

## BI-HAMILTONIAN STRUCTURE AS A SHADOW OF NON-NOETHER SYMMETRY

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**Abstract.** The correspondence between non-Noether symmetries and bi-Hamiltonian structures is discussed. We show that in regular Hamiltonian systems the presence of the global bi-Hamiltonian structure is caused by the symmetry of the space of solution. As an example, the well-known bi-Hamiltonian realization of the Korteweg–de Vries equation is discussed.

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The Noether theorem, Lutzky’s theorem, bi-Hamiltonian formalism and bi-differential calculi are often used in generating conservation laws and all these approaches are unified by the same idea – to construct conserved quantities using some invariant geometric object (a generator of the symmetry – a Hamiltonian vector field in the Noether theorem, non-Hamiltonian one in Lutzky’s approach, a closed 2-form in bi-Hamiltonian formalism and an auxiliary differential in the case of bidifferential calculi). There is a close relationship between these three approaches. Some aspects of this relationship were established in [3–4] and [6]. In the present paper it is discussed how the bi-Hamiltonian structure can be interpreted as a manifestation of the symmetry of space of solutions. A good candidate for this role is non-Noether symmetry. Such a symmetry is a group of transformations that maps the space of solutions of equations of motion onto itself, but unlike the Noether one, does not preserve action.

In the case of a regular Hamiltonian system the phase space is equipped with symplectic form  $\omega$  (closed  $d\omega = 0$  and nondegenerate  $i_X\omega = 0 \rightarrow X = 0$  2-form) and the time evolution is governed by Hamilton’s equation

$$i_{X_h}\omega + dh = 0, \tag{1}$$

where  $X_h$  is the vector field tangent to solutions  $X_h = \sum_i \dot{p}_i \partial_{p_i} + \dot{q}_i \partial_{q_i}$  and  $i_{X_h}\omega$  denotes the contraction of  $X_h$  and  $\omega$ . The vector field is said to be (locally) Hamiltonian if it preserves  $\omega$ . According to Liouville’s theorem,  $X_h$  defined by (1) automatically preserves  $\omega$  (indeed,  $L_{X_h}\omega = di_{X_h}\omega + i_{X_h}d\omega = -ddh = 0$ ).

One can show that the group of transformations of a phase space generated by any non-Hamiltonian vector field  $E$

$$g(a) = e^{aL_E}$$

does not preserve action

$$g_*(A) = g_* \left( \int pdq - hdt \right) = \int g_*(pdq - hdt) \neq 0$$

because  $d(L_E(pdq - hdt)) = L_E\omega - dE(h) \wedge dt \neq 0$  (the first term in the right-hand side does not vanish since  $E$  is non-Hamiltonian and as far as  $E$  is time independent  $L_E\omega$  and  $dE(h) \wedge dt$  are linearly independent 2-forms). As a result, every non-Hamiltonian vector field  $E$  commuting with  $X_h$  leads to non-Noether symmetry (since  $E$  preserves the vector field tangent to solutions  $L_E(X_h) = [E, X_h] = 0$ , it maps the space of solutions onto itself). Any such symmetry yields the following integrals of motion [1–2], [4–5]:

$$l_k = Tr(R^k), \quad k = 1, 2, \dots, n,$$

where  $R = \omega^{-1}L_E\omega$  and  $n$  is half-dimension of the phase space.

It is interesting that for any non-Noether symmetry, the triple  $(h, \omega, \omega_E)$  carries the bi-Hamiltonian structure (§4.12 in [7], [8–10]). Indeed,  $\omega_E$  is a closed ( $d\omega_E = dL_E\omega = L_E d\omega = 0$ ) and an invariant ( $L_{X_h}\omega_E = L_{X_h}L_E\omega = L_EL_{X_h}\omega = 0$ ) 2-form (but generic  $\omega_E$  is degenerate). So every non-Noether symmetry quite naturally endows a dynamical system with the bi-Hamiltonian structure.

Now let us discuss how non-Noether symmetry can be recovered from a bi-Hamiltonian system. The generic bi-Hamiltonian structure on a phase space consists of a Hamiltonian system  $h, \omega$  and an auxiliary closed 2-form  $\omega^\bullet$  satisfying  $L_{X_h}\omega^\bullet = 0$ . Let us call it the global bi-Hamiltonian structure whenever  $\omega^\bullet$  is exact (there exists a 1-form  $\theta^\bullet$  such that  $\omega^\bullet = d\theta^\bullet$ ) and  $X_h$  is (globally) a Hamiltonian vector field with respect to  $\omega^\bullet$  ( $i_{X_h}\omega^\bullet + dh^\bullet = 0$ ). In the local coordinates  $\theta^\bullet = \theta_i^\bullet dz^i$ . As far as  $\omega$  is nondegenerate, there exists a vector field  $E^\bullet = E^{\bullet i} \partial_{z^i}$  such that  $i_{E^\bullet}\omega = \theta^\bullet$  (in the local coordinates  $E^{\bullet i} = (\omega^{-1})^{ij}\theta_j^\bullet$ ). By construction,

$$L_{E^\bullet}\omega = \omega^\bullet.$$

Indeed,  $L_{E^\bullet}\omega = di_{E^\bullet}\omega + i_{E^\bullet}d\omega = d\theta^\bullet = \omega^\bullet$  and

$$i_{[E^\bullet, X_h]}\omega = L_{E^\bullet}(i_{X_h}\omega) - i_{X_h}L_{E^\bullet}\omega = -d(E^\bullet(h) - h^\bullet) = -dh^\bullet.$$

