morphism m is uniquely determined by the commutativity of (3.2), the star-multiplication $m*$ is uniquely determined by the commutativity of (4.2). However is it possible to "have $m*$ independently of m"? In other words, we introduce

Definition. Let $S = (A, B, \alpha, \beta, \gamma)$ be an internal reflexive graph in $\mathbb{C}$, and $k : X \to A$ a (fixed) kernel of α. The graph S is said to be star-*multiplicative* if there exists a (unique) morphism $s : A \times_B X \to X$ making the diagram

$$\begin{array}{ccccc} X & \xrightarrow{\langle k,0\rangle} & A \times_B X & \xleftarrow{\langle \beta\gamma k,1\rangle} & X \\ & \searrow\!\!\!= & \downarrow{\scriptstyle s} & =\!\!\!\swarrow & \\ & & X & & \end{array} \tag{4.3}$$

commute; here $A \times_B X$ is constructed as in (4.1).

And we are going to describe the star-multiplicative graphs, essentially following the arguments of Section 3:

4.3. Applying Theorem 1.3 to

$$X \xrightarrow{\langle k,0\rangle} A \times_B X \underset{\langle \beta\gamma k,1\rangle}{\overset{\mathrm{proj}_2}{\rightleftarrows}} X \tag{4.4}$$

(instead of (1.7)), we obtain

Lemma. *In the notation above, let* $\xi'' : X \flat X \to X$ *be the* X*-action on* X *corresponding to* (4.4), *i.e. the unique morphism making the diagram*

$$\begin{array}{ccc} X \flat X & \xrightarrow{\kappa_{X,X}} & X + X \\ {\scriptstyle \xi''}\downarrow & & \downarrow{\scriptstyle [\langle \beta\gamma k,1\rangle,\langle k,0\rangle]} \\ X & \xrightarrow[\langle k,0\rangle]{} & A \times_B X \end{array} \tag{4.5}$$

commute. Then the following conditions are equivalent:

(*a*) *the graph* $S = (A, B, \alpha, \beta, \gamma)$ *is star-multiplicative;*

(*b*) *the diagram*

$$\begin{array}{ccc} X \flat X & \xrightarrow{\kappa_{X,X}} & X + X \\ {\scriptstyle \xi''}\downarrow & & \downarrow{\scriptstyle [1,1]} \\ X & = & X \end{array} \tag{4.6}$$

commutes.

4.4. Let us compare diagrams (4.5) and (4.6). Since the commutativity of (4.5) implies (and in fact is equivalent to) $k\xi'' = [\beta\gamma k, k]\kappa_{X,X}$, Lemma 3.2 gives

Corollary. *In the notation above, the following conditions are equivalent:*
(a) *the graph $S = (A, B, \alpha, \beta, \gamma)$ is star-multiplicative;*
(b) *the diagram*

$$\begin{array}{ccc} X\flat X & \xrightarrow{\kappa_{X,X}} & X+X \\ {\scriptstyle\kappa_{X,X}}\downarrow & & \downarrow{\scriptstyle[k,k]} \\ X+X & \xrightarrow[{[\beta\gamma k,k]}]{} & A \end{array} \tag{4.7}$$

commutes.

4.5. The next step is the translation of the commutativity of (4.7) in terms of (B, X, ξ, f), which however is immediate unlike a similar translation in Section 3: (4.7) is the same (up to a canonical isomorphism) as

$$\begin{array}{ccc} X\flat X & \xrightarrow{\kappa_{X,X}} & X+X \\ {\scriptstyle\kappa_{X,X}}\downarrow & & \downarrow{\scriptstyle[\sigma_\xi\iota_2,\sigma_\xi\iota_2]=\sigma_\xi\iota_2[1,1]} \\ X+X & \xrightarrow[{[\iota_\xi\gamma\sigma_\xi\iota_2,\sigma_\xi\iota_1]=\sigma_\xi[\iota_2\gamma\sigma_\xi\iota_2,\iota_2]=\sigma_\xi[\iota_1 f,\iota_2]=\sigma_\xi(f+1)}]{} & B\ltimes(X,\xi) \end{array} \tag{4.8}$$

and so the commutativity of (4.7) is equivalent to $\sigma_\xi(f+1)\kappa_{X,X} = \sigma_\xi[\iota_2, \iota_2]\kappa_{X,X}$, which can also be expressed as an equality of the following two composites:

$$X\flat X \xrightarrow{\kappa_{X,X}} X+X \underset{f+1}{\overset{\iota_2[1,1]}{\rightrightarrows}} B+X \xrightarrow{\sigma_\xi} B\ltimes(X,\xi). \tag{4.9}$$

Lemma. *The internal reflexive graph corresponding to an internal precrossed module (B, X, ξ, f) under the equivalence* (2.2) *is star-multiplicative if and only if the two composites* (4.9) *coincide.*

4.6. Finally, we have to simplify our result as we did in 3.9, but only for the bottom composite in (4.9), using the commutative diagram

$$\begin{array}{ccccc} X\flat X & \xrightarrow{f\flat 1} & B\flat X & \xrightarrow{\xi} & X \\ {\scriptstyle\kappa_{X,X}}\downarrow & & \downarrow{\scriptstyle\kappa_{B,X}} & & \downarrow{\scriptstyle\sigma_\xi\iota_2} \\ X\flat X & \xrightarrow[f+1]{} & B+X & \xrightarrow[\sigma_\xi]{} & B\ltimes(X,\xi) \end{array} \tag{4.10}$$

which presents it as

$$\sigma_\xi(f+1)\kappa_{X,X} = \sigma_\xi\iota_2\xi(f\flat 1). \tag{4.11}$$

And again, since $\sigma_\xi\iota_2$ is a monomorphism, the equality of the two composites in (4.9) is equivalent to $[1,1]\kappa_{X,X} = \xi(f\flat 1)$, yielding the following

Theorem. *An internal precrossed module* (B, X, ξ, f) *in* $\mathbb{C}$ *corresponds to a star-multiplicative graph under the equivalence* (2.2) *if and only if the diagram*

$$\begin{array}{ccc} X\flat X & \xrightarrow{f\flat 1} & B\flat X \\ {\scriptstyle \kappa_{X,X}}\downarrow & & \downarrow{\scriptstyle \xi} \\ X+X & \xrightarrow[{[1,1]}]{} & X \end{array} \tag{4.12}$$

commutes.

4.7. *Remark.* In the case of the category of groups considered in Example 3.10, we have $[1,1]\kappa_{X,X}(y,x,-y) = y+x-y$ and $\xi(f\flat 1)(y,x,-y) = \xi(f(y),x,-f(y)) = f(y)x$, which tells us that the commutativity of (4.12) is equivalent to (3.16), and hence also to the commutativity of (3.14). Moreover, this suggests to replace (3.14) by the much simpler condition (4.12) in the definition of internal crossed module in a general semiabelian category $\mathbb{C}$ and the new definition will be equivalent to the one we use if and only if every star-multiplicative graph in $\mathbb{C}$ is multiplicative or, equivalently, if every star-multiplication (uniquely) extends to an internal category structure. It would be interesting to describe semiabelian categories with this extension property.

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