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BI-HAMILTONIAN STRUCTURE AS A SHADOW OF NON-NOETHER SYMMETRY

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Abstract. The correspondence between non-Noether symmetries and bi-Hamiltonian structures is discussed. We show that in regular Hamiltonian systems the presence of the global bi-Hamiltonian structure is caused by the symmetry of the space of solution. As an example, the well-known bi-Hamiltonian realization of the Korteweg–de Vries equation is discussed.

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The Noether theorem, Lutzky's theorem, bi-Hamiltonian formalism and bi-differential calculi are often used in generating conservation laws and all these approaches are unified by the same idea – to construct conserved quantities using some invariant geometric object (a generator of the symmetry – a Hamiltonian vector field in the Noether theorem, non-Hamiltonian one in Lutzky's approach, a closed 2-form in bi-Hamiltonian formalism and an auxiliary differential in the case of bidifferential calculi). There is a close relationship between these three approaches. Some aspects of this relationship were established in [3–4] and [6]. In the present paper it is discussed how the bi-Hamiltonian structure can be interpreted as a manifestation of the symmetry of space of solutions. A good candidate for this role is non-Noether symmetry. Such a symmetry is a group of transformations that maps the space of solutions of equations of motion onto itself, but unlike the Noether one, does not preserve action.

In the case of a regular Hamiltonian system the phase space is equipped with symplectic form ω (closed $d\omega = 0$ and nondegenerate $i_X\omega = 0 \to X = 0$ 2-form) and the time evolution is governed by Hamilton's equation

$$i_{X_h}\omega + dh = 0, \tag{1}$$

where X_h is the vector field tangent to solutions $X_h = \Sigma_i \dot{p}_i \partial_{p_i} + \dot{q}_i \partial_{q_i}$ and $i_{X_h}\omega$ denotes the contraction of X_h and ω. The vector field is said to be (locally) Hamiltonian if it preserves ω. According to Liouville's theorem, X_h defined by (1) automatically preserves ω (indeed, $L_{X_h}\omega = di_{X_h}\omega + i_{X_h}d\omega = -ddh = 0$).

One can show that the group of transformations of a phase space generated by any non-Hamiltonian vector field E

$$g(a) = e^{aL_E}$$

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does not preserve action

$$g_*(A) = g_*\left(\int pdq - hdt\right) = \int g_*(pdq - hdt) \neq 0$$

because $d(L_E(pdq - hdt)) = L_E\omega - dE(h) \wedge dt \neq 0$ (the first term in the right-hand side does not vanish since E is non-Hamiltonian and as far as E is time independent $L_E\omega$ and $dE(h) \wedge dt$ are linearly independent 2-forms). As a result, every non-Hamiltonian vector field E commuting with X_h leads to non-Noether symmetry (since E preserves the vector field tangent to solutions $L_E(X_h) = [E, X_h] = 0$, it maps the space of solutions onto itself). Any such symmetry yields the following integrals of motion [1–2], [4–5]:

$$l_k = Tr(R^k), \quad k = 1, 2, \ldots, n,$$

where $R = \omega^{-1}L_E\omega$ and n is half-dimension of the phase space.

It is interesting that for any non-Noether symmetry, the triple (h, ω, ω_E) carries the bi-Hamiltonian structure (§4.12 in [7], [8–10]). Indeed, ω_E is a closed ($d\omega_E = dL_E\omega = L_E d\omega = 0$) and an invariant ($L_{X_h}\omega_E = L_{X_h}L_E\omega = L_E L_{X_h}\omega = 0$) 2-form (but generic ω_E is degenerate). So every non-Noether symmetry quite naturally endows a dynamical system with the bi-Hamiltonian structure.

Now let us discuss how non-Noether symmetry can be recovered from a bi-Hamiltonian system. The generic bi-Hamiltonian structure on a phase space consists of a Hamiltonian system h, ω and an auxiliary closed 2- form $\omega^\bullet$ satisfying $L_{X_h}\omega^\bullet = 0$. Let us call it the global bi-Hamiltonian structure whenever $\omega^\bullet$ is exact (there exists a 1-form $\theta^\bullet$ such that $\omega^\bullet = d\theta^\bullet$) and X_h is (globally) a Hamiltonian vector field with respect to $\omega^\bullet$ ($i_{X_h}\omega^\bullet + dh^\bullet = 0$). In the local coordinates $\theta^\bullet = \theta^\bullet_i dz^i$. As far as ω is nondegenerate, there exists a vector field $E^\bullet = E^{\bullet i}\partial_{z^i}$ such that $i_{E^\bullet}\omega = \theta^\bullet$ (in the local coordinates $E^{\bullet i} = (\omega^{-1})^{ij}\theta^\bullet_j$). By construction,

$$L_{E^\bullet}\omega = \omega^\bullet.$$

Indeed, $L_{E^\bullet}\omega = di_{E^\bullet}\omega + i_{E^\bullet}d\omega = d\theta^\bullet = \omega^\bullet$ and

$$i_{[E^\bullet, X_h]}\omega = L_{E^\bullet}(i_{X_h}\omega) - i_{X_h}L_{E^\bullet}\omega = -d(E^\bullet(h) - h^\bullet) = -dh'.$$