In other words,  $[X_h, E^\bullet]$  is a Hamiltonian vector field, i.e.,  $[X_h, E] = X_{h'}$ . So  $E^\bullet$  is not a generator of symmetry since it does not commute with  $X_h$  but one can construct (locally) the Hamiltonian counterpart of  $E^\bullet$  (note that  $E^\bullet$  itself is non-Hamiltonian) –  $X_g$  with

$$g(z) = \int_0^t h' dt. \tag{2}$$

Here the integration along the solution of Hamilton's equation with fixed origin and end point in  $z(t) = z$  is assumed. Note that (2) defines  $g(z)$  only locally and, as a result,  $X_g$  is a locally Hamiltonian vector field satisfying, by construction, the same commutation relations as  $E^\bullet$  (namely,  $[X_h, X_g] = X_{h'}$ ). Finally,

one recovers the generator of non-Noether symmetry, i.e., the non-Hamiltonian vector field  $E = E^\bullet - X_g$  commuting with  $X_h$  and satisfying

$$L_E\omega = L_{E^\bullet}\omega - L_{X_g}\omega = L_{E^\bullet}\omega = \omega^\bullet$$

(thanks to Liouville's theorem  $L_{X_g}\omega = 0$ ). So in the case of a regular Hamiltonian system every global bi-Hamiltonian structure is naturally associated with the (non-Noether) symmetry of space of solutions.

**Example 1.** As a toy example one can consider a free particle

$$h = \frac{1}{2} \sum_i p_i^2, \quad \omega = \sum_i dp_i \wedge dq_i.$$

This Hamiltonian system can be extended to the bi-Hamiltonian one

$$h, \omega, \omega^\bullet = \sum_i p_i dp_i \wedge dq_i.$$

Clearly,  $d\omega^\bullet = 0$  and  $X_h = \sum_i p_i \partial_{q_i}$  preserves  $\omega^\bullet$ . The conserved quantities  $p_i$  are associated with this simple bi-Hamiltonian structure. This system can be obtained from the following (non-Noether) symmetry (infinitesimal form):

$$\begin{aligned} q_i &\rightarrow (1 + ap_i)q_i, \\ p_i &\rightarrow (1 + ap_i)p_i \end{aligned}$$

generated by  $E = \sum_i p_i q_i \partial_{q_i} + \sum_i p_i^2 \partial_{p_i}$

**Example 2.** The earliest and probably the most well-known bi-Hamiltonian structure is the one discovered by F. Magri and associated with the Korteweg–de Vries integrable hierarchy. The KdV equation

$$u_t + u_{xxx} + uu_x = 0$$

(zero boundary conditions for  $u$  and its derivatives are assumed) appears to be Hamilton's equation

$$i_{X_h}\omega + dh = 0,$$

where  $X_h = \int_{-\infty}^{+\infty} dx u_t \partial_u$  (here  $\partial_u$  denotes a variational derivative with respect to the field  $u(x)$ ) is the vector field tangent to the solutions,

$$\omega = \int_{-\infty}^{+\infty} dx du \wedge dv$$

is a symplectic form (here  $v = \partial_x^{-1}u$ ) and the function

$$h = \int_{-\infty}^{+\infty} dx \left( \frac{u^3}{3} - u_x^2 \right)$$

plays the role of a Hamiltonian. This dynamical system possesses a non-trivial symmetry – a one-parameter group of non-cannonical transformations  $g(a) =$

$e^{L_E}$  generated by the non-Hamiltonian vector field

$$E = \int_{-\infty}^{+\infty} dx \left( u_{xx} + \frac{1}{2} u^2 \right) \partial_u + X_F;$$

here the first term represents the non-Hamiltonian part of the generator of symmetry, while the second one is its Hamiltonian counterpart associated with

$$F = \int_{-\infty}^{+\infty} \left( \frac{1}{12} u^2 v + \frac{1}{4} \partial_x^{-1} \left( \frac{u^3}{3} - u_x^2 \right) + \frac{3}{4} v \frac{l_3}{l_2} \right) dx$$

( $l_{2,3}$  are defined in (3)). The physical origin of this symmetry is unclear, however the symmetry seems to be very important since it leads to the well-known infinite sequence of conservation laws in involution:

$$\begin{aligned} l_1 &= \int_{-\infty}^{+\infty} u dx, \\ l_2 &= \int_{-\infty}^{+\infty} u^2 dx, \\ l_3 &= \int_{-\infty}^{+\infty} \left( \frac{u^3}{3} - u_x^2 \right) dx, \\ l_4 &= \int_{-\infty}^{+\infty} \left( \frac{5}{36} u^4 - \frac{5}{3} u u_x^2 + u_{xx}^2 \right) dx, \\ &\dots \end{aligned} \tag{3}$$

and ensures the integrability of the KdV equation. The second Hamiltonian realization of the KdV equation discovered by F. Magri [8]

$$i_{X_h} \omega^\bullet + dh^\bullet = 0$$

(where  $\omega^\bullet = L_E \omega$  and  $h^\bullet = L_E h$ ) is a result of the invariance of the KdV equation under the aforementioned transformations  $g(a)$ .

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