In other words, $[X_h, E^\bullet]$ is a Hamiltonian vector field, i.e., $[X_h, E] = X_{h'}$. So $E^\bullet$ is not a generator of symmetry since it does not commute with X_h but one can construct (locally) the Hamiltonian counterpart of $E^\bullet$ (note that $E^\bullet$ itself is non-Hamiltonian) – X_g with

$$g(z) = \int_0^t h'dt. \tag{2}$$

Here the integration along the solution of Hamilton's equation with fixed origin and end point in $z(t) = z$ is assumed. Note that (2) defines $g(z)$ only locally and, as a result, X_g is a locally Hamiltonian vector field satisfying, by construction, the same commutation relations as $E^\bullet$ (namely, $[X_h, X_g] = X_{h'}$). Finally, one recovers the generator of non-Noether symmetry, i.e., the non-Hamiltonian vector field $E = E^{\bullet} - -X_g$ commuting with X_h and satisfying

$$L_E\omega = L_{E^{\bullet}}\omega - L_{X_g}\omega = L_{E^{\bullet}}\omega = \omega^{\bullet}$$

(thanks to Liouville's theorem $L_{X_g}\omega = 0$). So in the case of a regular Hamiltonian system every global bi-Hamiltonian structure is naturally associated with the (non-Noether) symmetry of space of solutions.

Example 1. As a toy example one can consider a free particle

$$h = \frac{1}{2}\sum_i p_i^2, \quad \omega = \sum_i dp_i \wedge dq_i.$$

This Hamiltonian system can be extended to the bi-Hamiltonian one

$$h, \omega, \omega^{\bullet} = \sum_i p_i dp_i \wedge dq_i.$$

Clearly, $d\omega^{\bullet} = 0$ and $X_h = \sum_i p_i \partial_{q_i}$ preserves $\omega^{\bullet}$. The conserved quantities p_i are associated with this simple bi-Hamiltonian structure. This system can be obtained from the following (non-Noether) symmetry (infinitesimal form):

$$\begin{aligned} q_i &\rightarrow (1 + ap_i)q_i, \\ p_i &\rightarrow (1 + ap_i)p_i \end{aligned}$$

generated by $E = \sum_i p_i q_i \partial_{q_i} + \sum_i p_i^2 \partial_{p_i}$

Example 2. The earliest and probably the most well-known bi-Hamiltonian structure is the one discovered by F. Magri and associated with the Korteweg–de Vries integrable hierarchy. The KdV equation

$$u_t + u_{xxx} + uu_x = 0$$

(zero boundary conditions for u and its derivatives are assumed) appears to be Hamilton's equation

$$i_{X_h}\omega + dh = 0,$$

where $X_h = \int_{-\infty}^{+\infty} dx\ u_t \partial_u$ (here ∂_u denotes a variational derivative with respect to the field $u(x)$) is the vector field tangent to the solutions,

$$\omega = \int_{-\infty}^{+\infty} dx\ du \wedge dv$$

is a symplectic form (here $v = \partial_x^{-1} u$) and the function

$$h = \int_{-\infty}^{+\infty} dx \left(\frac{u^3}{3} - u_x^2\right)$$

plays the role of a Hamiltonian. This dynamical system possesses a non-trivial symmetry – a one-parameter group of non-cannonical transformations $g(a) =$

e^{L_E} generated by the non-Hamiltonian vector field

$$E = \int_{-\infty}^{+\infty} dx \left(u_{xx} + \frac{1}{2}\, u^2 \right) \partial_u + X_F;$$

here the first term represents the non-Hamiltonian part of the generator of symmetry, while the second one is its Hamiltonian counterpart associated with

$$F = \int_{-\infty}^{+\infty} \left(\frac{1}{12}\, u^2 v + \frac{1}{4}\, \partial_x^{-1} \left(\frac{u^3}{3} - u_x^2 \right) + \frac{3}{4}\, v\, \frac{l_3}{l_2} \right) dx$$

($l_{2,3}$ are defined in (3)). The physical origin of this symmetry is unclear, however the symmetry seems to be very important since it leads to the well-known infinite sequence of conservation laws in involution:

$$\begin{aligned} l_1 &= \int_{-\infty}^{+\infty} u dx, \\ l_2 &= \int_{-\infty}^{+\infty} u^2 dx, \\ l_3 &= \int_{-\infty}^{+\infty} \left(\frac{u^3}{3} - u_x^2 \right) dx, \\ l_4 &= \int_{-\infty}^{+\infty} \left(\frac{5}{36}\, u^4 - \frac{5}{3}\, u u_x^2 + u_{xx}^2 \right) dx, \qquad (3) \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

and ensures the integrability of the KdV equation. The second Hamiltonian realization of the KdV equation discovered by F. Magri [8]

$$i_{X_{h^\bullet}} \omega^\bullet + dh^\bullet = 0$$

(where $\omega^\bullet = L_E \omega$ and $h^\bullet = L_E h$) is a result of the invariance of the KdV equation under the aforementioned transformations $g(a)$.

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